Discussiones Mathematicae Graph Theory 39 (2019) 567–573 doi:10.7151/dmgt.2096

SUFFICIENT CONDITIONS FOR MAXIMALLY EDGE-CONNECTED AND SUPER-EDGE-CONNECTED GRAPHS DEPENDING ON THE CLIQUE NUMBER

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Abstract

Let G be a connected graph with minimum degree δ and edge-connectivity λ . A graph is maximally edge-connected if $\lambda = \delta$, and it is super-edgeconnected if every minimum edge-cut is trivial; that is, if every minimum edge-cut consists of edges incident with a vertex of minimum degree. The clique number $\omega(G)$ of a graph G is the maximum cardinality of a complete subgraph of G. In this paper, we show that a connected graph G with clique number $\omega(G) \leq r$ is maximally edge-connected or super-edge-connected if the number of edges is large enough. These are generalizations of corresponding results for triangle-free graphs by Volkmann and Hong in 2017.

Keywords: edge-connectivity, clique number, maximally edge-connected graphs, super-edge-connected graphs.

2010 Mathematics Subject Classification: 05C40.

1. TERMINOLOGY AND INTRODUCTION

Let G be a finite and simple graph with vertex set V = V(G) and edge set E = E(G). The order and size of G are defined by n = n(G) = |V(G)| and m = m(G) = |E(G)|, respectively. If $N(v) = N_G(v)$ is the neighborhood of the vertex $v \in V(G)$, then we denote by $d(v) = d_G(v) = |N(v)|$ the degree of v and by $\delta = \delta(G)$ the minimum degree of the graph G. For a subset $X \subseteq V(G)$, let G[X] to denote the subgraph of G induced by X. For two subsets X and Y of V(G) let [X, Y] be the set of edges with one endpoint in X and the other one in Y. The clique number $\omega(G)$ of a graph G is the maximum cardinality of a

complete subgraph of G. An *edge-cut* of a connected graph G is a set of edges whose removal disconnects G. The *edge connectivity* $\lambda = \lambda(G)$ of a connected graph G is defined as the minimum cardinality of an edge-cut over all edge-cuts of G. An edge-cut S is a *minimum edge-cut* or a λ -cut if $|S| = \lambda(G)$. The inequality $\lambda(G) \leq \delta(G)$ is immediate. We call a connected graph *maximally edge-connected*, if $\lambda(G) = \delta(G)$. In 1981, Bauer *et al.* [1] proposed the concept of super-edge connectedness. A graph is called *super-edge-connected* or *super-\lambda* if every minimum edge-cut is trivial; that is, if every minimum edge-cut consists of edges incident with a vertex of minimum degree. Thus every super-edgeconnected graph is also maximally edge-connected.

Sufficient conditions for graphs to be maximally edge-connected or superedge-connected were given by several authors, see for example the survey paper by Hellwig and Volkmann [3]. The starting point was an article by Chartrand [2] in 1966. He observed that if δ is large enough, then the graph is maximally edge-connected. A similar condition for super-edge-connectivity was given by Kelmans [4] six years later. Over the years, these results have been strengthened many times and in many ways.

Recently, Volkmann and Hong [6] showed that a connected graph or a connected triangle-free graph is maximally edge-connected or super- λ if the number of edges is large enough. In particular, they received the following results.

Theorem 1. Let G be a connected triangle-free graph of order $n \ge 2$, size m, minimum degree δ and edge-connectivity λ . If

$$m > \left\lfloor \frac{n^2}{4} \right\rfloor - \delta(n - 1 - 2\delta) - 1,$$

then $\lambda = \delta$.

Theorem 2. Let G be a connected triangle-free graph of order n, size m, minimum degree $\delta \geq 3$ and edge-connectivity λ . If

$$m > \left\lfloor \frac{(n+1)^2}{4} \right\rfloor - \delta(n+1-2\delta),$$

then G is super- λ .

In this paper, we will generalize Theorems 1 and 2 to connected graphs with clique number $\omega(G) \leq r$ for $r \geq 2$. Examples will demonstrate that our results are sharp.

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2. Maximally Edge-Connected Graphs

The main tool of our article is the famous theorem of Turán [5].

