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# SUFFICIENT CONDITIONS FOR MAXIMALLY EDGE-CONNECTED AND SUPER-EDGE-CONNECTED GRAPHS DEPENDING ON THE CLIQUE NUMBER

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**Abstract**

Let $G$ be a connected graph with minimum degree $\delta$ and edge-connectivity $\lambda$. A graph is maximally edge-connected if $\lambda = \delta$, and it is super-edge-connected if every minimum edge-cut is trivial; that is, if every minimum edge-cut consists of edges incident with a vertex of minimum degree. The clique number $\omega(G)$ of a graph $G$ is the maximum cardinality of a complete subgraph of $G$. In this paper, we show that a connected graph $G$ with clique number $\omega(G) \leq r$ is maximally edge-connected or super-edge-connected if the number of edges is large enough. These are generalizations of corresponding results for triangle-free graphs by Volkmann and Hong in 2017.

**Keywords:** edge-connectivity, clique number, maximally edge-connected graphs, super-edge-connected graphs.

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## 1. Terminology and Introduction

Let $G$ be a finite and simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* and *size* of $G$ are defined by $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$, respectively. If $N(v) = N_G(v)$ is the neighborhood of the vertex $v \in V(G)$, then we denote by $d(v) = d_G(v) = |N(v)|$ the *degree* of $v$ and by $\delta = \delta(G)$ the *minimum degree* of the graph $G$. For a subset $X \subseteq V(G)$, let $G[X]$ to denote the subgraph of $G$ induced by $X$. For two subsets $X$ and $Y$ of $V(G)$ let $[X, Y]$ be the set of edges with one endpoint in $X$ and the other one in $Y$. The *clique number* $\omega(G)$ of a graph $G$ is the maximum cardinality of a complete subgraph of $G$. An *edge-cut* of a connected graph $G$ is a set of edges whose removal disconnects $G$. The *edge connectivity* $\lambda = \lambda(G)$ of a connected graph $G$ is defined as the minimum cardinality of an edge-cut over all edge-cuts of $G$. An edge-cut $S$ is a *minimum edge-cut* or a *$\lambda$-cut* if $|S| = \lambda(G)$. The inequality $\lambda(G) \leq \delta(G)$ is immediate. We call a connected graph *maximally edge-connected*, if $\lambda(G) = \delta(G)$. In 1981, Bauer *et al.* [1] proposed the concept of super-edge connectedness. A graph is called *super-edge-connected* or *super-$\lambda$* if every minimum edge-cut is trivial; that is, if every minimum edge-cut consists of edges incident with a vertex of minimum degree. Thus every super-edge-connected graph is also maximally edge-connected.

Sufficient conditions for graphs to be maximally edge-connected or super-edge-connected were given by several authors, see for example the survey paper by Hellwig and Volkmann [3]. The starting point was an article by Chartrand [2] in 1966. He observed that if $\delta$ is large enough, then the graph is maximally edge-connected. A similar condition for super-edge-connectivity was given by Kelmans [4] six years later. Over the years, these results have been strengthened many times and in many ways.

Recently, Volkmann and Hong [6] showed that a connected graph or a connected triangle-free graph is maximally edge-connected or super-$\lambda$ if the number of edges is large enough. In particular, they received the following results.

**Theorem 1.** *Let $G$ be a connected triangle-free graph of order $n \geq 2$, size $m$, minimum degree $\delta$ and edge-connectivity $\lambda$. If*

$$m > \left\lfloor \frac{n^2}{4} \right\rfloor - \delta(n - 1 - 2\delta) - 1,$$

*then $\lambda = \delta$.*

**Theorem 2.** *Let $G$ be a connected triangle-free graph of order $n$, size $m$, minimum degree $\delta \geq 3$ and edge-connectivity $\lambda$. If*

$$m > \left\lfloor \frac{(n+1)^2}{4} \right\rfloor - \delta(n + 1 - 2\delta),$$

*then $G$ is super-$\lambda$.*

In this paper, we will generalize Theorems 1 and 2 to connected graphs with clique number $\omega(G) \leq r$ for $r \geq 2$. Examples will demonstrate that our results are sharp.

## 2. MAXIMALLY EDGE-CONNECTED GRAPHS

The main tool of our article is the famous theorem of Turán [5].

