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ON DECOMPOSING THE COMPLETE SYMMETRIC DIGRAPH INTO ORIENTATIONS OF $K_4 - e$

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Abstract

Let D be any of the 10 digraphs obtained by orienting the edges of $K_4 - e$. We establish necessary and sufficient conditions for the existence of a (K_n^*, D)-design for 8 of these digraphs. Partial results as well as some nonexistence results are established for the remaining 2 digraphs.

Keywords: digraph decompositions, orientations of $K_4 - e$.

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1. Introduction

Let $\mathbb{Z}_m$ denote the group of integers modulo m. For integers a and b with $a \leq b$, let $[a, b] = \{a, a + 1, \ldots, b\}$. For a graph (or digraph) H, let $V(H)$ and $E(H)$ denote the vertex set of H and the edge (or arc) set of H, respectively. The *order* and the *size* of a graph (or digraph) H are $|V(H)|$ and $|E(H)|$, respectively.

We denote the complete multipartite graph with parts of sizes a_i for $1 \leq i \leq m$ by $K_{a_1,a_2,\ldots,a_m}$. If $a_i = a$ for all $i \in \{1, 2, \ldots, m\}$, then we use the notation $K_{m\times a}$. Additionally, $K_{m\times a,b}$ denotes the complete multipartite graph with m parts of size a and one part of size b.

Let H be a graph and let $\mathcal{G}$ be a set of subgraphs of H. We will refer to a graph $G \in \mathcal{G}$ as a *G-block*. A *$\mathcal{G}$-decomposition* of H is a set $\Delta = \{G_1, G_2, \ldots, G_r\}$

of pairwise edge-disjoint subgraphs of H such that for every $i \in [1,r]$, $G_i \cong G$ for some $G \in \mathcal{G}$ and such that $E(H) = \bigcup_{i=1}^{r} E(G_i)$. Of particular importance is when $\mathcal{G} = \{G\}$, in which case we write "G-decomposition of H" instead of "$\{G\}$-decomposition of H." A G-decomposition of H is also known as an (H,G)-*design*. The set of all n for which K_n admits a G-decomposition is called the *spectrum of* G. The spectrum has been determined for many classes of graphs, including for all graphs on at most 4 vertices [4] and all graphs on 5 vertices (see [3] and [10]). We direct the reader to [2] and [5] for recent surveys on graph decompositions.

By *blowing up* the vertices of a graph G by some positive integer t, we mean replacing every vertex of G with t independent vertices and replacing every edge in G by a $K_{t,t}$. For example, assume we have a $(K_{x\times 2}, K_3)$-design. After blowing up the vertices of $K_{x\times 2}$ by 5, our corresponding $(K_{x\times 2}, K_3)$-design becomes a $(K_{x\times 10}, K_{3\times 5})$-design.

Similar concepts to the ones defined above for undirected graphs can be defined for digraphs. First, we introduce additional notation. For an undirected graph G, let G^* denote the digraph obtained from G by replacing each edge $\{u,v\} \in E(G)$ with the arcs (u,v) and (v,u). Thus K_n^*, the *complete symmetric digraph of order* n, is the digraph on n vertices with the arcs (u,v) and (v,u) between every pair of distinct vertices u and v.

Let D and H be digraphs such that D is a subgraph of H. The *reverse orientation of* D, denoted $\mathrm{Rev}(D)$, is the digraph with vertex set $V(D)$ and arc set $\{(v,u) : (u,v) \in E(D)\}$. A D-*decomposition* of H is a set $\Delta = \{D_1, D_2, \ldots, D_r\}$ of pairwise arc-disjoint subgraphs of H each of which is isomorphic to D and such that $E(H) = \bigcup_{i=1}^{r} E(D_i)$. As with the undirected case, a D-decomposition of H is also known as an (H,D)-*design*, and the set of all n for which K_n^* admits a D-decomposition is called the *spectrum of* D. Furthermore, we say D is *self-complementary in* H if D is isomorphic to the digraph with arc set $E(H) \setminus E(D)$. That is, D is self-complementary in H if H has size $2 \cdot |E(D)|$ and there exists an (H,D)-design.

The spectra for several digraphs of small order have been determined. This includes the spectra for all digraphs on at most 3 vertices [11] and all bipartite digraphs on 4 vertices with up to 5 arcs [7].

In this paper, we extend the known results on small digraphs by determining the spectrum for 8 of the 10 digraphs obtained by orienting the edges of $K_4 - e$, the graph obtained from removing a single edge from K_4. Some nonexistence results are proven for the remaining 2 such digraphs. We use the naming convention found in *An Atlas of Graphs* [13] by Read and Wilson. The digraphs under investigation are shown in Figures 1 and 2 with a key that denotes a labeled copy for each of the 10 digraphs of interest. For example, D75$[w,x,y,z]$ refers to the digraph with vertex set $\{w,x,y,z\}$ and arc set $\big\{(w,x),(w,y),(w,z),(x,y),(z,y)\big\}$.

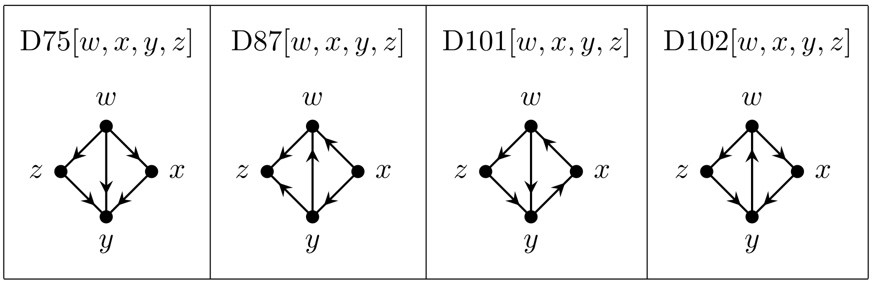

Figure 1. The four orientations of $K_4 - e$ that are self-complementary in $(K_4 - e)^*$.

