Discussiones Mathematicae
Graph Theory 39 (2019) 575–588
doi:10.7151/dmgt.2102

1-RESTRICTED OPTIMAL RUBBLING ON GRAPHS

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Abstract

Let G be a graph with vertex set V and a distribution of pebbles on the vertices of V. A pebbling move consists of removing two pebbles from a vertex and placing one pebble on a neighboring vertex, and a rubbling move consists of removing a pebble from each of two neighbors of a vertex v and placing a pebble on v. We seek an initial placement of a minimum total number of pebbles on the vertices in V, so that no vertex receives more than one pebble and for any given vertex $v \in V$, it is possible, by a sequence of pebbling and rubbling moves, to move at least one pebble to v. This minimum number of pebbles is the 1-restricted optimal rubbling number. We determine the 1-restricted optimal rubbling numbers for Cartesian products. We also present bounds on the 1-restricted optimal rubbling number.

Keywords: graph pebbling, graph rubbling, optimal rubbling, t-restricted optimal pebbling.

2010 Mathematics Subject Classification: 05C78.

1. Introduction

We consider a variation of graph rubbling. Let G be a graph with vertex set V. A placement of pebbles on the vertices such that each vertex of V is assigned a non-negative integer number of pebbles is called a *pebble distribution* on G. Figure 1 gives an example of a pebble distribution where two pebbles are placed on u, one pebble is placed on x, and no pebbles are placed on the other vertices of V.

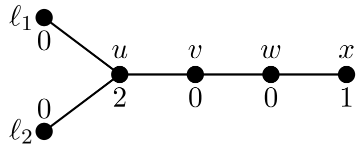

Figure 1. A pebble distribution.

Two moves, namely a *pebbling move* and a *rubbling move*, are defined as follows.

Definition. Let f be a pebble distribution on a graph G such that $f(u) \geq 2$ for some vertex $u \in V$, and let v be adjacent to u. Then a pebbling move, denoted $p(u \to v)$, removes two pebbles from u and places one on v. This defines a new pebble distribution f' such that $f'(u) = f(u) - 2$, $f'(v) = f(v) + 1$, and $f'(z) = f(z)$ for $z \in V \setminus \{u, v\}$.

Definition. Let w be a vertex of G, and let v and x be distinct vertices adjacent to w. Let f be a pebble distribution such that $f(v) \geq 1$ and $f(x) \geq 1$. Then a rubbling move, denoted $r(v, x \to w)$, removes one pebble from each of v and x and places one pebble on w. This defines a new pebble distribution f' such that $f'(v) = f(v) - 1$, $f'(x) = f(x) - 1$, $f'(w) = f(w) + 1$, and $f'(z) = f(z)$ for $z \in V \setminus \{v, x, w\}$.

For the graph in Figure 1, we can place a pebble on ℓ_1, ℓ_2, and v using the pebbling moves $p(u \to \ell_1)$, $p(u \to \ell_2)$, and $p(u \to v)$, respectively. After the move $p(u \to v)$, we can place a pebble on w using the rubbling move $r(v, x \to w)$.

In graph pebbling, only the pebbling move is allowed; while in graph rubbling both pebbling and rubbling moves are available. For graph rubbling, a vertex v

is *reachable* if there is a way to place a pebble on v using a sequence of pebbling and rubbling moves. If every vertex of a graph G is reachable under some pebble distribution f, then we say that f is a *rubbling configuration.* Thus, the example in Figure 1 is a rubbling configuration. The *rubbling number* of a graph G, denoted $\rho(G)$, is the smallest k such that every pebble distribution of k pebbles results in a rubbling configuration. The *optimal rubbling number* $\rho^*(G)$ is the smallest number of pebbles required for some rubbling configuration on a graph G. Rubbling and optimal rubbling were introduced in [2] and studied in [7, 8]. Optimal rubbling was generalized to t-restricted optimal rubbling in [1] as follows.

Definition. The *t-restricted optimal rubbling* number $\rho_t^*(G)$ of a graph G is the least k such that there exists some rubbling configuration f on k pebbles where for each $v \in V$, $f(v) \in \{0, 1, \ldots, t\}$. A rubbling configuration of G having $\rho_t^*(G)$ pebbles is called a ρ_t^*-configuration.

The focus in this paper is the 1-restricted rubbling number. This is the minimum number of pebbles required by a rubbling configuration which assigns each vertex in V either no pebbles or one pebble. Note that the restriction of at most one pebble per vertex applies only to the initial distribution of pebbles. Background and preliminary observations are given in Section 2. In Section 3, we determine bounds of the 1-restricted optimal rubbling numbers of Cartesian products of graphs. In Section 4, we give other bounds on the 1-restricted rubbling number.

Before proceeding, some additional terminology is needed. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u : uv \in E\}$, and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v, denoted $\deg(v)$, is $|N(v)|$. A vertex of degree 1 is called a *leaf* and its neighbor is a *support vertex.*

2. Background and Preliminaries

It follows immediately from the definition that assigning a pebble to each vertex is trivially a t-restricted optimal rubbling distribution for all $t \geq 1$. Thus, $\rho_t^*(G)$ is defined for all graphs G, and we have the following observation.

Observation 1 [1]**.** *For any graph G having n vertices,*

$$\rho^*(G) \leq \rho_t^*(G) \leq \rho_{t-1}^*(G) \leq \cdots \leq \rho_2^*(G) \leq \rho_1^*(G) \leq n.$$

The following bounds on the optimal rubbling number were given in [8].

