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ON THE TOTAL ROMAN DOMINATION IN TREES

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Abstract

A *total Roman dominating function* on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the following conditions: (i) every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$ and (ii) the subgraph of G induced by the set of all vertices of positive weight has no isolated vertex. The weight of a total Roman dominating function f is the value $f(V(G)) = \Sigma_{u \in V(G)} f(u)$. The *total Roman domination number* $\gamma_{tR}(G)$ is the minimum weight of a total Roman dominating function of G. Ahangar *et al.* in [H.A. Ahangar, M.A. Henning, V. Samodivkin and I.G. Yero, *Total Roman domination in graphs*, Appl. Anal. Discrete Math. 10 (2016) 501–517] recently showed that for any graph G without isolated vertices, $2\gamma(G) \le \gamma_{tR}(G) \le 3\gamma(G)$, where $\gamma(G)$ is the domination number of G, and they raised the problem of characterizing the graphs G achieving these upper and lower bounds. In this paper, we provide a constructive characterization of these trees.

Keywords: total Roman dominating function, total Roman domination number, trees.

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1. Introduction

In this paper, G is a simple graph without isolated vertices, with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* $|V|$ of G is denoted by $n = n(G)$.

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For every vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg(v) = \deg_G(v) = |N(v)|$. A *leaf* of T is a vertex of degree 1, a *support vertex* of T is a vertex adjacent to a leaf, a *strong support vertex* is a support vertex adjacent to at least two leaves and an *end support vertex* is a support vertex having at most one non-leaf neighbor. A *pendant path* P of a graph G is an induced path such that one of the end points has degree one in G, and its other end point is the only vertex of P adjacent to some vertex in $G - P$. The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest uv-path in G. The *diameter* of a graph G, denoted by $\mathrm{diam}(G)$, is the greatest distance between two vertices of G. For a vertex v in a (rooted) tree T, let $C(v)$ and $D(v)$ denote the set of children and descendants of v, respectively and let $D[v] = D(v) \cup \{v\}$. Also, the depth of v, $\mathrm{depth}(v)$, is the largest distance from v to a vertex in $D(v)$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. We write P_n for the *path* of order n. A *double star* is a tree with exactly two vertices that are not leaves. If $A \subseteq V(G)$ and f is a mapping from $V(G)$ into some set of numbers, then $f(A) = \sum_{x\in A} f(x)$. The sum $f(V(G))$ is called the *weight* $\omega(f)$ of f.

A vertex set S of a graph G is a dominating set if each vertex of G either belongs to S or is adjacent to a vertex in S. The domination number $\gamma(G)$ of G is the minimum cardinality over all dominating sets of G. A dominating set of G of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. The domination problem consists of finding the domination number of a graph. The domination problem has many applications and has attracted considerable attention [11, 15]. The literature on the subject of domination parameters in graphs has been surveyed and detailed in the two books [12, 13].

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (RDF) on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF is the value $f(V(G)) = \sum_{u\in V(G)} f(u)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF on G. Roman domination was introduced by Cockayne *et al.* in [10] and was inspired by the work of ReVelle and Rosing [17], Stewart [18]. It is worth mentioning that since 2004, a hundred papers have been published on this topic, where several new variations were introduced: weak Roman domination [14], Roman $\{2\}$-domination [9], maximal Roman domination [2], mixed Roman domination [4], double Roman domination [8] and recently total Roman domination introduced by Liu and Chang [16].

A *total Roman dominating function* of a graph G with no isolated vertex, abbreviated TRDF, is a Roman dominating function f on G with the additional property that the subgraph of G induced by the set of all vertices of positive weight under f has no isolated vertex. The *total Roman domination number*

$\gamma_{tR}(G)$ is the minimum weight of a TRDF on G. A TRDF of G with weight $\gamma_{tR}(G)$ is called a $\gamma_{tR}(G)$-function. The concept of the total Roman domination was introduced by Liu and Chang [16] and has been studied in [1, 3, 5–7].

Ahangar *et al.* [3] showed that for any graph G,

$$2\gamma(G) \leq \gamma_{tR}(G) \leq 3\gamma(G), \tag{1}$$

and they posed the following problems.

Problem 1. Characterize the graphs G satisfying $\gamma_{tR}(G) = 2\gamma(G)$.

Problem 2. Characterize the graphs G satisfying $\gamma_{tR}(G) = 3\gamma(G)$.

In this paper, we provide a constructive characterization of the trees T with $\gamma_{tR}(T) = 2\gamma(T)$ and $\gamma_{tR}(T) = 3\gamma(T)$ which settles the above problems for trees.

2. Preliminaries

In this section, we provide some results and definitions used throughout the paper. The proof of Observations 1 and 2 can be found in [6].

Observation 1 [6]**.** *If v is a strong support vertex in a graph G, then there exists a $\gamma_{tR}(G)$-function f such that $f(v) = 2$.*

Observation 2 [6]**.** *If u_1, u_2 are two adjacent support vertices in a graph G, then there exists a $\gamma_{tR}(G)$-function f such that $f(u_1) = f(u_2) = 2$.*

Observation 3. *If T is a double star, then $\gamma_{tR}(T) = 2\gamma(T)$.*

Observation 4. *Let H be a subgraph of a graph G such that G and H have no isolated vertex. If $\gamma_{tR}(H) = 3\gamma(H)$, $\gamma(G) \leq \gamma(H) + s$ and $\gamma_{tR}(G) \geq \gamma_{tR}(H) + 3s$ for some non-negative integer s, then $\gamma_{tR}(G) = 3\gamma(G)$.*

Proof. It follows from the assumptions and (1) that

$$\gamma_{tR}(G) \geq \gamma_{tR}(H) + 3s = 3\gamma(H) + 3s \geq 3\gamma(G) \geq \gamma_{tR}(G),$$

and this yields $\gamma_{tR}(G) = 3\gamma(G)$. ■

Observation 5. *Let H be a subgraph of a graph G such that G and H have no isolated vertex. If $\gamma_{tR}(G) = 3\gamma(G)$, $\gamma_{tR}(G) \leq \gamma_{tR}(H) + 3s$ and $\gamma(G) \geq \gamma(H) + s$ for some non-negative integer s, then $\gamma_{tR}(H) = 3\gamma(H)$.*

Proof. By (1) and the assumptions, we have

$$3\gamma(G) = \gamma_{tR}(G) \leq \gamma_{tR}(H) + 3s \leq 3\gamma(H) + 3s \leq 3\gamma(G),$$

and this leads to the result. ■

Similarly, we have the following results.