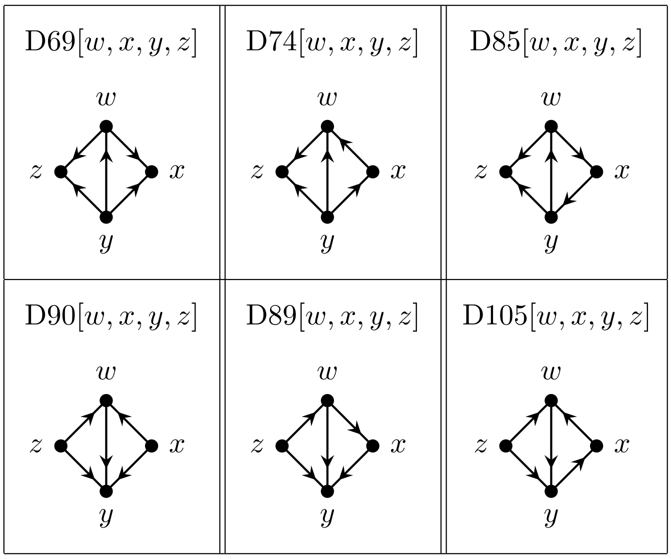

Figure 2. The six orientations of $K_4 - e$ that are not self-complementary in $(K_4 - e)^*$, shown paired with their reverse orientations.

Note that 6 of the digraphs of interest in this paper occur in pairs with respect to their reverse orientations (see Figure 2). Namely, D69 $\cong$ Rev(D90), D74 $\cong$ Rev(D89), and D85 $\cong$ Rev(D105). The remaining 4 digraphs of interest (see Figure 1) are isomorphic to their reverse orientations, e.g., D75 $\cong$ Rev(D75), which is shown in the proceeding section (see Lemma 4) to imply that these 4 digraphs are self-complementary in $(K_4 - e)^*$.

2. Some Basic Results

The necessary conditions for a digraph D to decompose K_n^* include

(A) $|V(D)| \leq n$,

(B) $|E(D)|$ divides $n(n-1)$, and
(C) both $\gcd\{\text{outdegree}(v) : v \in V(D)\}$ and $\gcd\{\text{indegree}(v) : v \in V(D)\}$ divide $n-1$.

Applying these necessary conditions to the 10 digraphs under consideration, we obtain the following necessary condition: For $D \in \{\text{D69}, \text{D74}, \text{D75}, \text{D85}, \text{D87}, \text{D89}, \text{D90}, \text{D101}, \text{D102}, \text{D105}\}$, a (K_n^*, D)-design exists only if $n \equiv 0$ or $1 \pmod 5$.

The following observation was stated in [6].

Observation 1. *Let D and H be digraphs. A D-decomposition of H exists if and only if a* $\text{Rev}(D)$*-decomposition of* $\text{Rev}(H)$ *exists.*

The fact that $K_n^* \cong \text{Rev}(K_n^*)$ leads to our next observation, also stated in [6].

Observation 2. *Let D be a digraph. A (K_n^*, D)-design exists if and only if a $(K_n^*, \text{Rev}(D))$-design exists.*

2.1. Results for self-complementary digraphs

We note that the existence of $(K_4 - e)$-decompositions of complete multigraphs (i.e., the spectrum of index λ) is known [12]. However, we present here the following theorem reduced to what is useful for characterizing the spectra of our 4 self-complementary digraphs.

Theorem 3 [4]**.** *There exists a $(K_4 - e)$-decomposition of K_n if and only if $n \equiv 0$ or $1 \pmod 5$ and $n \geq 6$.*

Since there does not exist a $(K_4 - e)$-decomposition of K_5, we must address decompositions of K_5^* (see Section 3). To make use of the known spectrum of $K_4 - e$, we present the following.

Lemma 4. *Let D be an orientation of a graph G. Then D is isomorphic to* $\text{Rev}(D)$ *if and only if D is self-complementary in G^*.*

Proof. Let D' be the digraph with vertex set $V(G^*)$ and arc set $E(G^*) \setminus E(D)$. Note that $E(D') = \{(v,u) : (u,v) \in E(D)\}$, which implies that D' is both the reverse orientation of D and the complement of D in G^*. The result then follows. ■

Since there exists a (K, G)-design if and only if a (K^*, G^*)-design exists, we arrive at the following corollary of the above lemma.

Corollary 5. *Let D be an orientation of a simple graph G such that D is self-complementary in G^*. If there exists a (K, G)-design, then there exists a (K^*, D)-design.*

In light of Corollary 5, we can combine Theorem 3 and Example 11 (see Section 3) to characterize the spectra of the digraphs that are self-complementary in $(K_4 - e)^*$, namely D75, D87, D101, and D102 (as seen in Figure 1).

Theorem 6. *Let* $D \in \{\mathrm{D75}, \mathrm{D87}, \mathrm{D101}, \mathrm{D102}\}$*. There exists a* (K_n^*, D)*-design if and only if* $n \equiv 0$ *or* $1 \pmod 5$ *and* $n \geq 5$*.*

2.2. Results for non-self-complementary digraphs

Our general constructions also use some basic results concerning decompositions of both complete graphs and complete multipartite graphs into complete graphs of orders 3 and 5. These are sometimes stated in the language of group divisible designs and/or pairwise balanced designs. Note that these background results concern graphs, as opposed to digraphs. Theorems 7, 8, and 9 can be found in the *Handbook of Combinatorial Designs* [8] (see [1] and [9]).