Theorem 2 [8]**.** *For every graph G on n vertices with diameter d, $\lceil \frac{d+2}{2} \rceil \leq \rho^*(G) \leq \lceil \frac{n+1}{2} \rceil$.*

Observation 1 implies the following useful corollary to Theorem 2.

Corollary 3. *For every graph G with diameter d, $\lceil \frac{d+2}{2} \rceil \leq \rho_1^*(G)$.*

As we shall see in Section 3, the upper bound of Theorem 2 also holds for $\rho_1^*(G)$. Let P_n and C_n be the path and cycle on n vertices, respectively. Belford, *et al.* [2] proved that $\rho^*(P_n) = \lceil \frac{n+1}{2} \rceil$ and $\rho^*(C_n) = \lceil \frac{n}{2} \rceil$. Moreover, they gave ρ^*-configurations that placed at most one pebble on any vertex, that is, ρ_1*-configurations. Hence, as noted in [1], the optimal rubbling and 1-restricted optimal rubbling numbers are equal for paths and cycles.

Proposition 4 [1]**.** *For a path P_n, $\rho^*(P_n) = \rho_1^*(P_n) = \lceil \frac{n+1}{2} \rceil$. For a cycle C_n, $\rho^*(C_n) = \rho_1^*(C_n) = \lceil \frac{n}{2} \rceil$.*

Thus, $\rho_1^*(G)$ and $\rho^*(G)$ are the same for some graphs. On the other hand, it was shown in [1] that the difference between $\rho_1^*(G)$ and $\rho_2^*(G)$ can be made arbitrarily large implying that the difference $\rho_1^*(G) - \rho^*(G)$ is also arbitrarily large. To see, this consider the tree G formed as follows. Begin with the path P_{4k-3} with vertex set $p_1, \ldots, p_{4k-3}$. Identify the center of the star K_{1,n_i}, where $n_i \geq 3$, with the vertex p_{4i+1} on the path for $i = 0, 1, \ldots, k-1$. For a ρ_1^*-configuration, place a pebble on the vertex p_{4i+1} and a pebble on each of two leaf neighbors of p_{4i+1}, and for a ρ_2^*-configuration, place two pebbles on the vertex p_{4i+1} for $i = 0, 1, \ldots, k-1$. Hence, $\rho_1^*(G) = 3k$ and $\rho_2^*(G) = 2k$, and so, $\rho_1^*(G) - \rho_2^*(G) = k$. Since $\rho^*(G) \leq \rho_2^*(G)$, the gap between $\rho_1^*(G)$ and $\rho^*(G)$ is at least k.

For our main results, we will need the 1-restricted optimal rubbling number for other classes of graphs. Let K_n denote the complete graph on n vertices. The star is the complete bipartite graph $K_{1,n}$, and the double star $S_{r,s}$ is a tree with exactly two non-leaf vertices, one of which is adjacent to r leaves and the other to s leaves. The following values were determined in [1].

Theorem 5 [1]**.** (i) *For the complete graph K_n, where $n \geq 3$, $\rho_1^*(K_n) = 2$.*

(ii) *For the star $K_{1,n}$, where $n \geq 3$, $\rho_1^*(K_{1,n}) = 3$.*

(iii) *Let $S_{r,s}$ be a double star with $1 \leq r \leq s$. Then $\rho_1^*(S_{r,s}) = 3$ if $r = 1$ and $s \leq 2$, and $\rho_1^*(S_{r,s}) = 4$ otherwise.*

It is natural to assume that if G is a proper subgraph of H, then $\rho_1^*(G) \leq \rho_1^*(H)$. This assumption proves to be false, even for connected graphs, as seen in Figure 2, where G is a proper subgraph of H, $\rho_1^*(G) = 5$, and $\rho_1^*(H) = 4$.

The following useful result was also given in [1].

Theorem 6 [1]**.** *If G is a graph of order n and maximum degree $\Delta(G)$, then $\rho_1^*(G) \leq n - \Delta(G) + 2$.*

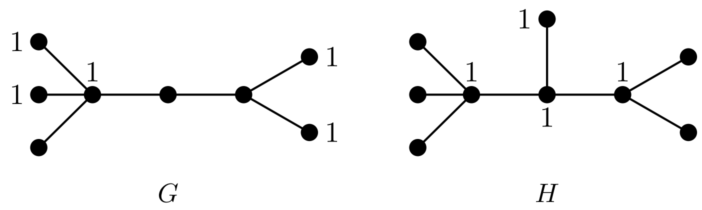

Figure 2. An example where $G \subset H$, but $\rho_1^*(G) > \rho_1^*(H)$.

We conclude this section by giving bounds on $\rho_1^*(G)$ in terms of the domination number and the 2-domination number. The *domination number* of a graph G, denoted $\gamma(G)$, is the minimum cardinality among all subsets S of V such that each vertex in $V \setminus S$ is adjacent to at least one vertex in S. The 2-*domination number* of a graph G, denoted $\gamma_2(G)$, is the minimum cardinality among all subsets S of V such that each vertex in $V \setminus S$ is adjacent to at least two vertices in S.