Observation 6. *Let H be a subgraph of a graph G such that G and H have no isolated vertex. If $\gamma_{tR}(H) = 2\gamma(H)$, $\gamma(G) \geq \gamma(H) + s$ and $\gamma_{tR}(G) \leq \gamma_{tR}(H) + 2s$ for some non-negative integer s, then $\gamma_{tR}(G) = 2\gamma(G)$.*

Observation 7. *Let H be a subgraph of a graph G such that G and H have no isolated vertex. If $\gamma_{tR}(G) = 2\gamma(G)$, $\gamma_{tR}(G) \geq \gamma_{tR}(H) + 2s$ and $\gamma(G) \leq \gamma(H) + s$ for some non-negative integer s, then $\gamma_{tR}(H) = 2\gamma(H)$.*

We close this section with some definitions.

Definition 8. Let v be a vertex of the graph G. A function $f : V(G) \to \{0, 1, 2\}$ is said to be a *nearly total Roman dominating function* (nearly TRDF) with respect to v, if the following three conditions are fulfilled:

(i) every vertex $x \in V(G) - \{v\}$ for which $f(x) = 0$ is adjacent to at least one vertex $y \in V(G)$ for which $f(y) = 2$,

(ii) every vertex $x \in V(G) - \{v\}$ for which $f(x) \geq 1$ is adjacent to at least one vertex $y \in V(G)$ for which $f(y) \geq 1$ and

(iii) $f(v) \geq 1$ or $f(v) + f(u) \geq 2$ for some $u \in N(v)$. Let

$$\gamma_{tR}(G; v) = \min\{\omega(f) \mid f \text{ is a nearly TRDF with respect to } v\}.$$

Observe that any total Roman dominating function on G is a nearly TRDF with respect to any vertex of G. Hence $\gamma_{tR}(G; v)$ is well defined and $\gamma_{tR}(G; v) \leq \gamma_{tR}(G)$ for each $v \in V(G)$. Define $W_G^1 = \{v \in V(G) \mid \gamma_{tR}(G; v) = \gamma_{tR}(G)\}$.

Definition 9. For a graph G and $v \in V(G)$, we say v has property P in G if there exists a $\gamma_{tR}(G)$-function f such that $f(v) = 2$. Assume that $W_G^2 = \{v \mid v$ has property P in $G\}$, $W_G^3 = \{v \mid v$ does not have property P in $G\}$.

We note that if a vertex $v \in V(G)$ satisfies the condition of Observations 1 or 2, then $v \in W_G^2$.

Definition 10. For a graph G and $v \in V(G)$, let

$$\gamma(G, v) = \min\{|S| : S \subseteq V(G) \text{ and each vertex } w \neq v \text{ is dominated by } S\}.$$

Clearly $\gamma(G, v) \leq \gamma(G)$ for each $v \in V(G)$. We define $W_G^4 = \{v \mid \gamma(G, v) = \gamma(G)\}$.

For a path $P_4 = v_1v_2v_3v_4$, we have $W_{P_4}^1 = W_{P_4}^2 = W_{P_4}^4 = \{v_2, v_3\}$, $W_{P_4}^3 = \{v_1, v_4\}$.

Definition 11. For a tree T, let $W_T^5 = \{v \mid$ there exists a function $f : V(T) \to \{0, 1, 2\}$ such that

(i) $\omega(f) = \gamma_{tR}(T) - 1$,

(ii) $f(v) = 1$,

(iii) every vertex $x \in V(T) - \{v\}$ for which $f(x) = 0$ is adjacent to at least one vertex $y \in V(T)$ for which $f(y) = 2$, and

(iiii) every vertex $x \in V(T) - \{v\}$ for which $f(x) \geq 1$ is adjacent to at least one vertex $y \in V(T)$ for which $f(y) \geq 1\}$.

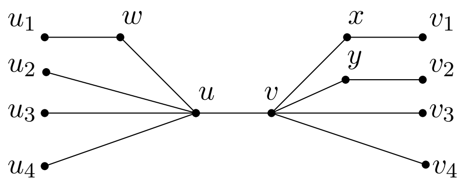


Figure 1. The graph H.

Let H be the graph illustrated in Figure 1. For any $\gamma_{tR}(H)$-function f, we have $f(u) = f(v) = 2$, $f(x) = 2$ or $f(x) = f(v_1) = 1$, $f(y) = 2$ or $f(y) = f(v_2) = 1$, $f(w) = 2$ or $f(w) = f(u_1) = 1$, and $f(z) = 0$ otherwise. It follows that $W_H^2 = \{u, v, x, y, w\}$ and $W_H^3 = \{u_i, v_i \mid i = 1, 2, 3, 4\}$. Now define $g : V(H) \to \{0, 1, 2\}$ by $g(u) = g(v) = g(x) = g(y) = 2$, $g(w) = 1$, and $g(z) = 0$ otherwise. Clearly, g is a nearly total Roman dominating function of H with respect to u_1 of weight $\gamma_{tR}(H) - 1$ yielding $u_1 \notin W_H^1$. Similarly, $v_1, v_2 \notin W_H^1$. It is easy to see that $W_H^1 = V(G) - \{u_1, v_1, v_2\}$.