Theorem 7. *If* n *is odd, then a* $\{K_3, K_5\}$*-decomposition of* K_n *exists.*

Theorem 8. *The necessary and sufficient conditions for the existence of a* K_3*-decomposition of* $K_{u\times m}$ *are* (i) $u \geq 3$, (ii) $(u-1)m \equiv 0 \pmod 2$, *and* (iii) $u(u-1)m^2 \equiv 0 \pmod 6$.

Theorem 9. *If* $u \geq 3$ *and* $u \equiv 0 \pmod 3$*, then there exists a* K_3*-decomposition of* $K_{u\times 2,4}$*.*

Our general constructions further rely on the following direct result of blowing up the vertices in the graphs of a decomposition. This well-known building block is a special case of Wilson's Fundamental Construction.

Lemma 10. *Let* m, r, s, t, u_1, $u_2, \ldots, u_m$ *all be positive integers. If there exists a* $\{K_r, K_s\}$*-decomposition of* $K_{u_1,u_2,\ldots,u_m}$*, then there exists a* $\{K_{r\times t}, K_{s\times t}\}$*-decomposition of* $K_{tu_1,tu_2,\ldots,tu_m}$*. In particular, if there exists a* $(K_{u_1,u_2,\ldots,u_m}, K_r)$*-design, then there exists a* $(K_{tu_1,tu_2,\ldots,tu_m}, K_{r\times t})$*-design.*

3. Examples of Small Designs

We now turn our attention to the designs of small order which will be used for the general constructions.

Given a digraph represented by the notation $D[a, b, c, d]$ and some $i \in \mathbb{Z}_n$, we define $D[a, b, c, d] + i = D[a + i, b + i, c + i, d + i]$ where all addition is performed in $\mathbb{Z}_n$. By convention, define $\infty + 1 = \infty$.

Example 11. There exists a (K_5^*, D)-design for $D \in \{\mathrm{D74}, \mathrm{D75}, \mathrm{D85}, \mathrm{D87}, \mathrm{D89}, \mathrm{D101}, \mathrm{D102}, \mathrm{D105}\}$.

Let $V(K_5^*) = \mathbb{Z}_4 \cup \{\infty\}$.
A $(K_5^*, \text{D74})$-design is given by $\{\text{D74}[0,\infty,2,1]+i : i \in \mathbb{Z}_4\}$.
A $(K_5^*, \text{D75})$-design is given by $\{\text{D75}[0,\infty,3,1]+i : i \in \mathbb{Z}_4\}$.
A $(K_5^*, \text{D85})$-design is given by $\{\text{D85}[0,\infty,2,1]+i : i \in \mathbb{Z}_4\}$.
A $(K_5^*, \text{D87})$-design is given by

$$\{\text{D87}[0,\infty,1,2], \text{D87}[0,3,2,\infty], \text{D87}[3,\infty,2,1], \text{D87}[3,0,1,\infty]\}.$$

A $(K_5^*, \text{D101})$-design is given by $\{\text{D101}[0,\infty,3,1]+i : i \in \mathbb{Z}_4\}$.
A $(K_5^*, \text{D102})$-design is given by

$$\{\text{D102}[0,\infty,1,2], \text{D102}[1,3,0,\infty], \text{D102}[2,\infty,3,0], \text{D102}[3,1,2,\infty]\}.$$

Applying Observation 2, we obtain the remaining designs.

Example 12. There exists a (K_6^*, D)-design for $D \in \{\text{D69}, \text{D74}, \text{D85}, \text{D89}, \text{D90}, \text{D105}\}$.

Let $V(K_6^*) = \mathbb{Z}_6$.
A $(K_6^*, \text{D69})$-design is given by $\{\text{D69}[0,2,1,4]+i : i \in \mathbb{Z}_6\}$.
A $(K_6^*, \text{D74})$-design is given by $\{\text{D74}[0,5,1,3]+i : i \in \mathbb{Z}_6\}$.
A $(K_6^*, \text{D85})$-design is given by $\{\text{D85}[0,3,5,4]+i : i \in \mathbb{Z}_6\}$.
Applying Observation 2, we obtain the remaining designs.

Example 13. There exists a (K_{10}^*, D)-design for $D \in \{\text{D74}, \text{D85}, \text{D89}, \text{D105}\}$.

Let $V(K_{10}^*) = \mathbb{Z}_9 \cup \{\infty\}$.
A $(K_{10}^*, \text{D74})$-design is given by

$$\{\text{D74}[0,\infty,4,6]+i : i \in \mathbb{Z}_9\} \cup \{\text{D74}[0,8,1,4]+i : i \in \mathbb{Z}_9\}.$$

A $(K_{10}^*, \text{D85})$-design is given by

$$\{\text{D85}[0,\infty,4,7]+i : i \in \mathbb{Z}_9\} \cup \{\text{D85}[0,6,1,2]+i : i \in \mathbb{Z}_9\}.$$

Applying Observation 2, we obtain the remaining designs.

Example 14. There exists a (K_{11}^*, D)-design for $D \in \{\text{D69}, \text{D74}, \text{D85}, \text{D89}, \text{D90}, \text{D105}\}$.

