Discussiones Mathematicae
Graph Theory 39 (2019) 605-608
doi:10.7151/dmgt. 2097
Note

# MINIMUM EDGE CUTS IN DIAMETER 2 GRAPHS 

Allan Bickle<br>Department of Mathematics<br>Penn State Altoona<br>3000 Ivyside Park<br>Altoona, PA 16601 USA<br>e-mail: allanbickle.wordpress.com<br>AND<br>Allen Schwenk<br>Department of Mathematics<br>Western Michigan University<br>1903 W. Michigan<br>Kalamazoo, MI 49008 USA<br>e-mail: schwenk@wmich.edu


#### Abstract

Plesnik proved that the edge connectivity and minimum degree are equal for diameter 2 graphs. We provide a streamlined proof of this fact and characterize the diameter 2 graphs with a nontrivial minimum edge cut.


Keywords: edge connectivity, diameter.
2010 Mathematics Subject Classification: 05C40.

Let $G$ be a graph. For $S, T \subseteq V(G)$, let $[S, T]$ be the set of edges with one end in $S$ and the other in $T$. An edge cut of a graph $G$ is a set $X=[S, T]$, of edges so that $G-X$ has more components than $G$. The edge connectivity $\lambda(G)$ of a connected graph is the smallest size of an edge cut. A disconnected graph has $\lambda(G)=0$. Often we can express an edge cut as $[S, \bar{S}]$, where $\bar{S}=V(G) \backslash S$.

Denote the minimum degree of $G$ by $\delta(G)$. It is well-known that $\lambda(G) \leq$ $\delta(G)$, since the edges incident with a vertex of minimum degree form an edge cut. Plesnik proved that this is an equality for diameter 2 graphs. We present a shorter proof.

Theorem 1 [3]. If $G$ has diameter 2, then $\lambda(G)=\delta(G)$.
Proof. Let $[S, \bar{S}]$ be a minimum edge cut. Now $S$ and $\bar{S}$ cannot both have vertices $u$ and $v$ that are not incident with $[S, \bar{S}]$, for then $\operatorname{diam}(G) \geq d(u, v) \geq$ 3. Say $S$ has every vertex incident with $[S, \bar{S}]$. Thus $|S| \leq|[S, \bar{S}]|=\lambda(G) \leq$ $\delta(G)$. Each vertex in $S$ is incident with at most $|S|-1$ edges in $G[S]$, and so at least $\delta(G)-|S|+1$ edges in $[S, \bar{S}]$. Thus

$$
\lambda(G)=|[S, \bar{S}]| \geq|S|(\delta(G)-|S|+1)
$$

This last expression attains its minimum value of $\delta(G)$ when $|S|=1$ or $|S|=$ $\delta(G)$. In both cases we have $\lambda(G) \geq \delta(G)$, so $\lambda(G)=\delta(G)$.

The following corollary follows from the proof of this theorem.
Corollary 2 [1]. If $G$ has diameter 2 , then one of the subgraphs on one side of a minimum edge cut is either $K_{1}$ or $K_{\delta(G)}$.

A trivial edge cut is an edge cut whose deletion isolates a single vertex. To study those diameter 2 graphs with a nontrivial minimum edge cut, we define the following set of graphs.

Definition. Let $\mathbb{G}$ be the set of graphs that contains the Cartesian product $K_{\frac{n}{2}} \square K_{2}, n \geq 4$, and those graphs that can be constructed as follows. Let $H_{1}$ be a graph with order $d>1$ and $\delta\left(H_{1}\right) \geq d-r-1$ and $H_{2}$ be a graph with order $r$. Add a perfect matching between $K_{d}$ and $H_{1}$ and join all the vertices of $H_{1}$ and $H_{2}$ (see Figure 1).


Figure 1. A graph in $\mathbb{G}$ with $d=3, H_{1}=P_{3}$, and $H_{2}=2 K_{1}$.

Theorem 3. A graph has diameter 2 and contains a non-trivial minimum edge cut if and only if it is in set $\mathbb{G}$.