Observation 7. *For every graph G, $\rho_1^*(G) \leq \gamma_2(G)$.*

We note that for the path P_3, $\rho_1^*(P_3) = \gamma_2(P_3) = 2$. On the other hand, the difference $\rho_1^*(G) - \gamma_2(G)$ can be made arbitrarily large with stars $K_{1,n}$ for $n \geq 3$, because $\rho_1^*(K_{1,n}) = 3$ and $\gamma_2(K_{1,n}) = n - 1$.

For a set S of vertices and a vertex $v \in S$, the vertices in $N(v) \cap (V \setminus S)$ whose only neighbor in S is v are called the *external private neighbors* of v (with respect to S). Let $epn(v, S)$ denote the set of external private neighbors of v with respect to S.

Theorem 8. *For every graph G, $\rho_1^*(G) \leq 3\gamma(G)$.*

Proof. Let S be a minimum dominating set of G. Place a pebble on each vertex in S. Let $v \in S$. If $|epn(v, S)| = 1$, then place a pebble on the external private neighbor of v. If $|epn(v, S)| \geq 2$, then choose exactly two external private neighbors of v and place a pebble on each. This distribution of at most $3\gamma(G)$ pebbles is 1-restricted rubbling configuration. To see this, let $x \in V$ be a vertex with no pebble. Since S is a dominating set of G, x has a neighbor in S. If x has at least two neighbors in S, then a rubbling move from these two neighbors places a pebble on x. Thus, we can assume $x \in epn(v, S)$ for some $v \in S$. The way we distributed the pebbles implies that $|epn(v, S)| \geq 3$ and two vertices, say w and y, in $epn(v, S) \setminus \{x\}$ have one pebble each. Then the rubbling move $r(w, y \to v)$ results in two pebbles on v, and a pebbling move $p(v \to x)$ places a pebble on x, and so, $\rho_1^*(G) \leq 3\gamma(G)$. ■

Interestingly, while stars $K_{1,n}$ for $n \geq 3$ provide a case where $\rho_1^*(K_{1,n})$ is smaller than $\gamma_2(K_{1,n})$, $\rho_1^*(K_{1,n}) = 3\gamma(K_{1,n})$.

3. Cartesian Products

The Cartesian product $G \Box H$ of graphs G and H has vertex set $V(G) \times V(H)$ and (g, h) is adjacent to (g', h') if and only if $g = g'$ and $hh' \in E(H)$ or $h = h'$ and $gg' \in E(G)$. The optimal rubbling number of a prism $C_n \Box P_2$ is determined in [7].

Theorem 9 [7]**.** *The optimal rubbling number for the prism $C_n \Box P_2$ is as follows $\rho^*(C_3 \Box P_2) = 3$ and $\rho^*(C_n \Box P_2) = \left\lceil \frac{2n}{3} \right\rceil$ for $n \geq 4$.*

In the proof to Theorem 9, a ρ^*-configuration is given for $C_n \Box P_2$ that places at most one pebble on any vertex when $n \equiv 0, 2 \pmod 3$. Hence, the optimal rubbling and 1-restricted optimal rubbling numbers are equal for these cases. Our next result shows that, with the exception $n = 4$, equality holds when $n \equiv 1 \pmod 3$ as well.

Theorem 10. *The 1-restricted optimal rubbling number for the prism $C_n \Box P_2$ is as follows $\rho_1^*(C_3 \Box P_2) = 3$, $\rho_1^*(C_4 \Box P_2) = 4$, and $\rho_1^*(C_n \Box P_2) = \left\lceil \frac{2n}{3} \right\rceil$ for $n \geq 5$.*

Proof. It is straightforward to check that $\rho_1^*(C_3 \Box P_2) = 3$ and $\rho_1^*(C_4 \Box P_2) = 4$. As previously mentioned, the proof of Theorem 9 gives the result for $n \equiv 0, 2 \pmod 3$. Hence, we may assume that $n \equiv 1 \pmod 3$ and $n \geq 7$. Since $\rho^*(G) \leq \rho_1^*(G)$, we have that $\rho^*(C_n \Box P_2) = \left\lceil \frac{2n}{3} \right\rceil \leq \rho_1^*(C_n \Box P_2)$. It suffices to give a 1-rubbling configuration of $C_n \Box P_2$ using $\left\lceil \frac{2n}{3} \right\rceil$ pebbles. We note that this configuration works for $n \equiv 0, 2 \pmod 3$ as well.

For ease of discussion, let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of one copy of C_n and let $u_0, u_1, \ldots, u_{n-1}$ be the vertices of the other copy of C_n such that $u_i v_i \in E(G)$ for $0 \leq i \leq n-1$. Assume that all computations on the indices are done modulo three. Place a pebble on each v_i for $i \equiv 0 \pmod 3$ and each u_j for $j \equiv 1 \pmod 3$. Then the number of pebbles placed is $\left\lceil \frac{2n}{3} \right\rceil$. We claim that this distribution is a 1-restricted rubbling configuration. Consider a vertex u_i that has no pebble. Then $i \equiv 0, 2 \pmod 3$. If $i \equiv 0 \pmod 3$ and $i \neq n-1$, then u_{i+1} and v_i begin with a pebble and a rubbling move $r(u_{i+1}, v_i \to u_i)$ will reach u_i. If $i \equiv 0 \pmod 3$ and $i = n-1$, then the rubbling move $r(v_0, u_1 \to u_0)$ followed by the rubbling move $r(u_0, v_i \to u_i)$ reaches u_i.