Let $V(K_{11}^*) = \mathbb{Z}_{11}$.
A $(K_{11}^*, \text{D69})$-design is given by

$$\{\text{D69}[0,2,7,5]+i : i \in \mathbb{Z}_{11}\} \cup \{\text{D69}[0,1,4,3]+i : i \in \mathbb{Z}_{11}\}.$$

A $(K_{11}^*, \text{D74})$-design is given by

$$\{\text{D74}[0,7,1,3]+i : i \in \mathbb{Z}_{11}\} \cup \{\text{D74}[0,4,10,8]+i : i \in \mathbb{Z}_{11}\}.$$

A $(K_{11}^*, \text{D85})$-design is given by

$$\{\text{D85}[0,6,10,7]+i : i \in \mathbb{Z}_{11}\} \cup \{\text{D85}[0,9,8,2]+i : i \in \mathbb{Z}_{11}\}.$$

Applying Observation 2, we obtain the remaining designs.

Example 15. There exists a (K_{20}^*, D)-design for $D \in \{\mathrm{D74}, \mathrm{D85}, \mathrm{D89}, \mathrm{D105}\}$.

Let $V(K_{20}^*) = \mathbb{Z}_{19} \cup \{\infty\}$.
A $(K_{20}^*, \mathrm{D74})$-design is given by

$$\{\mathrm{D74}[0, \infty, 13, 2] + i : i \in \mathbb{Z}_{19}\} \cup \{\mathrm{D74}[0, 12, 1, 10] + i : i \in \mathbb{Z}_{19}\}$$
$$\cup \{\mathrm{D74}[0, 14, 16, 12] + i : i \in \mathbb{Z}_{19}\} \cup \{\mathrm{D74}[0, 15, 18, 13] + i : i \in \mathbb{Z}_{19}\}.$$

A $(K_{20}^*, \mathrm{D85})$-design is given by

$$\{\mathrm{D85}[0, \infty, 2, 16] + i : i \in \mathbb{Z}_{19}\} \cup \{\mathrm{D85}[0, 12, 1, 6] + i : i \in \mathbb{Z}_{19}\}$$
$$\cup \{\mathrm{D85}[0, 13, 17, 7] + i : i \in \mathbb{Z}_{19}\} \cup \{\mathrm{D85}[0, 3, 18, 10] + i : i \in \mathbb{Z}_{19}\}.$$

Applying Observation 2, we obtain the remaining designs.

Example 16. There exists a (K_{21}^*, D)-design for $D \in \{\mathrm{D69}, \mathrm{D74}, \mathrm{D85}, \mathrm{D89}, \mathrm{D90}, \mathrm{D105}\}$.

Let $V(K_{21}^*) = \mathbb{Z}_{21}$.
A $(K_{21}^*, \mathrm{D69})$-design is given by

$$\{\mathrm{D69}[0, 4, 1, 12] + i : i \in \mathbb{Z}_{21}\} \cup \{\mathrm{D69}[0, 7, 2, 16] + i : i \in \mathbb{Z}_{21}\}$$
$$\cup \{\mathrm{D69}[0, 6, 19, 13] + i : i \in \mathbb{Z}_{21}\} \cup \{\mathrm{D69}[0, 17, 20, 9] + i : i \in \mathbb{Z}_{21}\}.$$

A $(K_{21}^*, \mathrm{D74})$-design is given by

$$\{\mathrm{D74}[0, 13, 4, 14] + i : i \in \mathbb{Z}_{21}\} \cup \{\mathrm{D74}[0, 2, 5, 20] + i : i \in \mathbb{Z}_{21}\}$$
$$\cup \{\mathrm{D74}[0, 18, 16, 1] + i : i \in \mathbb{Z}_{21}\} \cup \{\mathrm{D74}[0, 8, 17, 7] + i : i \in \mathbb{Z}_{21}\}.$$

A $(K_{21}^*, \mathrm{D85})$-design is given by

$$\{\mathrm{D85}[0, 18, 1, 15] + i : i \in \mathbb{Z}_{21}\} \cup \{\mathrm{D85}[0, 17, 2, 7] + i : i \in \mathbb{Z}_{21}\}$$
$$\cup \{\mathrm{D85}[0, 16, 19, 11] + i : i \in \mathbb{Z}_{21}\} \cup \{\mathrm{D85}[0, 12, 20, 9] + i : i \in \mathbb{Z}_{21}\}.$$

Applying Observation 2, we obtain the remaining designs.

Example 17. There exists a (K_{25}^*, D)-design for $D \in \{\mathrm{D69}, \mathrm{D90}\}$.

Let $V(K_{25}^*) = \mathbb{Z}_5 \times \mathbb{Z}_5$. A $(K_{25}^*, \mathrm{D69})$-design is given by

$$\{\mathrm{D69}[(1, i), (0, 1+i), (1, 2+i), (0, i)] : i \in \mathbb{Z}_5\}$$
$$\cup \{\mathrm{D69}[(2, i), (0, 1+i), (2, 2+i), (0, i)] : i \in \mathbb{Z}_5\}$$
$$\cup \{\mathrm{D69}[(3, i), (0, 1+i), (3, 2+i), (0, i)] : i \in \mathbb{Z}_5\}$$
$$\cup \{\mathrm{D69}[(0, 1+i), (1, i), (4, i), (0, i)] : i \in \mathbb{Z}_5\}$$
$$\cup \{\mathrm{D69}[(0, 2+i), (1, i), (4, 3+i), (0, i)] : i \in \mathbb{Z}_5\}$$
$$\cup \{\mathrm{D69}[(1, 3+i), (1, i), (0, 3+i), (0, i)] : i \in \mathbb{Z}_5\}$$
$$\cup \{\mathrm{D69}[(2, 1+i), (1, 1+i), (2, 4+i), (1, i)] : i \in \mathbb{Z}_5\}$$
$$\cup \{\mathrm{D69}[(3, i), (1, 1+i), (3, 3+i), (1, i)] : i \in \mathbb{Z}_5\}$$