To determine W_H^4, first we note that $\gamma(H) = 5$. Obviously, $\{u, v, x, y\}$ dominates all vertices in $V(H) - \{u_1\}$ and so $\gamma(H, u_1) \leq 4$ yielding $u_1 \notin W_H^4$. Similarly, $v_1, v_2 \notin W_H^4$. It is not hard to see that $W_H^4 = V(G) - \{u_1, v_1, v_2\}$.

Now, we determine W_H^5. The function $h : V(H) \to \{0, 1, 2\}$ defined by $h(u_1) = 1$, $h(u) = h(v) = h(x) = h(y) = 2$ and $h(x) = 0$ otherwise, is a function of weight $\gamma_{tR}(H) - 1$ satisfying the conditions of Definition 11 and hence $u_1 \in W_H^5$. Similarly, we have $v_1, v_2 \in W_H^5$. It is easy to verify that $W_H^5 = \{u_1, v_1, v_2\}$.

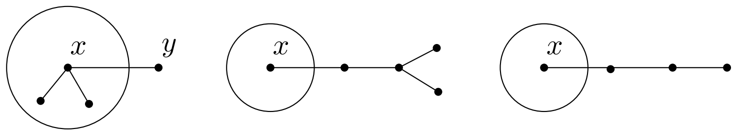
3. A Characterization of Trees T with $\gamma_{tR}(T) = 3\gamma(T)$

In this section we provide a constructive characterization of all trees T with $\gamma_{tR}(T) = 3\gamma(T)$. In order to do this, let $\mathcal{T}$ be the family of unlabeled trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_m$ $(m \geq 1)$ of trees such that T_1 is a path P_3, and, if $m \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the three operations $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ for $1 \leq i \leq m - 1$.

Operation $\mathcal{O}_1$. If $x \in V(T_i)$ and x is a strong support vertex, then Operation $\mathcal{O}_1$ adds a new vertex y and an edge xy to obtain T_{i+1}.

Operation $\mathcal{O}_2$. If $x \in W^1_{T_i}$, then Operation $\mathcal{O}_2$ adds a star $K_{1,3}$ and joins x to a leaf of it to obtain T_{i+1}.

Operation $\mathcal{O}_3$. If $x \in W^1_{T_i} \cap W^3_{T_i}$, then Operation $\mathcal{O}_3$ adds a path P_3 and joins x to a leaf of P_3 to obtain T_{i+1}.



Operation $\mathcal{O}_1$ Operation $\mathcal{O}_2$ Operation $\mathcal{O}_3$

Figure 2. The operations $\mathcal{O}_1, \mathcal{O}_2$ and $\mathcal{O}_3$.

Lemma 12. *If T_i is a tree with $\gamma_{tR}(T_i) = 3\gamma(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_1$, then $\gamma_{tR}(T_{i+1}) = 3\gamma(T_{i+1})$.*

Proof. Clearly $\gamma(T_{i+1}) = \gamma(T_i)$ and $\gamma_{tR}(T_{i+1}) = \gamma_{tR}(T_i)$ and so $\gamma_{tR}(T_{i+1}) = 3\gamma(T_{i+1})$. ■

Lemma 13. *If T_i is a tree with $\gamma_{tR}(T_i) = 3\gamma(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_2$, then $\gamma_{tR}(T_{i+1}) = 3\gamma(T_{i+1})$.*

Proof. Let $\mathcal{O}_2$ add a star $K_{1,3}$ with vertex set $\{y, y_1, y_2, y_3\}$ centered in y and join x to y_1. Obviously adding y to any $\gamma(T_i)$-set yields a dominating set of T_{i+1} and so $\gamma(T_{i+1}) \le \gamma(T_i) + 1$. Let now f be a $\gamma_{tR}(T_{i+1})$-function such that $f(y)$ is as large as possible. By Observation 1 we have $f(y) = 2$. Since f is a TRDF of G, we may assume that $f(y_1) \ge 1$. If $f(x) \ge 1$, then the function f, restricted to T_i is a nearly TRDF of T_i of weight at most $\gamma_{tR}(T_{i+1}) - 3$ and we deduce from $x \in W^1_{T_i}$ that $\gamma_{tR}(T_{i+1}) - 3 \ge \omega(f|_{T_i}) \ge \gamma_{tR}(T_i)$. If $f(x) = 0$ and $f(y_1) = 1$, then the function f, restricted to T_i is a TRDF of T_i of weight $\gamma_{tR}(T_{i+1}) - 3$ and so $\gamma_{tR}(T_{i+1}) - 3 \ge \omega(f|_{T_i}) \ge \gamma_{tR}(T_i)$. If $f(x) = 0$ and $f(y_1) = 2$, then the function $g : V(T_i) \to \{0, 1, 2\}$ defined by $g(x) = 1$ and $g(u) = f(u)$ for each $u \in V(T_i) - \{x\}$ is a nearly TRDF of T_i of weight $\gamma_{tR}(T_{i+1}) - 3$ and since $x \in W^1_{T_i}$ we have $\gamma_{tR}(T_{i+1}) - 3 \ge \omega(f|_{T_i}) \ge \gamma_{tR}(T_i)$. Hence, in all cases $\gamma_{tR}(T_{i+1}) \ge \gamma_{tR}(T_i) + 3$ and we conclude from Observation 4 that $\gamma_{tR}(T_{i+1}) = 3\gamma(T_{i+1})$. ■