**Theorem 3.** *Let $r \geq 1$ be an integer, and let $G$ be a graph of order $n$. If the clique number $\omega(G) \leq r$, then*

$$|E(G)| \leq \left\lfloor \frac{r-1}{2r} \cdot n^2 \right\rfloor.$$

**Theorem 4.** *Let $r \geq 2$ be an integer, and let $G$ be a connected graph of order $n$, size $m$, minimum degree $\delta \geq 1$, edge-connectivity $\lambda$ and clique number $\omega(G) \leq r$. If*

$$m > \left\lfloor \frac{r-1}{2r} \left( n^2 + 2 \left\lfloor \frac{r\delta}{r-1} \right\rfloor^2 - 2n \left\lfloor \frac{r\delta}{r-1} \right\rfloor \right) \right\rfloor + \delta - 1,$$

*then $\lambda = \delta$.*

***Proof.*** If $\delta = 1$, then $\lambda = \delta$ in every case. Thus assume in the following that $\delta \geq 2$. Suppose to the contrary that $\lambda \leq \delta - 1$. Then there exist two disjoint sets $X, Y \subset V(G)$ with $X \cup Y = V(G)$ and $|[X,Y]| = \lambda$. Assume, without loss of generality, that $|X| \leq |Y|$.

We first show that $X$ contains at least $\delta + 1$ vertices. Otherwise, suppose that $X$ contains at most $\delta$ vertices. Then we obtain

$$\delta |X| \leq \sum_{x \in X} d_G(x) \leq |X|(|X| - 1) + \lambda \leq \delta(|X| - 1) + \delta - 1.$$

Obviously, this is a contradiction and thus $|X| \geq \delta + 1$. Using Theorem 3, we conclude that

$$|E(G[X])| \leq \left\lfloor \frac{(r-1)|X|^2}{2r} \right\rfloor \tag{1}$$

and

$$|E(G[Y])| \leq \left\lfloor \frac{(r-1)|Y|^2}{2r} \right\rfloor. \tag{2}$$

Next we show that $|X| \geq \lfloor (r\delta)/(r-1) \rfloor$. Suppose to the contrary that $|X| \leq \lfloor (r\delta)/(r-1) \rfloor - 1$. Since $2\left|E\left(G[X]\right)\right| = \sum_{x \in X} d_G(x) - \lambda$, (1) implies that

$$\begin{aligned}
|X|\delta &\leq \sum_{x \in X} d_G(x) \leq 2 \left\lfloor \frac{(r-1)|X|^2}{2r} \right\rfloor + \lambda \leq \frac{(r-1)|X|^2}{r} + \delta - 1 \\
&\leq |X| \frac{r-1}{r} \left( \left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1 \right) + \delta - 1 \leq |X| \frac{r-1}{r} \left( \frac{r\delta}{r-1} - 1 \right) + \delta - 1 \\
&= |X|\delta - \frac{r-1}{r}|X| + \delta - 1
\end{aligned}$$

and thus $|X| \leq \frac{r(\delta-1)}{r-1}$. Using this argument once more, we arrive at

$$\begin{aligned} |X|\delta &\leq \frac{(r-1)|X|^2}{r} + \delta - 1 \leq |X|\frac{r-1}{r} \cdot \frac{r(\delta-1)}{r-1} + \delta - 1 \\ &= |X|(\delta-1) + \delta - 1 \end{aligned}$$

and thus $|X| \leq \delta - 1$, which contradicts the fact that $|X| \geq \delta + 1$. Hence $|X| \geq \lfloor (r\delta)/(r-1) \rfloor$. Since $|X| + |Y| = n$ and $|X| \leq n/2$, the inequalities (1) and (2) lead to

$$\begin{aligned} m &= |E(G[X])| + |E(G[Y])| + \lambda \\ &\leq \left\lfloor \frac{(r-1)|X|^2}{2r} \right\rfloor + \left\lfloor \frac{(r-1)|Y|^2}{2r} \right\rfloor + \delta - 1 \\ &= \left\lfloor \frac{(r-1)|X|^2}{2r} \right\rfloor + \left\lfloor \frac{(r-1)(n-|X|)^2}{2r} \right\rfloor + \delta - 1 \\ &\leq \left\lfloor \frac{(r-1)}{2r} \left( |X|^2 + (n-|X|)^2 \right) \right\rfloor + \delta - 1 \\ &= \left\lfloor \frac{(r-1)}{2r} \left( n^2 + 2(|X|^2 - n|X|) \right) \right\rfloor + \delta - 1 \\ &\leq \left\lfloor \frac{(r-1)}{2r} \left( n^2 + 2\left\lfloor \frac{r\delta}{r-1} \right\rfloor^2 - 2n \left\lfloor \frac{r\delta}{r-1} \right\rfloor \right) \right\rfloor + \delta - 1, \end{aligned}$$