$$\cup \{\mathrm{D69}[(0,4+i),(2,1+i),(2,2+i),(1,i)] : i \in \mathbb{Z}_5\}$$
$$\cup \{\mathrm{D69}[(1,1+i),(2,i),(4,2+i),(1,i)] : i \in \mathbb{Z}_5\}$$
$$\cup \{\mathrm{D69}[(0,2+i),(2,2+i),(0,1+i),(2,i)] : i \in \mathbb{Z}_5\}$$
$$\cup \{\mathrm{D69}[(1,i),(2,1+i),(3,1+i),(2,i)] : i \in \mathbb{Z}_5\}$$
$$\cup \{\mathrm{D69}[(3,3+i),(2,1+i),(4,i),(2,i)] : i \in \mathbb{Z}_5\}$$
$$\cup \{\mathrm{D69}[(3,4+i),(4,i),(1,2+i),(2,i)] : i \in \mathbb{Z}_5\}$$
$$\cup \{\mathrm{D69}[(0,3+i),(4,i),(3,1+i),(3,i)] : i \in \mathbb{Z}_5\}$$
$$\cup \{\mathrm{D69}[(0,4+i),(4,2+i),(4,1+i),(3,i)] : i \in \mathbb{Z}_5\}$$
$$\cup \{\mathrm{D69}[(1,1+i),(4,i),(1,i),(3,i)] : i \in \mathbb{Z}_5\}$$
$$\cup \{\mathrm{D69}[(1,4+i),(4,1+i),(4,3+i),(3,i)] : i \in \mathbb{Z}_5\}$$
$$\cup \{\mathrm{D69}[(2,4+i),(4,3+i),(1,2+i),(3,i)] : i \in \mathbb{Z}_5\}$$
$$\cup \{\mathrm{D69}[(2,1+i),(4,1+i),(2,i),(3,i)] : i \in \mathbb{Z}_5\}$$
$$\cup \{\mathrm{D69}[(2,3+i),(4,1+i),(4,4+i),(3,i)] : i \in \mathbb{Z}_5\}$$
$$\cup \{\mathrm{D69}[(2,2+i),(4,4+i),(4,i),(3,i)] : i \in \mathbb{Z}_5\}$$
$$\cup \{\mathrm{D69}[(3,4+i),(4,4+i),(0,i),(3,i)] : i \in \mathbb{Z}_5\}$$
$$\cup \{\mathrm{D69}[(3,3+i),(4,1+i),(0,i),(4,i)] : i \in \mathbb{Z}_5\}.$$

Applying Observation 2, we obtain a $(K_{25}^*, \mathrm{D90})$-design.

Example 18. There exists a (K_{30}^*, D)-design for $D \in \{\mathrm{D69}, \mathrm{D90}\}$.

Let $V(K_{30}^*) = \mathbb{Z}_5 \times \mathbb{Z}_6$.
A $(K_{30}^*, \mathrm{D69})$-design is given by

$$\{\mathrm{D69}[(1,i),(0,1+i),(1,2+i),(0,i)] : i \in \mathbb{Z}_6\}$$
$$\cup \{\mathrm{D69}[(1,4+i),(0,1+i),(2,i),(0,i)] : i \in \mathbb{Z}_6\}$$
$$\cup \{\mathrm{D69}[(2,2+i),(0,1+i),(2,4+i),(0,i)] : i \in \mathbb{Z}_6\}$$
$$\cup \{\mathrm{D69}[(3,i),(0,1+i),(3,2+i),(0,i)] : i \in \mathbb{Z}_6\}$$
$$\cup \{\mathrm{D69}[(3,4+i),(0,1+i),(4,i),(0,i)] : i \in \mathbb{Z}_6\}$$
$$\cup \{\mathrm{D69}[(0,1+i),(1,i),(4,2+i),(0,i)] : i \in \mathbb{Z}_6\}$$
$$\cup \{\mathrm{D69}[(0,i),(1,1+i),(0,3+i),(1,i)] : i \in \mathbb{Z}_6\}$$
$$\cup \{\mathrm{D69}[(0,4+i),(2,i),(4,1+i),(1,i)] : i \in \mathbb{Z}_6\}$$
$$\cup \{\mathrm{D69}[(1,1+i),(2,i),(2,1+i),(1,i)] : i \in \mathbb{Z}_6\}$$
$$\cup \{\mathrm{D69}[(1,4+i),(2,i),(2,3+i),(1,i)] : i \in \mathbb{Z}_6\}$$
$$\cup \{\mathrm{D69}[(1,3+i),(2,i),(3,i),(1,i)] : i \in \mathbb{Z}_6\}$$
$$\cup \{\mathrm{D69}[(1,5+i),(2,i),(3,4+i),(1,i)] : i \in \mathbb{Z}_6\}$$