Lemma 14. *If T_i is a tree with $\gamma_{tR}(T_i) = 3\gamma(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_3$, then $\gamma_{tR}(T_{i+1}) = 3\gamma(T_{i+1})$.*

Proof. Let $\mathcal{O}_3$ add a path yzw and the edge xy. Obviously any $\gamma(T_i)$-set can be extended to a dominating set of T_{i+1} by adding z and so $\gamma(T_{i+1}) \leq \gamma(T_i)+1$. Now assume f is a $\gamma_{tR}(T_{i+1})$-function such that $f(y)$ is as large as possible. Clearly $f(z) + f(w) \geq 2$. If $f(y) + f(z) + f(w) \geq 3$, then we may assume that $f(z) = 2$ and $f(y) \geq 1$ and by using an argument similar to that described in the proof of Lemma 13 we obtain $\gamma_{tR}(T_{i+1}) = 3\gamma(T_{i+1})$. Now let $f(y)+f(z)+f(w) = 2$. Then we must have $f(z) = f(w) = 1$ and $f(y) = 0$. Then the function f, restricted to T_i is a TRDF of T_i of weight $\gamma_{tR}(T_{i+1}) - 2$ with $f(x) = 2$. Since $x \in W^3_{T_i}$, we obtain $\gamma_{tR}(T_{i+1}) - 2 = \omega(f|_{T_i}) \geq \gamma_{tR}(T_i) + 1$ and so $\gamma_{tR}(T_{i+1}) \geq \gamma_{tR}(T_i) + 3$. Now the result follows by Observation 4. ■

Theorem 15. *If $T \in \mathcal{T}$, then $\gamma_{tR}(T) = 3\gamma(T)$.*

Proof. Let $T \in \mathcal{T}$. Then there exists a sequence of trees $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ such that T_1 is P_3, and if $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by one of the Operations $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ for $i = 1, 2, \ldots, k-1$.

We proceed by induction on the number of operations applied to construct T. If $k = 1$, then $T = P_3 \in \mathcal{T}$. Suppose that the result is true for each tree $T \in \mathcal{T}$ which can be obtained from a sequence of operations of length $k-1$ and let $T' = T_{k-1}$. By the induction hypothesis, we have $\gamma_{tR}(T') = 3\gamma(T')$. Since $T = T_k$ is obtained by one of the Operations $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ from T', we conclude from Lemmas 12, 13 and 14 that $\gamma_{tR}(T) = 3\gamma(T)$. ■

Now we are ready to prove the main result of this section.

Theorem 16. *Let T be a tree of order $n \geq 3$. Then $\gamma_{tR}(T) = 3\gamma(T)$ if and only if $T \in \mathcal{T}$.*

Proof. By Theorem 15, we only need to prove the necessity. Let T be a tree with $\gamma_{tR}(T) = 3\gamma(T)$. The proof is by induction on n. If $n = 3$, then the only tree T of order 3 with $\gamma_{tR}(T) = 3\gamma(T)$ is $P_3 \in \mathcal{T}$. Let $n \geq 4$ and let the statement hold for all trees T of order less than n and $\gamma_{tR}(T) = 3\gamma(T)$. Assume that T is a tree of order n with $\gamma_{tR}(T) = 3\gamma(T)$ and let f be a $\gamma_{tR}(T)$-function. By Observation 3 we have $\operatorname{diam}(T) \neq 3$. If $\operatorname{diam}(T) = 2$, then T is a star and T can be obtained from P_3 iterative application of Operation $\mathcal{O}_1$ and so $T \in \mathcal{T}$. Hence we assume $\operatorname{diam}(T) \geq 4$.

Let $v_1 v_2 \cdots v_k$ $(k \geq 5)$ be a diametrical path in T and root T at v_k. If $\deg(v_2) \geq 4$, then clearly $\gamma_{tR}(T) = \gamma_{tR}(T - v_1)$ and $\gamma(T) = \gamma(T - v_1)$ and hence $\gamma_{tR}(T - v_1) = 3\gamma(T - v_1)$. By the induction hypothesis we have $T - v_1 \in \mathcal{T}$. Now, T can be obtained from $T - v_1$ by Operation $\mathcal{O}_1$ and so $T \in \mathcal{T}$. Suppose that $\deg(v_2) \leq 3$. We consider two cases.

Case 1. $\deg(v_2) = 3$. We claim that $\deg(v_3) = 2$. Suppose, to the contrary, that $\deg(v_3) \geq 3$. Then each child of v_3 is a leaf or a support vertex. If v_3

has a children other than v_2 which is a leaf or a strong support vertex, then let $T' = T - T_{v_2}$. It is not hard to see that $\gamma(T) = \gamma(T')+1$ and $\gamma_{tR}(T) \leq \gamma_{tR}(T')+2$. Then $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 2 \leq 3\gamma(T') + 2 = 3\gamma(T) - 1$ which is a contradiction. Assume that each child of v_3 except v_2, is a support vertex of degree 2. Let $v_3z_2z_1$ be a pendant path in T. Suppose $T' = T - \{z_1, z_2\}$. As above we can see that $\gamma_{tR}(T) \leq 3\gamma(T) - 1$, a contradiction again. Thus $\deg(v_3) = 2$.