a contradiction to the hypothesis. Thus $\lambda = \delta$. ■

Theorem 1 is the special case $r = 2$ of Theorem 4. The next family of graphs shows that Theorem 4 is best possible in the sense that

$$m = \left\lfloor \frac{r-1}{2r} \left( n^2 + 2\left\lfloor \frac{r\delta}{r-1} \right\rfloor^2 - 2n \left\lfloor \frac{r\delta}{r-1} \right\rfloor \right) \right\rfloor + \delta - 1$$

does not guarantee $\lambda = \delta$.

**Example 5.** Let $r \geq 2$ and $q \geq 1$ be integers. Let $H_1$ and $H_2$ be two disjoint copies of the complete $r$-partite graph with $q$ vertices in each partite set. Define $H$ as the union of $H_1$ and $H_2$ by adding $\delta - 1 = q(r-1) - 1$ edges between $H_1$ and $H_2$ such that $\omega(H) \leq r$. Then $H$ has order $n = 2qr$, minimum degree $\delta = q(r-1)$ such that

$$\begin{aligned} m(H) &= q^2 r(r-1) + q(r-1) - 1 \\ &= \left\lfloor \frac{r-1}{2r} \left( n^2 + 2\left\lfloor \frac{r\delta}{r-1} \right\rfloor^2 - 2n \left\lfloor \frac{r\delta}{r-1} \right\rfloor \right) \right\rfloor + \delta - 1, \end{aligned}$$

but obviously, $\lambda(H) = \delta(H) - 1$.

## 3. Super Edge-Connected Graphs

**Theorem 6.** *Let $r \geq 2$ be an integer, and let $G$ be a connected graph of order $n$, size $m$, minimum degree $\delta \geq 2$, edge-connectivity $\lambda$ and $\omega(G) \leq r$. If $\delta \geq 3$ or $r \geq 3$ and*

$$m > \left\lfloor \frac{r-1}{2r} \left( n^2 + 2\left( \left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1 \right)^2 - 2n \left( \left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1 \right) \right) \right\rfloor + \delta,$$

*then $G$ is super-$\lambda$.*

***Proof.*** Suppose to the contrary that $G$ is not super-$\lambda$. Then there exist two disjoint sets $X, Y \subset V(G)$ such that $X \cup Y = V(G)$, $|X|, |Y| \geq 2$ and $|[X, Y]| = \lambda$. Assume, without loss of generality, that $2 \leq |X| \leq |Y|$.

We first show that $X$ contains at least $\delta$ vertices. Otherwise, suppose that $X$ contains at most $\delta - 1$ vertices. Then we obtain

$$\delta|X| \leq \sum_{x \in X} d_G(x) \leq |X|\big(|X| - 1\big) + \lambda \leq (\delta - 1)\big(|X| - 1\big) + \delta,$$

which implies that $|X| \leq 1$, contradicting that $|X| \geq 2$. Thus $|X| \geq \delta$.

Next we show that $|X| \geq \lfloor (r\delta)/(r-1) \rfloor - 1$. If $\delta = 2$ and $r \geq 3$, then

$$|X| \geq \delta = 2 \geq \lfloor (2r)/(r-1) \rfloor - 1 = \lfloor (r\delta)/(r-1) \rfloor - 1.$$

Let now $\delta \geq 3$. Suppose to the contrary that $X$ contains at most $\lfloor (r\delta)/(r-1) \rfloor - 2$ vertices. Since $2|E(G[X])| = \sum_{x \in X} d_G(x) - \lambda$, we conclude from (1) that

$$\begin{aligned}
|X|\delta &\leq \sum_{x \in X} d_G(x) \leq 2 \left\lfloor \frac{(r-1)|X|^2}{2r} \right\rfloor + \lambda \leq \frac{(r-1)|X|^2}{r} + \delta \\
&\leq |X| \frac{r-1}{r} \left( \left\lfloor \frac{r\delta}{r-1} \right\rfloor - 2 \right) + \delta \leq |X| \frac{r-1}{r} \left( \frac{r\delta}{r-1} - 2 \right) + \delta \\
&= |X|\delta - \frac{2(r-1)}{r}|X| + \delta
\end{aligned}$$