$$\begin{aligned}
&\cup \{\text{D69}[(2,4+i),(2,i),(3,1+i),(1,i)] : i \in \mathbb{Z}_6\}\\
&\cup \{\text{D69}[(3,2+i),(2,i),(4,i),(1,i)] : i \in \mathbb{Z}_6\}\\
&\cup \{\text{D69}[(4,3+i),(2,i),(4,4+i),(1,i)] : i \in \mathbb{Z}_6\}\\
&\cup \{\text{D69}[(0,i),(2,1+i),(0,2+i),(2,i)] : i \in \mathbb{Z}_6\}\\
&\cup \{\text{D69}[(3,5+i),(3,i),(2,5+i),(2,i)] : i \in \mathbb{Z}_6\}\\
&\cup \{\text{D69}[(0,i),(3,1+i),(0,4+i),(3,i)] : i \in \mathbb{Z}_6\}\\
&\cup \{\text{D69}[(0,2+i),(4,1+i),(4,i),(3,i)] : i \in \mathbb{Z}_6\}\\
&\cup \{\text{D69}[(1,4+i),(4,i),(4,3+i),(3,i)] : i \in \mathbb{Z}_6\}\\
&\cup \{\text{D69}[(2,4+i),(4,1+i),(0,1+i),(3,i)] : i \in \mathbb{Z}_6\}\\
&\cup \{\text{D69}[(2,1+i),(4,3+i),(1,3+i),(3,i)] : i \in \mathbb{Z}_6\}\\
&\cup \{\text{D69}[(2,2+i),(4,3+i),(4,1+i),(3,i)] : i \in \mathbb{Z}_6\}\\
&\cup \{\text{D69}[(2,3+i),(4,3+i),(4,5+i),(3,i)] : i \in \mathbb{Z}_6\}\\
&\cup \{\text{D69}[(3,1+i),(4,2+i),(1,1+i),(3,i)] : i \in \mathbb{Z}_6\}\\
&\cup \{\text{D69}[(3,3+i),(4,5+i),(1,2+i),(3,i)] : i \in \mathbb{Z}_6\}\\
&\cup \{\text{D69}[(0,5+i),(4,2+i),(0,4+i),(4,i)] : i \in \mathbb{Z}_6\}\\
&\cup \{\text{D69}[(2,2+i),(4,1+i),(1,2+i),(4,i)] : i \in \mathbb{Z}_6\}\\
&\cup \{\text{D69}[(3,3+i),(4,1+i),(3,1+i),(4,i)] : i \in \mathbb{Z}_6\}.
\end{aligned}$$

Applying Observation 2, we obtain a $(K^*_{30}, \text{D90})$-design.

Example 19. There exists a $(K^*_{3\times 5}, D)$-design for $D \in \{\text{D69}, \text{D74}, \text{D85}, \text{D89}, \text{D90}, \text{D105}\}$.

Let $V(K^*_{3\times 5}) = \mathbb{Z}_{15}$ with vertex partition $\{V_i : i \in \mathbb{Z}_3\}$, where $V_i = \{j \in \mathbb{Z}_{15} : j \equiv i \pmod 3\}$.
A $(K^*_{3\times 5}, \text{D69})$-design is given by

$$\{\text{D69}[0,8,10,11] + i : i \in \mathbb{Z}_{15}\} \cup \{\text{D69}[0,4,5,7] + i : i \in \mathbb{Z}_{15}\}.$$

A $(K^*_{3\times 5}, \text{D74})$-design is given by

$$\{\text{D74}[0,2,10,11] + i : i \in \mathbb{Z}_{15}\} \cup \{\text{D74}[0,13,5,4] + i : i \in \mathbb{Z}_{15}\}.$$

A $(K^*_{3\times 5}, \text{D85})$-design is given by

$$\{\text{D85}[0,7,5,1] + i : i \in \mathbb{Z}_{15}\} \cup \{\text{D85}[0,8,10,14] + i : i \in \mathbb{Z}_{15}\}.$$

Applying Observation 2, we obtain the remaining designs.

Example 20. There exists a $(K^*_{5\times 5}, D)$-design for $D \in \{\text{D69}, \text{D74}, \text{D85}, \text{D89}, \text{D90}, \text{D105}\}$.

First, let $V(K^*_{5\times 5}) = \mathbb{Z}_{25}$ with vertex partition $\{V_i : i \in \mathbb{Z}_5\}$, where $V_i = \{j \in \mathbb{Z}_{25} : j \equiv i \pmod 5\}$.
A $(K^*_{5\times 5}, \mathrm{D69})$-design is given by

$$\{\mathrm{D69}[1, 14, 0, 24] + i : i \in \mathbb{Z}_{25}\} \cup \{\mathrm{D69}[2, 8, 0, 18] + i : i \in \mathbb{Z}_{25}\}$$
$$\cup \{\mathrm{D69}[3, 12, 0, 22] + i : i \in \mathbb{Z}_{25}\} \cup \{\mathrm{D69}[4, 11, 0, 21] + i : i \in \mathbb{Z}_{25}\}.$$

A $(K^*_{5\times 5}, \mathrm{D90})$-design follows from Observation 2.
Next, let $D \in \{\mathrm{D74}, \mathrm{D89}, \mathrm{D105}, \mathrm{D85}\}$. A $(K_{5\times 5}, K_5)$-design can be obtained by removing one parallel class from an affine plane of order 5. Thus, there exists a $(K^*_{5\times 5}, K^*_5)$-design. Since a (K^*_5, D)-design exists by Example 11, the desired $(K^*_{5\times 5}, D)$-design exists.

4. Main Results

We first show some nonexistence results for $(\mathrm{D69}, K^*_n)$- and $(\mathrm{D90}, K^*_n)$-designs. Interestingly for $n \equiv 0 \pmod 5$, these designs do not exist for $n \in \{5, 10, 15, 20\}$ (see Theorem 21) but do exist for $n \in \{25, 30\}$ (see Examples 17 and 18). By Wilson's Theorem [14], there exists an integer n_0 such that for all $n \geq n_0$ that satisfy the necessary conditions there exists both a $(\mathrm{D69}, K^*_n)$-design and a $(\mathrm{D90}, K^*_n)$-design. We conjecture that $n_0 = 25$ for this pair of digraphs.