Proof. $(\Leftarrow)$ It is readily checked that a graph $G \in \mathbb{G}$ has diameter $2, \delta(G)=$ $d=\lambda(G)$, and contains a nontrivial minimum edge cut.
$(\Rightarrow)$ Let $G$ have diameter 2 and contain a non-trivial minimum edge cut $[S, \bar{S}]$, and let $d=\delta(G)$. Then (say) $S=K_{d}$, and the order of $\bar{S}$ is at least $d$. If it is exactly $d$, then $G=K_{\frac{n}{2}} \square K_{2}$. If not, then $\bar{S}$ contains vertices not adjacent to any vertex of $K_{d}$. Let ${ }_{2}^{2}$ be the subgraph induced by these vertices and $H_{1}=\bar{S}-H_{2}$. Then each vertex of $H_{2}$ is adjacent to each vertex of $H_{1}$ since otherwise $G$ would not have diameter 2 . Since $G$ has minimum degree $d, H_{1}$ must have minimum degree at least $d-r-1$.

Corollary 4. If $G \in \mathbb{G}$, then it has between $d$ and $\max \{n-d, 3 d-1\}$ trivial minimum edge cuts.

Proof. The number of trivial minimum edge cuts is the number of vertices of minimum degree. All the vertices of $K_{d}$ have minimum degree, so this is at least $d$. Now $K_{\frac{n}{2}} \square K_{2}$ has $n=2 d$ such vertices. If $G$ is regular, then it has at most $d+d+(d-1)$ vertices since each vertex in $H_{1}$ has degree at least $1+n\left(H_{2}\right)$. If $n\left(H_{2}\right) \geq d$ then each vertex in $H_{1}$ has degree more than $d$, so there are at most $n-d$ minimum degree vertices.

Corollary 5. All graphs in set $\mathbb{G}$ have a single non-trivial minimum edge cut except for $C_{4}$ and $C_{5}$.

Proof. Let $G \in \mathbb{G}$, so $\delta(G) \geq 2$. If $\delta(G)=2$, then $C_{4}$ and $C_{5}$ have two and five nontrivial edge cuts, respectively. Now $C_{5}+e$ has a single non-trivial minimum edge cut. Let $u$ and $v$ be the vertices in $H_{1}$. If there are at least two vertices in $H_{2}$, then $G$ has a spanning subgraph with $n-4 u-v$ paths of length 2 and one $u-v$ path of length 3 . Hence the result holds for $\delta(G)=2$.

Let $d=\delta(G)>2$. Assume the result holds for graphs with minimum degree $d-1$. Then no nontrivial minimum edge cut separates vertices in $K_{d}$. Now $H=G-K_{d}$ has $\operatorname{diam}(H) \leq 2$ and $\delta(H) \geq d-1$. Now $H$ is not $C_{4}$ or $C_{5}$, so it has at most one nontrivial minimum edge cut. If it has such a cut, then there are at least $d-1$ vertices on each side of it, so $n\left(H_{2}\right) \geq d-2$. Then $H$ contains spanning subgraph $K_{d, n\left(H_{2}\right)}$. But this graph has no nontrivial minimum edge cut, so neither does $H$. Then $G$ has no other nontrivial minimum edge cut.

Finally, we consider the nature of minimum edge cuts in almost all graphs.
Theorem 6. Almost all graphs have a single minimum edge cut, which is trivial.
Proof. In random graph theory, it is known that almost all graphs have diameter 2 [1]. This implies that $\lambda(G)=\delta(G)$ for almost all graphs. Erdős and Wilson
[2] showed that almost all graphs have a unique vertex of maximum degree. By symmetry, almost all graphs have a unique vertex of minimum degree.

Those graphs with a minimum non-trivial edge cut have the structure described in Theorem 3, including at least $\delta(G)>1$ vertices of minimum degree. Hence almost all graphs have a single minimum edge cut, which is trivial.

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Received 12 June 2017
Accepted 18 November 2017