and thus $|X| \leq \frac{r\delta}{2(r-1)}$. Using this argument once more, we arrive at

$$|X|\delta \leq \frac{(r-1)|X|^2}{r} + \delta \leq |X| \frac{r-1}{r} \cdot \frac{r\delta}{2(r-1)} + \delta = \frac{|X|\delta}{2} + \delta$$

and thus $|X| \leq 2$, which contradicts the fact that $3 \leq \delta \leq |X|$. Hence we have shown that $|X| \geq \lfloor (r\delta)/(r-1) \rfloor - 1$ when $\delta \geq 3$ or $r \geq 3$. Since $|X| + |Y| = n$

and $|X| \le n/2$, the inequalities (1) and (2) lead to

$$\begin{aligned} m &= \left|E\left(G[X]\right)\right| + \left|E\left(G[Y]\right)\right| + \lambda \le \left\lfloor \frac{(r-1)|X|^2}{2r} \right\rfloor + \left\lfloor \frac{(r-1)|Y|^2}{2r} \right\rfloor + \delta \\ &= \left\lfloor \frac{(r-1)|X|^2}{2r} \right\rfloor + \left\lfloor \frac{(r-1)(n-|X|)^2}{2r} \right\rfloor + \delta \\ &\le \left\lfloor \frac{(r-1)}{2r}\left(|X|^2 + (n-|X|)^2\right) \right\rfloor + \delta \\ &= \left\lfloor \frac{(r-1)}{2r}\left(n^2 + 2(|X|^2 - n|X|)\right) \right\rfloor + \delta \\ &\le \left\lfloor \frac{(r-1)}{2r}\left(n^2 + 2\left(\left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1\right)^2 - 2n\left(\left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1\right)\right) \right\rfloor + \delta, \end{aligned}$$

a contradiction to the hypothesis. Thus $G$ is super-$\lambda$. ■

Theorem 2 is the special case $r = 2$ of Theorem 6. The next family of graphs shows that Theorem 6 is best possible in the sense that

$$m = \left\lfloor \frac{r-1}{2r}\left(n^2 + 2\left(\left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1\right)^2 - 2n\left(\left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1\right)\right) \right\rfloor + \delta$$

does not guarantee that the graph is super-$\lambda$.

**Example 7.** Let $r \ge 2$ and $q \ge 3$ be integers. Let $H_1$ be the complete $r$-partite graph with $q-1$ vertices in one partite set and $q$ vertices in $r-1$ partite sets, and let $H_2$ be the complete $r$-partite graph with $q$ vertices in each partite set. Define $H$ as the union of $H_1$ and $H_2$ by adding $\delta = q(r-1)$ edges between $H_1$ and $H_2$ such that $\omega(H) \le r$ and $\delta(H) = \delta = q(r-1)$. Then $H$ has order $n = 2qr - 1$, minimum degree $\delta = q(r-1)$ such that

$$\begin{aligned} m(H) &= q^2r^2 - q^2r \\ &= \left\lfloor \frac{r-1}{2r}\left(n^2 + 2\left(\left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1\right)^2 - 2n\left(\left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1\right)\right) \right\rfloor + \delta \end{aligned}$$

but obviously, $H$ is not super-$\lambda$.

Our last example demonstrates that Theorem 6 is not valid for $\delta = 2$ and $r = 2$ in general.

**Example 8.** Let $q \ge 2$ be an integer, and let $K_{q,q}$ be the complete bipartite graph with the partite sets $X = \{x_1, x_2, \ldots, x_q\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$, and let $u$ and $v$ be two further vertices. Define the graph $H$ as the union of $K_{q,q}$, $u$ and

$v$ together with the edges $uv$, $ux_1$ and $vx_2$. Then $H$ has order $n(H) = 2q + 2$, minimum degree $\delta(H) = 2$ and $\omega(H) \leq 2$. Furthermore,

$$\begin{aligned} m(H) &= q^2 + 3 > q^2 - q + 4 \\ &= \left\lfloor \frac{r-1}{2r} \left( n^2 + 2 \left( \left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1 \right)^2 - 2n \left( \left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1 \right) \right) \right\rfloor + \delta \end{aligned}$$

but $H$ is not super-$\lambda$.

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