Theorem 21. *There does not exist a* D69- *or* D90-*decomposition of* K^*_{5k} *for* $1 \leq k \leq 4$.

Proof. We prove by contradiction that a D69-decomposition of K^*_{5k} cannot exist. Note that by Observation 2, a D90-decomposition must also not exist.

Let Δ be a D69-decomposition of K^*_{5k}. Given a vertex $v \in V(K^*_{5k})$, let $n_w(v)$ denote the number of D69-blocks in Δ where vertex w in $\mathrm{D69}[w, x, y, z]$ is identified with vertex v. Define $n_x(v)$, $n_y(v)$, and $n_z(v)$ similarly. Thus, the following must hold:

$$1n_w(v) + 2n_x(v) + 0n_y(v) + 2n_z(v) = 5k - 1,$$
$$2n_w(v) + 0n_x(v) + 3n_y(v) + 0n_z(v) = 5k - 1.$$

Substituting $\bar{n}(v) = n_x(v) + n_z(v)$, the above equations can be parameterized as

$$n_w(v) = 5k - 1 - 2\bar{n}(v), \tag{1}$$
$$n_y(v) = -\tfrac{1}{3}(5k - 1 - 4\bar{n}(v)). \tag{2}$$

Since $\bar{n}(v)$, $n_w(v)$, and $n_y(v)$ must all be nonnegative integers, we have that

$$\left.\begin{array}{l} 0 \leq 5k - 1 - 2\bar{n}(v) \\ 0 \leq -\frac{1}{3}(5k - 1 - 4\bar{n}(v)) \end{array}\right\} \implies \tfrac{1}{4}(5k - 1) \leq \bar{n}(v) \leq \tfrac{1}{2}(5k - 1).$$

Furthermore, equation (2) implies that $5k - 1 - 4\bar{n}(v)$ must be a multiple of 3; hence, $k + \bar{n}(v) + 1 \equiv 0 \pmod 3$.

Next, consider the case when $k = 1$. The above conditions require that for every $v \in V(K_5^*)$, we have $1 \le \bar{n}(v) \le 2$ and $\bar{n}(v) \equiv 1 \pmod 3$. Thus, $\bar{n}(v)$ can only equal 1, and by equation (1), $n_w(v) = 2$ for every $v \in V(K_5^*)$. However, this would imply $|\Delta| = 10$, which is a contradiction (because $|\Delta| = 4$ when $k = 1$). Similarly if k is 2, 3, or 4, then $\bar{n}(v)$ can only equal 3, 5, or 7, respectively, which further yields only one value for $n_w(v)$: 3, 4, or 5, respectively, for every $v \in V(K_{5k}^*)$. However, this would imply $|\Delta|$ is a multiple of $5k$, which is a contradiction because

$$|\Delta| = \frac{|E(K_{5k}^*)|}{|E(\text{D69})|} = \frac{5k(5k-1)}{5} = k(5k-1),$$

which is not divisible by $5k$. ■

Next we turn our attention to developing the general constructions needed to piece together the small designs presented in Section 3 and show sufficiency of the necessary conditions for the remaining four non-self-complementary digraphs.

Theorem 22. *Let $D \in \{\text{D74}, \text{D85}, \text{D89}, \text{D105}\}$. If $n \equiv 0 \pmod 5$ with $n \ge 5$, then a (K_n^*, D)-design exists.*

Proof. Let $D \in \{\text{D74}, \text{D85}, \text{D89}, \text{D105}\}$ and let $n \equiv 0$ or $5 \pmod{10}$.

Case 1. $n \equiv 0 \pmod{10}$. Let $n = 10x = 5(2x)$ for some positive integer x. When x is 1 or 2 the result follows from Examples 13 and 15, respectively, so we now consider when $x \ge 3$. Let $H_1, H_2, \ldots, H_x$ be disjoint sets of 2 vertices each.

Subcase 1a. $x \equiv 0$ or $1 \pmod 3$. Let $K_{x\times 2}$ have vertex partition $\{H_i : 1 \le i \le x\}$. By Theorem 8, a $(K_{x\times 2}, K_3)$-design exists. Therefore, by Lemma 10 a $(K_{x\times 10}, K_{3\times 5})$-design exists. Let H_i' be the set obtained from H_i after blowing up each vertex in $K_{x\times 2}$ by 5. Now consider K_n^* to have vertex set $\bigcup_{i=1}^{x} H_i'$ where each H_i' induces a K_{10}^*. Thus, K_n^* decomposes into copies of K_{10}^* and $K_{3\times 5}^*$. Since both a (K_{10}^*, D)-design and a $(K_{3\times 5}^*, D)$-design exist by Examples 13 and 19, respectively, we have our desired (K_n^*, D)-design.

Subcase 1b. $x \equiv 2 \pmod 3$. Let $H_0 = H_{x-1} \cup H_x$ and let $K_{(x-2)\times 2,4}$ have vertex partition $\{H_i : 0 \le i \le x-2\}$. By Theorem 9, a $(K_{(x-2)\times 2,4}, K_3)$-design exists. Therefore, by Lemma 10 a $(K_{(x-2)\times 10,20}, K_{3\times 5})$-design exists. Let H_i' be the set obtained from H_i after blowing up each vertex in $K_{(x-2)\times 2,4}$ by 5. Now consider K_n^* to have vertex set $\bigcup_{i=0}^{x-2} H_i'$ where H_0' induces a K_{20}^* and, for $1 \le i \le x-2$, each H_i' induces a K_{10}^*. Thus, K_n^* decomposes into copies of K_{10}^*, K_{20}^*, and $K_{3\times 5}^*$. Since a (K_{10}^*, D)-design, a (K_{20}^*, D)-design, and a $(K_{3\times 5}^*, D)$-design exist by Examples 13, 15, and 19, respectively, we have our desired (K_n^*, D)-design.