If $i \equiv 2 \pmod 3$ and $i \neq n-2$, then the rubbling move $r(u_{i+2}, v_{i+1} \to u_{i+1})$ places a pebble on u_{i+1}. A subsequent rubbling move $r(u_{i-1}, u_{i+1} \to u_i)$ will reach u_i. If $i \equiv 2 \pmod 3$ and $i = n-2$, then the following sequence of rubbling moves result in a pebble placed on u_i: $r(v_0, u_1 \to u_0)$, $r(u_0, v_{i+1} \to u_{i+1})$, and $r(u_{i-1}, u_{i+1} \to u_i)$.

Next consider a vertex v_i that has no pebble. Then $i \equiv 1, 2 \pmod 3$. If $i \equiv 1 \pmod 3$, then the rubbling move $r(v_{i-1}, u_i \to v_i)$ places a pebble on v_i. If $i \equiv 2 \pmod 3$, then the rubbling move $r(v_{i-2}, u_{i-1} \to v_{i-1})$, followed by the rubbling move $r(v_{i-1}, v_{i+1} \to v_i)$ reaches v_i. ■

Graham's Conjecture asserts that the pebbling number of a Cartesian product is at most the product of the pebbling numbers of the graphs in that product. This conjecture has received a lot of attention in the literature [3, 4, 6, 9], but remains unresolved. It is shown in [2] that the rubbling analog of the conjecture does not hold. On the other hand, in [5], it is proven that the conjecture holds for optimal pebbling. Since optimal pebbling is optimal rubbling sans the rubbling move, it follows that Graham's conjecture holds for optimal rubbling. We show that the conjecture holds even with the added restriction of at most one pebble per vertex in the initial rubbling configuration.

Let (g,h) be the vertex of $G\Box H$ corresponding to $g \in V(G)$ and $h \in V(H)$. For $h \in V(H)$, let G_h denote the copy of G in $G\Box H$ induced by the set of vertices $\{(g,h) : g \in V(G)\}$. Analogously, H_g denotes the copy of H in $G\Box H$ induced by the set of vertices $\{(g,h) : h \in V(H)\}$.

Theorem 11. *For graphs G and H, $\rho_1^*(G\Box H) \le \rho_1^*(G)\rho_1^*(H)$.*

Proof. Let $\rho_1^*(G) = k$ and $\rho_1^*(H) = m$. Let f_G be a ρ_1^*-configuration of G and f_H be a ρ_1^*-configuration of H. Let $\{g_1, \ldots, g_k\}$ and $\{h_1, \ldots, h_m\}$ be the vertices containing a pebble under f_G and f_H, respectively.

Consider the pebble distribution f on $G\Box H$ obtained by placing one pebble on each vertex in the set $\{(g_i, h_j) : i = 1, \ldots, k, j = 1, \ldots, m\}$. Clearly, every vertex in G_{h_j} is reachable for all j as $\{(g_i, h_j) : i = 1, \ldots, k\}$ is a rubbling configuration on G_{h_j}. Similarly, every vertex in H_{g_i} is reachable for all i.

Suppose that $(g,h) \in V(G\Box H)$ such that (g,h) has no pebble under f. Since f_H is a rubbling configuration on H, we can perform rubbling and pebbling moves on H_{g_i} to place a pebble on (g_i, h) for each $i = 1, \ldots, k$. Since f_G is a rubbling configuration on G, we can perform rubbling and pebbling moves on G_h to place a pebble on (g,h).

Hence, f is a rubbling configuration of $G\Box H$ containing $km = \rho_1^*(G)\rho_1^*(H)$ pebbles, and so, $\rho_1^*(G\Box H) \le \rho_1^*(G)\rho_1^*(H)$. ■

Note that $\rho_1^*(P_2) = 2$, and by Proposition 4, $\rho_1^*(C_n) = \lceil \frac{n}{2} \rceil$. From Theorem 10, the bound of Theorem 11 is sharp for $C_4\Box P_2$. However, by Theorem 10, $\rho_1^*(C_n\Box P_2) = \lceil 2n/3 \rceil$ for $n \ge 5$. Thus, for $n \ge 5$, $\rho_1^*(P_2)\rho_1^*(C_n) - \rho_1^*(P_2\Box C_n) = 2\lceil \frac{n}{2} \rceil - \lceil \frac{2n}{3} \rceil \approx \frac{n}{3}$. Hence, this difference can be made arbitrarily large.

Our next result deals with prisms. Recall that $\gamma(G)$ is the domination number of G.

Theorem 12. *For any graph G, $\rho_1^*(G\Box P_2) \le \gamma(G) + \rho_1^*(G)$.*

Proof. Label the two copies of G created by $G\Box P_2$ as G_1 and G_2. Place pebbles on G_1 in a ρ_1^*-configuration of G. Then each vertex in G_1 is reachable. Now place one pebble on each vertex in some minimum dominating set S of G_2. Let v be a vertex in G_2 having no pebble and let v' be the neighbor of v in G_1. Then v' is reachable by some sequence of pebbling and rubbling moves using only pebbles on vertices in G_1. Moreover, since S is a dominating set of G_2, v has a neighbor, say u, in S. Hence, after v' is reached in G_1, the rubbling move $r(u, v' \to v)$ reaches v. Thus, this distribution of pebbles is a rubbling configuration using $\gamma(G) + \rho_1^*(G)$ pebbles, and so, $\rho_1^*(G \Box P_2) \leq \gamma(G) + \rho_1^*(G)$. ■

4. Bounds

We begin with an upper bound on the 1-restricted rubbling number of a tree in terms of its order and number of leaves. A non-leaf vertex of a tree is called an *internal vertex*.