Assume $T' = T - T_{v_3}$. Let S be a $\gamma(T)$-set containing support vertices, and define $S' = S - \{v_2\}$ if $v_3 \notin S$ and $S' = (S - \{v_2, v_3\}) \cup \{v_4\}$ when $v_3 \in S$. Clearly, S' is a dominating set of T' and so $\gamma(T') \leq |S'| = \gamma(T) - 1$. On the other hand, any $\gamma_{tR}(T')$-function can be extended to a TRDF of T by assigning 1 to v_3, 2 to v_2 and 0 to the leaves adjacent to v_2. This yields $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 3$. It follows from Observation 5 that $\gamma_{tR}(T') = 3\gamma(T')$ and by the induction hypothesis we have $T' \in \mathcal{T}$. If $v_4 \notin W^1_{T'}$, then let g be a nearly TRDF of T' with respect to v_4 of weight at most $\gamma_{tR}(T') - 1$ and define $h : V(T) \to \{0, 1, 2\}$ by $h(u) = g(u)$ for $u \in V(T')$, $h(v_3) = 1, h(v_2) = 2$ and $h(u) = 0$ otherwise. Clearly h is a TRDF of T of weight $\gamma_{tR}(T') + 2$ which leads to a contradiction. Hence $v_4 \in W^1_{T'}$ and T can be obtained from T' by Operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$ in this case.

Case 2. $\deg(v_2) = 2$. Considering Case 1, we may assume that each child of v_3 is a support vertex of degree 2. If $\deg(v_3) \geq 3$, then let $T' = T - T_{v_3}$. Any $\gamma(T')$-set can be extended to a dominating set of T by adding $C(v_3)$ and so $\gamma(T) \leq \gamma(T') + |C(v_3)|$. On the other hand, let S be a $\gamma(T)$-set containing no leaves. To dominate the leaves of T_{v_3}, we must have $C(v_3) \subseteq S$. Then the set $S' = S \setminus C(v_3)$ if $v_3 \notin S$ and $S' = (S - (C(v_3) \cup \{v_3\})) \cup \{v_4\}$ if $v_3 \in S$, is a dominating set set of T' and this implies that $\gamma(T') \leq \gamma(T) - |C(v_3)|$. Hence $\gamma(T) = \gamma(T') + |C(v_3)|$.

Also, any $\gamma_{tR}(T')$-function can be extended to a TRDF of T by assigning 1 to v_3, 2 to the children of v_3 and 0 to all leaves of T_{v_3}, and so

$$\begin{aligned}
\gamma_{tR}(T) &\leq \gamma_{tR}(T') + 2|C(v_3)| + 1 \\
&\leq 3\gamma(T') + 2|C(v_3)| + 1 \\
&= 3(\gamma(T') + |C(v_3)|) - |C(v_3)| + 1 \\
&= 3\gamma(T) - |C(v_3)| + 1 \\
&< 3\gamma(T) \qquad (\text{since } |C(v_3)| \geq 2),
\end{aligned}$$

a contradiction. Henceforth, we assume $\deg(v_3) = 2$. Suppose $T' = T - T_{v_3}$. Clearly, $\gamma(T) = \gamma(T') + 1$. Analogously as in Case 1, we can see that $\gamma_{tR}(T') = 3\gamma(T')$ and $v_4 \in W^1_{T'}$. Thus $T' \in \mathcal{T}$ by the induction hypothesis. If $v_4 \notin W^3_{T'}$, then let g be a $\gamma_{tR}(T')$-function with $g(v_4) = 2$ and define $h : V(T) \to \{0, 1, 2\}$ by $h(u) = g(u)$ for $u \in V(T')$ and $h(v_3) = 0, h(v_2) = h(v_1) = 1$. Clearly h is an TRDF of T of weight $\gamma_{tR}(T') + 2$ which leads to a contradiction. Hence $v_4 \in W^3_{T'}$

and T can be obtained from T' by Operation $\mathcal{O}_3$. It follows that $T \in \mathcal{T}$ and the proof is complete. ■

4. A Characterization of Trees T with $\gamma_{tR}(T) = 2\gamma(T)$

In this section we present a constructive characterization of all trees T with $\gamma_{tR}(T) = 2\gamma(T)$.

Let $\mathcal{F}$ be the family of unlabeled trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_m$ ($m \geq 1$) of trees such that T_1 is a path P_2 or P_4, and, if $m \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the following four operations for $1 \leq i \leq m-1$.

Operation $\mathcal{T}_1$. If $x \in W^2_{T_i}$ is a support vertex, then the Operation $\mathcal{T}_1$ adds a new vertex y and an edge xy to obtain T_{i+1}.

Operation $\mathcal{T}_2$. If $x \in V(T_i)$ is at distance 2 from a leaf w, then the Operation $\mathcal{T}_2$ adds a path yz and joins x to y to obtain T_{i+1}.

Operation $\mathcal{T}_3$. If $x \in W^4_{T_i}$, then the Operation $\mathcal{T}_3$ adds a path $z_4z_3z_2z_1$ and joins x to z_3 to obtain T_{i+1}.

Operation $\mathcal{T}_4$. If $x \in W^2_{T_i} \cup W^5_{T_i}$, then the Operation $\mathcal{T}_4$ adds a path $P_3 = zyw$ and joins x to z to obtain T_{i+1}.

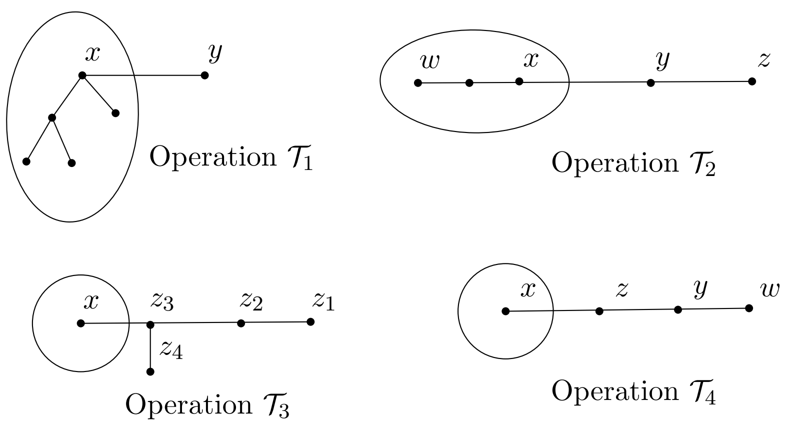


Figure 3. The operations $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ and $\mathcal{T}_4$.