Case 2. $n \equiv 5 \pmod{10}$. Let $n = 10x + 5 = 5(2x + 1)$ for some positive integer x. Let $H_1, H_2, \ldots, H_{2x+1}$ be sets consisting of a single vertex each. By Theorem 7 a $\{K_3, K_5\}$-decomposition of K_{2x+1} exists. Therefore, by Lemma 10 a $\{K_{3\times 5}, K_{5\times 5}\}$-decomposition of $K_{(2x+1)\times 5}$ exists. Let H_i' be the set obtained from H_i after blowing up each vertex in K_{2x+1} by 5. Now consider K_n^* to have vertex set $\bigcup_{i=1}^{2x+1} H_i'$ where each H_i' induces a K_5^*. Thus, K_n^* decomposes into copies of K_5^*, $K_{3\times 5}^*$, and $K_{5\times 5}^*$. Since a (K_5^*, D)-design, a $(K_{3\times 5}^*, D)$-design, and a $(K_{5\times 5}^*, D)$-design all exist by Examples 11, 19, and 20, respectively, we have our desired (K_n^*, D)-design. ■

Theorem 23. *Let* $D \in \{\mathrm{D69}, \mathrm{D74}, \mathrm{D85}, \mathrm{D89}, \mathrm{D90}, \mathrm{D105}\}$. *If* $n \equiv 1 \pmod{5}$ *with* $n \geq 6$*, then a* (K_n^*, D)*-design exists.*

Proof. Let $D \in \{\mathrm{D69}, \mathrm{D74}, \mathrm{D85}, \mathrm{D89}, \mathrm{D90}, \mathrm{D105}\}$ and let $n \equiv 1$ or $6 \pmod{10}$.

Case 1. $n \equiv 1 \pmod{10}$. Let $n = 10x+1 = 5(2x)+1$ for some positive integer x. When x is 1 or 2 the result follows from Examples 14 and 16, respectively, so we now consider when $x \geq 3$.

Subcase 1a. $x \equiv 0$ or $1 \pmod{3}$. Here we can consider $V(K_n^*) = \left(\bigcup_{i=1}^{x} H_i'\right) \cup \{\infty\}$, where each H_i' is defined as in Subcase 1a of the proof of Theorem 22 with the modification that each $H_i' \cup \{\infty\}$ induces a K_{11}^*. Similarly to the proof of that Subcase 1a, the desired (K_n^*, D)-design can be constructed using (K_{11}^*, D)-designs—in place of (K_{10}^*, D)-designs—along with $(K_{3\times 5}^*, D)$-designs, which exist by Examples 14 and 19, respectively.

Subcase 1b. $x \equiv 2 \pmod{3}$. Here we can consider $V(K_n^*) = \left(\bigcup_{i=0}^{x-2} H_i'\right) \cup \{\infty\}$, where each H_i' is defined as in Subcase 1b of the proof of Theorem 22 with the modifications that $H_0' \cup \{\infty\}$ induces a K_{21}^* and, for $1 \leq i \leq x-2$, each $H_i' \cup \{\infty\}$ induces a K_{11}^*. Similarly to the proof of that Subcase 1b, the desired (K_n^*, D)-design can be constructed using (K_{11}^*, D)-designs and a (K_{21}^*, D)-design—in place of (K_{10}^*, D)- and (K_{20}^*, D)-designs—along with $(K_{3\times 5}^*, D)$-designs, which exist by Examples 14, 16, and 19, respectively.

Case 2. $n \equiv 6 \pmod{10}$. Here we can consider $V(K_n^*) = \left(\bigcup_{i=1}^{2x+1} H_i'\right) \cup \{\infty\}$, where each H_i' is defined as in Case 2 of the proof of Theorem 22 with the modification that $H_i' \cup \{\infty\}$ induces a K_6^*. Similarly to the proof of that Case 2, the desired (K_n^*, D)-design can be constructed by using (K_6^*, D)-designs—in place of (K_5^*, D)-designs—along with $(K_{3\times 5}^*, D)$-designs and $(K_{5\times 5}^*, D)$-designs, which exist by Examples 12, 19, and 20. ■

Our results from this section along with those in Theorem 6 are summarized in the following two main theorems.

Theorem 24. *Let* $D \in \{\mathrm{D74}, \mathrm{D75}, \mathrm{D85}, \mathrm{D87}, \mathrm{D89}, \mathrm{D101}, \mathrm{D102}, \mathrm{D105}\}$. *There exists a* (K_n^*, D)*-design if and only if* $n \equiv 0$ *or* $1 \pmod 5$ *and* $n \geq 5$.

Theorem 25. *Let* $D \in \{\mathrm{D69}, \mathrm{D90}\}$. *There exists a* (K_n^*, D)*-design if* $n \equiv 1 \pmod 5$ *and* $n \geq 6$.

Finally, we formally state our conjecture regarding the open results for the $\{\mathrm{D69}, \mathrm{D90}\}$ pair of digraphs.

Conjecture 26. *Let* $D \in \{\mathrm{D69}, \mathrm{D90}\}$. *There exists a* (K_n^*, D)*-design if* $n \equiv 0 \pmod 5$ *and* $n \geq 25$.

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