Theorem 13. *If T is a tree on n vertices with ℓ leaves, then $\rho_1^*(T) \leq n - \ell + 2$. Further, this bound is sharp.*

Proof. Let T be a tree of order n with ℓ leaves. If T is the trivial graph or the P_2, the bound clearly holds. If T is the path P_n for $n \geq 3$, then by Proposition 4, $\rho_1^*(T) = \lceil \frac{n+1}{2} \rceil < n = n - \ell + 2$. Hence, we may assume that $T \neq P_n$, $n \geq 4$, and $\ell \geq 3$. If T is a star of order $n \geq 4$, then by Theorem 5, $\rho_1^*(T) = 3 = n - (n-1) + 2 = n - \ell + 2$. Thus, this bound is sharp for stars. We may assume that T is not a star, that is, T has at least two internal vertices. We build a 1-restricted rubbling configuration of T as follows. Begin by placing a pebble on each internal vertex. We consider two cases.

Case 1. T has a support vertex v that is adjacent to two internal vertices.

Let x and y be two internal vertex neighbors of v, and let w be a support vertex of T different from v. Complete the initial configuration by placing a pebble on a leaf neighbor, say u, of v. Since there is a unique path via internal vertices from v to any other support vertex, there exists a vw-path that does not include at least one of x and y, say x. Hence, a rubbling move $r(u, x \to v)$ places two pebbles on v. Then a sequence of pebbling moves on the $v-w$-path will result in two pebbles on w. Hence, the leaves adjacent to w are reachable. Since w is an arbitrary support vertex, it follows that this is a 1-rubbling configuration of T. Hence, $\rho_1^*(T) \leq n - \ell + 1 < n - \ell + 2$.

Case 2. Every support vertex is adjacent to exactly one internal vertex.

First assume that T has a vertex v that is adjacent to at least two leaves. In this case, complete the initial configuration by placing one pebble on each of two leaf neighbors of v. Then a rubbling move between these two leaves results in two pebbles on v. Repeating the argument in case one shows that this is a 1-rubbling configuration. Hence, $\rho_1^*(T) \leq n - \ell + 2$.

Henceforth, we can assume that every support vertex of T is adjacent to exactly one leaf and exactly one internal vertex, that is, every support vertex of T has degree two. Since $T \neq P_n$, it follows that T has at least one internal vertex, say x, with degree three or more. Note that x is not a support vertex by assumption. Hence, every neighbor of x is an internal vertex of T. Recall that there is a unique path from x to any support vertex, say v. Since x has degree three or more, at least two neighbors of x, say y and z, are not on the $x-v$-path. Moreover, y and z are internal vertices, so they each have one pebble. A rubbling move $r(y, z \to x)$ places two pebbles on x, and a sequence of pebbling moves results in two pebbles on v. Hence, the leaf neighbors of v can be pebbled. Since v is an arbitrary support vertex, it follows that this is a 1-rubbling configuration of T. Hence, $\rho_1^*(T) \leq n - \ell < n - \ell + 2$. ∎

We next characterize the trees achieving the bound of Theorem 13. To do this, we define two families of graphs. Let $S'_{r,s}$ be the tree formed from the double star $S_{r,s}$ by subdividing the edge between the two non-leaf vertices exactly once, and let $S''_{r,s}$ be the tree formed from the double star $S_{r,s}$ by subdividing the edge between the two non-leaf vertices exactly twice. Let $\mathcal{F}$ be the family of trees consisting of non-trivial stars $K_{1,m}$ for $m \neq 2$, double stars $S_{r,s}$ for $1 \leq r \leq s$ and $r + s \geq 4$, the trees $S'_{r,s}$ for $r \geq 2$ and $s \geq 3$, and the trees $S''_{r,s}$ for $3 \leq r \leq s$.

Theorem 14. *A tree T with $n \geq 2$ vertices and ℓ leaves has $\rho_1^*(T) = n - \ell + 2$ if and only if $T \in \mathcal{F}$.*

Proof. Assume that T is a tree of order $n \geq 2$ with ℓ leaves having $\rho_1^*(T) = n - \ell + 2$. First if $n = 2$, then $T \in \mathcal{F}$. We note that $n \neq 3$, for otherwise, $T = P_3$ and $\rho_1^*(T) = 2 < n - 2 + 2$. Hence, we can assume that $n \geq 4$.

The proof of Theorem 13 implies that every internal, non-support vertex of T has degree two. Moreover, every support vertex of T is adjacent to at most one internal vertex. If $T = K_{1,m}$ for $m \geq 3$, then $\rho_1^*(T) = 3 = n - (n-1) + 2$ and $T \in \mathcal{F}$. Thus, we can assume that every support vertex is adjacent to exactly one internal vertex. It follows that T is a tree with exactly two support vertices, say x and y, connected by the $x-y$-path where every vertex except possibly x and y on this path has degree two in T. Let $x = u_0, u_1, \ldots, u_k = y$ be the vertices on the $x-y$-path in T.