Lemma 17. *If T_i is a tree with $\gamma_{tR}(T_i) = 2\gamma(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{T}_1$, then $\gamma_{tR}(T_{i+1}) = 2\gamma(T_{i+1})$.*

Proof. It is easy to see that $\gamma(T_{i+1}) = \gamma(T_i)$ and $\gamma_{tR}(T_{i+1}) = \gamma_{tR}(T_i)$ and so $\gamma_{tR}(T_{i+1}) = 2\gamma(T_{i+1})$. ■

Lemma 18. *If T_i is a tree with $\gamma_{tR}(T_i) = 2\gamma(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{T}_2$, then $\gamma_{tR}(T_{i+1}) = 2\gamma(T_{i+1})$.*

Proof. Let w' be the support vertex of w. If S is a $\gamma(T_{i+1})$-set, then clearly $y, w' \in S$ and $S - \{y\}$ is a dominating set of T_i yielding $\gamma(T_{i+1}) \geq \gamma(T_i) + 1$. Also, if f is a $\gamma_{tR}(T_i)$-function such that $f(x) \geq 1$, then f can be extended to a TRDF of T_{i+1} by assigning the weight 1 to y, z. Hence $\gamma_{tR}(T_{i+1}) \leq \gamma_{tR}(T_i) + 2$. Now the result follows by Observation 6. ■

Lemma 19. *If T_i is a tree with $\gamma_{tR}(T_i) = 2\gamma(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{T}_3$, then $\gamma_{tR}(T_{i+1}) = 2\gamma(T_{i+1})$.*

Proof. If S is a $\gamma(T_{i+1})$-set containing no leaves, then $z_3, z_2 \in S$ and we deduce from $x \in W^4_{T_i}$ that $|S - \{z_3, z_2\}| \geq \gamma(T_i)$ yielding $\gamma(T_{i+1}) \geq \gamma(T_i) + 2$. On the other hand, any $\gamma_{tR}(T_i)$-function can be extended to a TRDF of T by assigning the weight 2 to z_3, z_2 and the weight 0 to z_1, z_4 and so $\gamma_{tR}(T_{i+1}) \leq \gamma_{tR}(T_i) + 4$. It follows from Observation 6 that $\gamma_{tR}(T_{i+1}) = 2\gamma(T_{i+1})$. ■

Lemma 20. *If T_i is a tree with $\gamma_{tR}(T_i) = 2\gamma(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{T}_4$, then $\gamma_{tR}(T_{i+1}) = 2\gamma(T_{i+1})$.*

Proof. Let $\mathcal{T}_4$ add a path zyw and joins x to z. If S is a $\gamma(T_{i+1})$-set, then $y \in S$ and the set $S' = S - \{y\}$ if $z \notin S$ and $S' = (S - \{y, z\}) \cup \{x\}$ if $z \in S$, is a dominating set of T_i yielding $\gamma(T_{i+1}) \geq \gamma(T_i) + 1$. Now we show that $\gamma_{tR}(T_{i+1}) \leq \gamma_{tR}(T_i) + 2$. If $x \in W^2_{T_i}$, then let f be a $\gamma_{tR}(T_i)$-function with $f(x) = 2$. Clearly f can be extended to an TRDF of T_{i+1} by assigning the weight 1 to w, y and the weight 0 to z and so $\gamma_{tR}(T_{i+1}) \leq \gamma_{tR}(T_i) + 2$. If $x \in W^5_{T_i}$, then let f be a function satisfying the conditions of Definition 11. Clearly f can be extended to a TRDF of T_{i+1} by assigning the weight 1 to z, y, w and so $\gamma_{tR}(T_{i+1}) \leq \gamma_{tR}(T_i) + 2$. Now the result follows by Observation 6. ■

Theorem 21. *If $T \in \mathcal{F}$, then $\gamma_{tR}(T) = 2\gamma(T)$.*

Proof. Let $T \in \mathcal{F}$. Then there exists a sequence of trees $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ such that T_1 is P_2 or P_4, and if $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by one of the Operations $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{T}_4$ for $i = 1, 2, \ldots, k-1$.

We proceed by induction on the number of operations used to construct T. If $k = 1$, then $T = P_2$ or P_4 and the result is trivial. Suppose the statement holds for each tree $T \in \mathcal{F}$ which can be obtained from a sequence of operations of length $k-1$ and let $T' = T_{k-1}$. By the induction hypothesis, we have $\gamma_{tR}(T') = 2\gamma(T')$. Since $T = T_k$ is obtained by one of the Operations $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{T}_4$ we conclude from previous lemmas that $\gamma_{tR}(T) = 2\gamma(T)$. ■

Now we prove the main result of this section.

Theorem 22. *Let T be a tree of order $n \geq 2$. Then $\gamma_{tR}(T) = 2\gamma(T)$ if and only if $T \in \mathcal{F}$.*

Proof. According to Theorem 21, we only need to prove the necessity. Let T be a tree with $\gamma_{tR}(T) = 2\gamma(T)$. Since $\gamma_{tR}(K_{1,s}) = 3 = 3\gamma(K_{1,s})$ for $s \geq 2$, T is not a star of order $n(T) \geq 3$. We proceed by induction on n. If $n \in \{2, 4\}$, then the only trees T of order 2 or 4 with $\gamma_{tR}(T) = 2\gamma(T)$ are $P_2, P_4 \in \mathcal{F}$. Assume $n \geq 5$ and let the statement hold for all trees T of order less than n and $\gamma_{tR}(T) = 2\gamma(T)$. Assume that T is a tree of order n with $\gamma_{tR}(T) = 2\gamma(T)$ and let f be a $\gamma_{tR}(T)$-function. Since T is not a star, we have $\operatorname{diam}(T) \geq 3$. If $\operatorname{diam}(T) = 3$, then T is a double star and T can be obtained from P_4 by iterative application of Operation $\mathcal{T}_1$ because the support vertices of P_4 belong to $W^2_{P_4}$ and so $T \in \mathcal{F}$. Hence we assume $\operatorname{diam}(T) \geq 4$.

