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Note

MINIMUM EDGE CUTS IN DIAMETER 2 GRAPHS

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Abstract

Plesnik proved that the edge connectivity and minimum degree are equal for diameter 2 graphs. We provide a streamlined proof of this fact and characterize the diameter 2 graphs with a nontrivial minimum edge cut.

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Let G be a graph. For S, $T \subseteq V(G)$, let $[S,T]$ be the set of edges with one end in S and the other in T. An edge cut of a graph G is a set $X = [S,T]$, of edges so that $G - X$ has more components than G. The edge connectivity $\lambda(G)$ of a connected graph is the smallest size of an edge cut. A disconnected graph has $\lambda(G) = 0$. Often we can express an edge cut as $\left[S, \overline{S}\right]$, where $\overline{S} = V(G) \setminus S$.

Denote the minimum degree of G by $\delta(G)$. It is well-known that $\lambda(G) \leq \delta(G)$, since the edges incident with a vertex of minimum degree form an edge cut. Plesnik proved that this is an equality for diameter 2 graphs. We present a shorter proof.

Theorem 1 [3]**.** *If G has diameter 2, then $\lambda(G) = \delta(G)$.*

Proof. Let $[S, \overline{S}]$ be a minimum edge cut. Now S and $\overline{S}$ cannot both have vertices u and v that are not incident with $[S, \overline{S}]$, for then $diam(G) \geq d(u, v) \geq 3$. Say S has every vertex incident with $[S, \overline{S}]$. Thus $|S| \leq \left|[S, \overline{S}]\right| = \lambda(G) \leq \delta(G)$. Each vertex in S is incident with at most $|S| - 1$ edges in $G[S]$, and so at least $\delta(G) - |S| + 1$ edges in $[S, \overline{S}]$. Thus

$$\lambda(G) = \left|\, [S, \overline{S}] \,\right| \geq |S| \left(\delta(G) - |S| + 1\right).$$

This last expression attains its minimum value of $\delta(G)$ when $|S| = 1$ or $|S| = \delta(G)$. In both cases we have $\lambda(G) \geq \delta(G)$, so $\lambda(G) = \delta(G)$. ■

The following corollary follows from the proof of this theorem.

Corollary 2 [1]**.** *If G has diameter 2, then one of the subgraphs on one side of a minimum edge cut is either K_1 or $K_{\delta(G)}$.*

A trivial edge cut is an edge cut whose deletion isolates a single vertex. To study those diameter 2 graphs with a nontrivial minimum edge cut, we define the following set of graphs.

Definition. Let $\mathbb{G}$ be the set of graphs that contains the Cartesian product $K_{\frac{n}{2}} \square K_2$, $n \geq 4$, and those graphs that can be constructed as follows. Let H_1 be a graph with order $d > 1$ and $\delta(H_1) \geq d - r - 1$ and H_2 be a graph with order r. Add a perfect matching between K_d and H_1 and join all the vertices of H_1 and H_2 (see Figure 1).

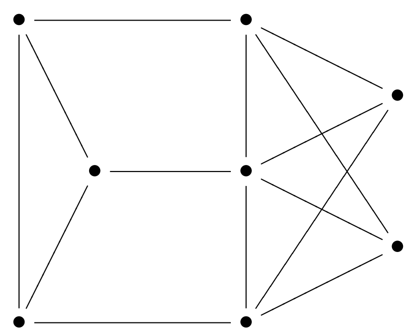

Figure 1. A graph in $\mathbb{G}$ with $d = 3$, $H_1 = P_3$, and $H_2 = 2K_1$.

Theorem 3. *A graph has diameter 2 and contains a non-trivial minimum edge cut if and only if it is in set $\mathbb{G}$.*

Proof. (⇐) It is readily checked that a graph $G \in \mathbb{G}$ has diameter 2, $\delta(G) = d = \lambda(G)$, and contains a nontrivial minimum edge cut.

(⇒) Let G have diameter 2 and contain a non-trivial minimum edge cut $[S, \overline{S}]$, and let $d = \delta(G)$. Then (say) $S = K_d$, and the order of $\overline{S}$ is at least d. If it is exactly d, then $G = K_{\frac{n}{2}} \,\square\, K_2$. If not, then $\overline{S}$ contains vertices not adjacent to any vertex of K_d. Let H_2 be the subgraph induced by these vertices and $H_1 = \overline{S} - H_2$. Then each vertex of H_2 is adjacent to each vertex of H_1 since otherwise G would not have diameter 2. Since G has minimum degree d, H_1 must have minimum degree at least $d - r - 1$. ■

Corollary 4. *If $G \in \mathbb{G}$, then it has between d and* $\max\{n-d, 3d-1\}$ *trivial minimum edge cuts.*

Proof. The number of trivial minimum edge cuts is the number of vertices of minimum degree. All the vertices of K_d have minimum degree, so this is at least d. Now $K_{\frac{n}{2}} \,\square\, K_2$ has $n = 2d$ such vertices. If G is regular, then it has at most $d + d + (d-1)$ vertices since each vertex in H_1 has degree at least $1 + n(H_2)$. If $n(H_2) \geq d$ then each vertex in H_1 has degree more than d, so there are at most $n - d$ minimum degree vertices. ■

Corollary 5. *All graphs in set $\mathbb{G}$ have a single non-trivial minimum edge cut except for C_4 and C_5.*

Proof. Let $G \in \mathbb{G}$, so $\delta(G) \geq 2$. If $\delta(G) = 2$, then C_4 and C_5 have two and five nontrivial edge cuts, respectively. Now $C_5 + e$ has a single non-trivial minimum edge cut. Let u and v be the vertices in H_1. If there are at least two vertices in H_2, then G has a spanning subgraph with $n - 4$ $u - v$ paths of length 2 and one $u - v$ path of length 3. Hence the result holds for $\delta(G) = 2$.

Let $d = \delta(G) > 2$. Assume the result holds for graphs with minimum degree $d - 1$. Then no nontrivial minimum edge cut separates vertices in K_d. Now $H = G - K_d$ has $\mathit{diam}(H) \leq 2$ and $\delta(H) \geq d - 1$. Now H is not C_4 or C_5, so it has at most one nontrivial minimum edge cut. If it has such a cut, then there are at least $d - 1$ vertices on each side of it, so $n(H_2) \geq d - 2$. Then H contains spanning subgraph $K_{d,n(H_2)}$. But this graph has no nontrivial minimum edge cut, so neither does H. Then G has no other nontrivial minimum edge cut. ■

Finally, we consider the nature of minimum edge cuts in almost all graphs.

Theorem 6. *Almost all graphs have a single minimum edge cut, which is trivial.*

Proof. In random graph theory, it is known that almost all graphs have diameter 2 [1]. This implies that $\lambda(G) = \delta(G)$ for almost all graphs. Erdős and Wilson

[2] showed that almost all graphs have a unique vertex of maximum degree. By symmetry, almost all graphs have a unique vertex of minimum degree.

Those graphs with a minimum non-trivial edge cut have the structure described in Theorem 3, including at least $\delta(G) > 1$ vertices of minimum degree. Hence almost all graphs have a single minimum edge cut, which is trivial. ■

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