By Theorem 4, $\rho^*(P_n) = \rho_1^*(P_n) = \left\lceil \frac{n+1}{2} \right\rceil < n = n - 2 + 2 = n - \ell + 2$. Thus we can assume that at least one of x and y has degree three or more. If $k = 1$, then x and y are adjacent and T is the double star $S_{r,s}$ for $1 \leq r \leq s$. By Theorem 5, $\rho_1^*(S_{r,s}) = 3 < n - \ell + 2$ if $r = 1$ and $s \leq 2$, and $\rho_1^*(S_{r,s}) = 4 = n - \ell + 2$ otherwise, so the result holds. Hence, we may assume that $k \geq 2$. If $k = 2$, then T is the tree $S'_{r,s}$. It is straightforward to check that $\rho_1^*(S'_{r,s}) = 5 = n - \ell + 2$ if and only if $r \geq 2$ and $s \geq 3$. Hence, we can assume that $k \geq 3$. If $k \geq 4$, then placing one pebble on each u_i except for $i = 2$, one pebble on a leaf adjacent to

$u_0 = x$ and one on a leaf adjacent to y gives a 1-restricted rubbling configuration having at most $n - \ell - 1 + 2 = n - \ell + 1 < n - \ell + 2$ pebbles. Therefore, we can assume that $k = 3$. Thus, T is the tree $S''_{r,s}$. It is straightforward to check that $\rho_1^*(S''_{r,s}) = 6 = n - \ell + 2$ if and only if $3 \leq r \leq s$. ■

Our next result involves trees T with a given diameter, denoted diam(T), and with centers having large degree.

Theorem 15. *Let T be a tree with center u if* $\mathrm{diam}(T)$ *is even, and centers u and v if* $\mathrm{diam}(T)$ *is odd. Let $k = \lfloor \mathrm{diam}(T)/2 \rfloor$.*

(1) *If* $\mathrm{diam}(T) = 2k$ *and* $\deg(u) \geq 2^{k+1} - 5$*, then* $\rho_1^*(T) \leq 2^{k+1} - 1$.

(2) *If* $\mathrm{diam}(T) = 2k + 1$, $\deg(u) \geq 2^{k+1} - 5$*, and* $\deg(v) \geq 2^k - 2$*, then* $\rho_1^*(T) \leq 2^{k+1} + 2^k - 2$.

Proof. (1) Let T be a tree with diameter $2k$. If $\mathrm{diam}(T) = 2$, then T is a star and the result holds, so we can assume that $k \geq 2$. Then the center of T is a unique vertex u. Let $\deg(u) = n$ and the neighbors of u be $u_1,\ldots,u_n$.

Suppose that $2^{k+1} - 5 \leq n \leq 2^{k+1} - 2$. For our initial configuration, place one pebble on each of u, $u_1,\ldots,u_n$. Let w be a vertex in T having no pebble. Without loss of generality, assume that the shortest path from u to w passes through u_n. Using rubbling moves to move pebbles from $u_1,\ldots,u_{n-1}$ onto u, it is possible to move an additional $\lfloor \frac{n-1}{2} \rfloor$ pebbles to u. This results in $\lfloor \frac{n+1}{2} \rfloor$ pebbles on u. Using pebbling moves from u to u_n can result in $\lfloor \frac{n+1}{4} \rfloor + 1$ pebbles on u_n. Since $n \geq 2^{k+1} - 5$, it follows that $\lfloor \frac{n+1}{4} \rfloor + 1 \geq 2^{k-1}$. Hence, we can reach any vertex on the branch rooted at u_n. It follows that this placement of pebbles is a 1-restricted optimal rubbling configuration of T using $1 + n \leq 1 + 2^{k+1} - 2 = 2^{k+1} - 1$ pebbles. Therefore, $\rho_1^*(T) \leq 2^{k+1} - 1$.

Suppose that $n \geq 2^{k+1} - 1$. For our initial configuration, place a pebble on each of u, $u_1,\ldots,u_{2^{k+1}-2}$. Again, using rubbling move, u can obtain 2^k pebbles. Hence, it is possible to reach any vertex in the graph using a series of pebbling moves. Again, we used $2^{k+1} - 1$ pebbles in our initial configuration, so the bound holds for trees having even diameter.

(2) Suppose that T is a tree of diameter $2k + 1$ with center vertices u and v such that $\deg(u) \geq 2^{k+1} - 5$ and $\deg(v) \geq 2^k - 2$. Note that the bound $\rho_1^*(T) \leq 2^{k+1} + 2^k - 2$ was shown to hold true for trees of diameter three (i.e., double stars) in [1]. For this reason, we will assume that $k \geq 2$. Suppose that the non-center neighbors of u are $u_1,\ldots,u_n$, and that the non-center neighbors of v are $v_1,\ldots,v_m$. If $2^{k+1} - 6 \leq n \leq 2^{k+1} - 2$ and $2^k - 3 \leq m \leq 2^k - 2$, then begin by placing a pebble on each of u, v, $u_1,\ldots,u_n$, $v_1,\ldots,v_m$. Let w be a vertex having no pebble. If w is a vertex in the branch rooted at u_n, then move the needed pebbles from $u_1,\ldots,u_{n-1}$, and v to u using rubbling moves. This results in u having $\lfloor \frac{n+2}{2} \rfloor$ pebbles. Move the needed pebbles from u to u_n using pebbling