Let $v_1 v_2 \cdots v_k$ ($k \geq 5$) be a diametrical path in T such that $\deg(v_2)$ is as large as possible and root T at v_k. First let $\deg(v_2) \geq 3$. Clearly $\gamma_{tR}(T) \geq \gamma_{tR}(T - v_1)$ and $\gamma(T) = \gamma(T - v_1)$. If $\gamma_{tR}(T) \geq \gamma_{tR}(T - v_1) + 1$, then we have

$$2\gamma(T) = \gamma_{tR}(T) \geq \gamma_{tR}(T - v_1) + 1 \geq 2\gamma(T - v_1) + 1 = 2\gamma(T) + 1$$

which is a contradiction. Thus $\gamma_{tR}(T) = \gamma_{tR}(T - v_1)$. By Observation 1, there exists a $\gamma_{tR}(T)$-function f such that $f(v_2) = 2$. Then clearly f is a $\gamma_{tR}(T - v_1)$-function yielding $v_2 \in W^2_{T-v_1}$. Now, T can be obtained from $T - v_1$ by Operation $\mathcal{T}_1$ and so $T \in \mathcal{F}$. Suppose that $\deg(v_2) = 2$.

Consider the following cases.

Case 1. $\deg(v_3) = 2$. Let $T' = T - T_{v_3}$. Clearly

$$\gamma(T') = \gamma(T) - 1. \tag{2}$$

Now let f be a $\gamma_{tR}(T)$-function. Clearly $f(v_1) + f(v_2) \geq 2$. If $f(v_1) + f(v_2) \geq 3$, then clearly $f(v_3) = 0$ and the function f, restricted to T' is a TRDF of T' yielding $\gamma_{tR}(T) \geq \gamma_{tR}(T') + 3$. But then

$$2\gamma(T) = \gamma_{tR}(T) \geq \gamma_{tR}(T') + 3 \geq 2\gamma(T') + 3 = 2(\gamma(T) - 1) + 3 = 2\gamma(T) + 1,$$

a contradiction. Thus $f(v_1) + f(v_2) = 2$. If $f(v_3) = 1$ and $f(v_4) = 0$, then we get a contradiction as above. If $f(v_3) = 1$ and $f(v_4) \geq 1$, then the function $g : V(T') \to \{0, 1, 2\}$ defined by $g(v_5) = \min\{2, f(v_5) + 1\}$ and $g(u) = f(u)$ otherwise, is a TRDF of T' of weight $\gamma_{tR}(T) - 2$. Assume that $f(v_3) \neq 1$. If $f(v_3) = 2$, then $f(v_4) = 0$ and the function $g : V(T') \to \{0, 1, 2\}$ defined by $g(v_4) = 1, g(v_5) = \min\{2, f(v_5) + 1\}$ and $g(u) = f(u)$ otherwise, is a TRDF of T' of weight $\gamma_{tR}(T) - 2$. We conclude from

$$2\gamma(T) = \gamma_{tR}(T) \geq \gamma_{tR}(T') + 2 \geq 2\gamma(T') + 2 \geq 2(\gamma(T) - 1) + 2 = 2\gamma(T)$$

that

$$\gamma_{tR}(T) = \gamma_{tR}(T') + 2. \tag{3}$$

By (2) and (3), we obtain $\gamma_{tR}(T') = 2\gamma(T')$ and by the induction hypothesis we have $T' \in \mathcal{F}$. Now we show that $v_4 \in W^2_{T'} \cup W^5_{T'}$. Let f be a $\gamma_{tR}(T)$-function. As above we can see that $f(v_1) + f(v_2) = 2$. If $f(v_3) = 0$, then the function f restricted to T' is a $\gamma_{tR}(T')$-function with $f(v_4) = 2$ implying that $v_4 \in W^2_{T'}$. If $f(v_3) = 2$ and v_4 has a neighbor with positive weight under f, then the function $g : V(T') \to \{0,1,2\}$ defined by $g(v_4) = 1$ and $g(x) = f(x)$ otherwise, is a TRDF of T' of weight $\gamma_{tR}(T) - 3$ contradicting (3). If $f(v_3) = 2$ and v_4 has no neighbor other than v_3 with positive weight under f, then the function $g : V(T') \to \{0,1,2\}$ defined by $g(v_4) = 1$ and $g(x) = f(x)$ otherwise, is a function of weight $\gamma_{tR}(T) - 3 = \gamma_{tR}(T') - 1$ satisfying the conditions of Definition 11 and so $v_4 \in W^5_{T'}$. Suppose that $f(v_3) = 1$. We can see as above that $f(v_4) \geq 1$. If $f(v_4) = 2$, then the function $g : V(T') \to \{0,1,2\}$ defined by $g(v_5) = \min\{2, f(v_5) + 1\}$ and $g(x) = f(x)$ otherwise, is a $\gamma_{tR}(T')$-function with $g(v_4) = 2$ implying that $v_4 \in W^2_{T'}$. If $f(v_4) = 1$ and v_4 has a neighbor different from v_3 with positive weight under f, then the function f restricted to T' is a TRDF of T' of weight $\gamma_{tR}(T) - 3$ which contradicts (3). Finally if $f(v_4) = 1$ and v_4 has no neighbor other than v_3 with positive weight, then the function f restricted to T' fulfilled the conditions of Definition 11 and so $v_4 \in W^5_{T'}$. Thus $v_4 \in W^2_{T'} \cup W^5_{T'}$ and T can be obtained from T' by operation $\mathcal{T}_4$ and so $T \in \mathcal{F}$.