moves. This results in $\lfloor \frac{n+6}{4} \rfloor$ pebbles on u_n. Since $n \geq 2^{k+1}-6$, there are at least 2^{k-1} pebbles on u_n. Hence, w can be reached by pebbling moves. Similarly, if w is the branch rooted at v_m, we move pebbles from $u_1,\ldots,u_n$ to u using rubbling moves. We then move pebbles from u to v using pebbling moves. We also move pebbles from $v_1,\ldots,v_{m-1}$ (and possibly u) to v using rubbling moves. This results in $\lfloor \frac{n+2m+4}{4} \rfloor$ pebbles on v. Moving pebbles from v to v_m using pebbling moves results in $\lfloor \frac{n+2m+12}{8} \rfloor$ pebbles on v_m. Since $n \geq 2^{k+1}-6$ and $m \geq 2^k - 3$, there are at least 2^{k-1} pebbles on v_m. Hence, we can reach any vertex in the branch rooted at v_m. Note that we used $2+n+m$ pebbles in our initial configuration. Since $n \leq 2^{k+1}-2$ and $m \leq 2^k - 2$, it follows that $\rho_1^*(T) \leq 2^{k+1}+2^k-2$.

A similar argument gives the remaining cases. For this reason, we simply provide the required initial rubbling configurations and leave the details of the proof to the reader. If $n \geq 2^{k+1}-1$ and $2^k-3 \leq m \leq 2^k-2$, then place one pebble on each of u, v, $u_1,\ldots,u_{2^{k+1}-2}$, $v_1,\ldots,v_m$ in the initial configuration. If $2^{k+1}-6 \leq n \leq 2^{k+1}-5$ and $m = 2^k-1$, then place one pebble on each of u, v, $u_1,\ldots,u_n$, $v_1,\ldots,v_m$ in the initial configuration. If $2^{k+1}-6 \leq n \leq 2^{k+1}-5$ and $m \geq 2^k$, then place one pebble on each of u, v, $u_1,\ldots,u_n$, $v_1,\ldots,v_{2^k}$ in the initial configuration. If $2^{k+1}-4 \leq n \leq 2^{k+1}-2$ and $m = 2^k-1$, then place one pebble on each of u, v, $u_1,\ldots,u_{2^{k+1}-4}$, $v_1,\ldots,v_m$ in the initial configuration. If $2^{k+1}-4 \leq n \leq 2^{k+1}-2$ and $m \geq 2^k$, then place one pebble on each of u, v, $u_1,\ldots,u_{2^{k+1}-4}$, $v_1,\ldots,v_{2^k}$ in the initial configuration. If $n \geq 2^{k+1}-1$ and $m \geq 2^k-1$, then place one pebble on each of u, v, $u_1,\ldots,u_{2^{k+1}-2}$, $v_1,\ldots,v_{2^k-2}$ in the initial configuration. ■

In their work on optimal rubbling, Katona and Sieben [8] proved that for any connected graph G of order n, $\rho^*(G) \leq \lceil \frac{n+1}{2} \rceil$. They showed that the path requires the most pebbles for optimal rubbling amongst the graphs with a given number of vertices. However, their proof uses rubbling configurations that initially place more than one pebble on each the vertex, so it does not extend to give the upper bound for 1-restricted rubbling. But our next result shows the bound does indeed hold for the 1-restricted rubbling number of connected graphs G. Since deleting an edge cannot decrease the 1-restricted optimal rubbling number, it suffices to prove the result for trees.

Theorem 16. *For any tree T on n vertices, $\rho_1^*(T) \leq \lceil \frac{n+1}{2} \rceil$.*

Proof. Let T be a tree on n vertices. We proceed by induction on n. If $n \leq 3$, the result holds. By Theorem 5, the claim holds for stars and double stars with at least four vertices. Hence, we may assume that the diameter of T is at least four and $n \geq 5$.

Assume for any tree T' with order $n' < n$, that $\rho_1^*(T') \leq \left\lceil \frac{n'+1}{2} \right\rceil$. Root T at some leaf r, and let leaf w be of maximum distance from r. Let v be the parent of w, and let u be the parent of v. We consider three cases.

Case 1. Vertex v has exactly one child, w. Let $T' = T - \{v, w\}$. Then $n' = n - 2$. From our inductive hypothesis, there is a ρ_1^*-configuration $f(T')$ of T', such that $|f(T')| \le \left\lceil \frac{(n-2)+1}{2} \right\rceil$. In this rubbling configuration of T', u can obtain a pebble. Thus, $f(T')$ can be extended to a rubbling configuration of T by placing a pebble on w. Note that a rubbling move $r(u, w \to v)$ places a pebble on v. Thus, $\rho_1^*(T) \le \left\lceil \frac{(n-2)+1}{2} + 1 \right\rceil = \left\lceil \frac{n+1}{2} \right\rceil$.

Case 2. Vertex v has exactly two children, say x and w. By our choice of w, x is a leaf in T. Define $T' = T - \{w, x\}$. Then from our inductive hypothesis, there exists a ρ_1^*-configuration $f(T')$ of T' such that $f(T') \le \left\lceil \frac{(n-2)+1}{2} \right\rceil$. We are now presented with two subcases.

(a) In the configuration $f(T')$, v begins with a pebble. Place pebbles on T in the form of $f(T')$. Remove the pebble from v and place one pebble on each of w and x. A rubbling move $r(w, x \to v)$ places a pebble on v, giving the previous configuration $f(T')$. Hence, this is a rubbling configuration for T, implying that $\rho_1^*(T) \le \left\lceil \frac{(n-2)+1}{2} + 1 \right\rceil = \left\lceil \frac{n+1}{2} \right\rceil$.

(b) In $f(T')$, v does not begin with a pebble. Thus, there must exist some sequence of moves using only the vertices of T' that places a pebble on v. Place pebbles on T in the form of $f(T')$ and place one additional pebble on v. Then v can receive a second pebble, and a pebbling move from v reaches w and x. Clearly, this is a rubbling configuration of T, and so, $\rho_1^*(T) \le \left\lceil \frac{(n-2)+1}{2} + 1 \right\rceil = \left\lceil \frac{n+1}{2} \right\rceil$.

Case 3. Vertex v has three or more children. Let $w = \ell_1, \ldots, \ell_k$ be the leaves adjacent to v, where $k \ge 3$. Define $T' = T - \{v, \ell_1, \ldots, \ell_k\}$. By inductive hypothesis, there exists some ρ_1^*-configuration $f(T')$, such that $\left|f(T')\right| \le \left\lceil \frac{(n-k-1)+1}{2} \right\rceil$. Note that $n' \le n - 4$.

We place pebbles on T' in the form of $f(T')$ and place one pebble on each of v and ℓ_1. Note that we can place a pebble on u using pebbling and rubbling moves on T'. Then a rubbling move $r(u, \ell_1 \to v)$ places a second pebble on v. It follows that every other leaf neighbor of v is reachable via a pebbling move. Thus, this is a rubbling configuration on T. Hence, $\rho_1^*(T) \le \left\lceil \frac{(n-k-1)+1}{2} + 2 \right\rceil \le \left\lceil \frac{n+1}{2} \right\rceil$.

We have now shown inductively that $\rho_1^*(T) \le \left\lceil \frac{n+1}{2} \right\rceil$ for every tree T. ■

Corollary 17. *For any connected graph G on n vertices,* $\rho_1^*(G) \le \left\lceil \frac{n+1}{2} \right\rceil$.

Our final result will show that for each integer value $2 \le b \le \left\lceil \frac{n+1}{2} \right\rceil$, there exists a graph of order n whose 1-restricted optimal rubbling number equals b. To do so, we define the *double pencil* $P_{n,x}$ having order n as follows: begin with the path $p_1, \ldots, p_{n-2}$, along with two additional vertices u and v each adjacent to $p_1, \ldots, p_x$ for some x, where $1 \le x \le n - 2$. Therefore, $\deg(u) = \deg(v) = x$. For an example, see Figure 3.

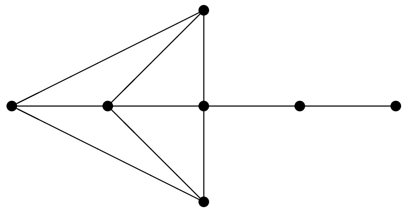

Figure 3. The graph $P_{7,3}$.

Theorem 18. *For any ordered pair (n, b) of positive integers, where $n \geq 3$ and $2 \leq b \leq \lceil \frac{n+1}{2} \rceil$, there exists a graph G of order n having $\rho_1^*(G) = b$.*

Proof. By Proposition 4, the path P_n is a graph of order n with $\rho_1^*(P_n) = \lceil \frac{n+1}{2} \rceil$. Further, if $n = 3$, then $\lceil \frac{n+1}{2} \rceil = 2$. Hence, we may assume that $n \geq 4$ and $b < \lceil \frac{n+1}{2} \rceil$.

Given positive integers n and b, where $n \geq 4$ and $2 \leq b < \lceil \frac{n+1}{2} \rceil$, consider the double pencil $P_{n,x}$ with $x = n - 2b + 2$. Clearly, $P_{n,x}$ has n vertices. We show that $\rho_1^*(P_{n,x}) = b$.

Note first that the diameter d of $P_{n,x}$ is $n - x$, except for the case, where $n \geq 6$ is even and $x = 2$, for which $d = n - 3$. By Corollary 3, $\lceil \frac{d+2}{2} \rceil \leq \rho_1^*(P_{n,x})$. In either case, $\lceil \frac{d+2}{2} \rceil = \left\lceil \frac{(n-x)+2}{2} \right\rceil$, so $\left\lceil \frac{(n-x)+2}{2} \right\rceil = \left\lceil \frac{n-(n-2b+2)+2}{2} \right\rceil = b \leq \rho_1^*(P_{n,x})$.

To see that $\rho_1^*(P_{n,x}) \leq b$, we give a rubbling configuration using b pebbles as follows. Begin by placing one pebble on each of u and v. If $b = 2$, then we are finished. Hence, assume that $b \geq 3$, and so, $x \neq n - 2$. For this case, complete the configuration by placing a pebble on p_{n-2} and p_{x+2i}, for $i = 1, \ldots, \lfloor (n-x-3)/2 \rfloor$. A rubbling move will reach each vertex in $N(u) \cup N(v)$, and every vertex on the $p_{x+2} - p_{n-2}$ path is reachable. Finally, p_{x+1} is reachable by the rubbling move $r(p_x, p_{x+2} \to p_{x+1})$. Thus, $\rho_1^*(P_{n,x}) \leq 3 + \lfloor \frac{n-x-3}{2} \rfloor = b$. ∎

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Received 1 May 2017
Revised 23 October 2017
Accepted 24 October 2017