Case 2. $\deg(v_3) \geq 3$. By the choice of diametrical path, we may assume that all the children of v_3 with depth one have degree 2. We consider three subcases.

Subcase 2.1. v_3 is a support vertex and is at distance 2 from some leaves different from v_1. Let $T' = T - \{v_1, v_2\}$. Then clearly $\gamma(T) = \gamma(T') + 1$ and $\gamma_{tR}(T) \geq \gamma_{tR}(T') + 2$. Hence $\gamma_{tR}(T') = 2\gamma(T')$ by Observation 7. By the induction hypothesis we have $T' \in \mathcal{F}$ and hence T can be obtained from T' by Operation $\mathcal{T}_2$ and so $T \in \mathcal{F}$.

Subcase 2.2. All children of v_3 have degree 2. Let $v_3z_2z_1$ be a pendant path and let $T' = T - \{v_1, v_2\}$. Clearly $\gamma(T) = \gamma(T') + 1$. Now let f be a $\gamma_{tR}(T)$-function. Then $f(v_2) \geq 1$, $f(v_1) + f(v_2) \geq 2$ and $f(z_1) + f(z_2) \geq 2$. If $f(v_3) \geq 1$ or $f(v_3) = 0$ and $f(v_2) = 1$, then the function f restricted to T' is a TRDF of T' of weight $\omega(f) - 2$ and so $\gamma_{tR}(T) \geq \gamma_{tR}(T') + 2$. Assume that $f(v_3) = 0$ and $f(v_2) = 2$. Since f is a TRDF of T, we have $f(v_1) = 1$. Then the function $g : V(T) \to \{0,1,2\}$ defined by $g(v_3) = g(v_2) = g(v_1) = 1$ and $g(x) = f(x)$ otherwise, is a $\gamma_{tR}(T)$-function and as above we obtain $\gamma_{tR}(T) \geq \gamma_{tR}(T') + 2$. Hence $\gamma_{tR}(T') = 2\gamma(T')$ by Observation 7. By the induction hypothesis we have $T' \in \mathcal{F}$ and so T can be obtained from T' by Operation $\mathcal{T}_2$. Thus $T \in \mathcal{F}$.

Subcase 2.3. All children of v_3 except v_2 are leaves. Let w be a leaf adjacent to v_3. First let v_3 be a strong support vertex. It is easy to see that $\gamma(T) = \gamma(T - w)$

and $\gamma_{tR}(T) = \gamma_{tR}(T - w)$ yielding $\gamma_{tR}(T - w) = 2\gamma(T - w)$. By the induction hypothesis we have $T - w \in \mathcal{F}$ and by Observation 2 we obtain $v_3 \in W^2_{T-w}$. Thus T can be obtained from $T - w$ by Operation $\mathcal{T}_1$ and so $T \in \mathcal{F}$. Suppose next that v_3 is not a strong support vertex. Then by the assumption we have $\deg(v_3) = 3$. Consider the following.

(a) v_4 is a support vertex. Let $T' = T - T_{v_2}$. It is easy to see that $\gamma_{tR}(T) = \gamma_{tR}(T') + 2$ and $\gamma(T) = \gamma(T') + 1$. It follows that $\gamma_{tR}(T') = 2\gamma(T')$ and by the induction hypothesis we have $T' \in \mathcal{F}$. Then T can be obtained from T' by Operation $\mathcal{T}_2$ and so $T \in \mathcal{F}$.

(b) v_4 has a child z_2 with depth 1. As above we may assume that $\deg(z_2) = 2$. Let z_1 be the leaf adjacent to z_2 and let $T' = T - \{z_1, z_2\}$. Clearly $\gamma(T) = \gamma(T') + 1$. By Observation 2, there exists a $\gamma_{tR}(T)$-function f such that $f(v_2) = f(v_3) = 2$. Also we have $f(z_1) + f(z_2) \geq 2$. Obviously the function f restricted to T' is a TRDF of T' and so $\gamma_{tR}(T) \geq \gamma_{tR}(T') + 2$. We conclude from $2\gamma(T) = \gamma_{tR}(T) \geq \gamma_{tR}(T') + 2 \geq 2\gamma(T') + 2 = 2\gamma(T)$ that $\gamma_{tR}(T') = 2\gamma(T')$ and by the induction hypothesis we have $T' \in \mathcal{T}$. Now T can be obtained from T' by Operation $\mathcal{T}_2$ and so $T \in \mathcal{F}$.

(c) v_4 has a child z_3 with depth 2. Let $v_4z_3z_2z_1$ be a path in T. Using the above argument we may assume that $\deg(z_2) = 2$ and either $\deg(z_3) = 2$ or $\deg(z_3) = 3$ and z_3 is a support vertex. If $\deg(z_3) = 2$, then as in Case 1 we can see that $T \in \mathcal{F}$.

Let $\deg(z_3) = 3$ and z_3 is a support vertex. Let $T' = T - T_{z_3}$. It is not hard to see that $\gamma(T) = \gamma(T') + 2$ and $\gamma_{tR}(T) = \gamma_{tR}(T') + 4$. This implies that $\gamma_{tR}(T') = 2\gamma(T')$ and by the induction hypothesis we have $T' \in \mathcal{F}$. Since v_4 is adjacent to a support vertex, we deduce that $v_4 \in W^4_{T'}$. Now T can be obtained from T' by Operation $\mathcal{T}_3$ and so $T \in \mathcal{F}$.

This completes the proof. ∎

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