Theorem 3. Let $r \ge 1$ be an integer, and let G be a graph of order n. If the clique number $\omega(G) \le r$, then

$$|E(G)| \le \left\lfloor \frac{r-1}{2r} \cdot n^2 \right\rfloor.$$

Theorem 4. Let $r \ge 2$ be an integer, and let G be a connected graph of order n, size m, minimum degree $\delta \ge 1$, edge-connectivity λ and clique number $\omega(G) \le r$. If

$$m > \left\lfloor \frac{r-1}{2r} \left(n^2 + 2 \left\lfloor \frac{r\delta}{r-1} \right\rfloor^2 - 2n \left\lfloor \frac{r\delta}{r-1} \right\rfloor \right) \right\rfloor + \delta - 1,$$

then $\lambda = \delta$.

Proof. If $\delta = 1$, then $\lambda = \delta$ in every case. Thus assume in the following that $\delta \geq 2$. Suppose to the contrary that $\lambda \leq \delta - 1$. Then there exist two disjoint sets $X, Y \subset V(G)$ with $X \cup Y = V(G)$ and $|[X,Y]| = \lambda$. Assume, without loss of generality, that $|X| \leq |Y|$.

We first show that X contains at least $\delta + 1$ vertices. Otherwise, suppose that X contains at most δ vertices. Then we obtain

$$\delta|X| \le \sum_{x \in X} d_G(x) \le |X|(|X|-1) + \lambda \le \delta(|X|-1) + \delta - 1.$$

Obviously, this is a contradiction and thus $|X| \ge \delta + 1$. Using Theorem 3, we conclude that

(1)
$$|E(G[X])| \le \left\lfloor \frac{(r-1)|X|^2}{2r} \right\rfloor$$

and

(2)
$$|E(G[Y])| \le \left\lfloor \frac{(r-1)|Y|^2}{2r} \right\rfloor.$$

Next we show that $|X| \ge \lfloor (r\delta)/(r-1) \rfloor$. Suppose to the contrary that $|X| \le \lfloor (r\delta)/(r-1) \rfloor - 1$. Since $2 |E(G[X])| = \sum_{x \in X} d_G(x) - \lambda$, (1) implies that

$$\begin{split} |X|\delta &\leq \sum_{x \in X} d_G(x) \leq 2\left\lfloor \frac{(r-1)|X|^2}{2r} \right\rfloor + \lambda \leq \frac{(r-1)|X|^2}{r} + \delta - 1 \\ &\leq |X|\frac{r-1}{r} \left(\left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1 \right) + \delta - 1 \leq |X|\frac{r-1}{r} \left(\frac{r\delta}{r-1} - 1 \right) + \delta - 1 \\ &= |X|\delta - \frac{r-1}{r}|X| + \delta - 1 \end{split}$$

and thus $|X| \leq \frac{r(\delta-1)}{r-1}$. Using this argument once more, we arrive at

$$\begin{split} |X|\delta &\leq \frac{(r-1)|X|^2}{r} + \delta - 1 \leq |X|\frac{r-1}{r} \cdot \frac{r(\delta-1)}{r-1} + \delta - 1 \\ &= |X|(\delta-1) + \delta - 1 \end{split}$$

and thus $|X| \leq \delta - 1$, which contradicts the fact that $|X| \geq \delta + 1$. Hence $|X| \geq \lfloor (r\delta)/(r-1) \rfloor$. Since |X| + |Y| = n and $|X| \leq n/2$, the inequalities (1) and (2) lead to

$$\begin{split} m &= |E(G[X])| + |E(G[Y])| + \lambda \\ &\leq \left\lfloor \frac{(r-1)|X|^2}{2r} \right\rfloor + \left\lfloor \frac{(r-1)|Y|^2}{2r} \right\rfloor + \delta - 1 \\ &= \left\lfloor \frac{(r-1)|X|^2}{2r} \right\rfloor + \left\lfloor \frac{(r-1)(n-|X|)^2}{2r} \right\rfloor + \delta - 1 \\ &\leq \left\lfloor \frac{(r-1)}{2r} \left(|X|^2 + (n-|X|)^2 \right) \right\rfloor + \delta - 1 \\ &= \left\lfloor \frac{(r-1)}{2r} \left(n^2 + 2(|X|^2 - n|X|) \right) \right\rfloor + \delta - 1 \\ &\leq \left\lfloor \frac{(r-1)}{2r} \left(n^2 + 2 \left\lfloor \frac{r\delta}{r-1} \right\rfloor^2 - 2n \left\lfloor \frac{r\delta}{r-1} \right\rfloor \right) \right\rfloor + \delta - 1, \end{split}$$

a contradiction to the hypothesis. Thus $\lambda = \delta$.

Theorem 1 is the special case r = 2 of Theorem 4. The next family of graphs shows that Theorem 4 is best possible in the sense that

$$m = \left\lfloor \frac{r-1}{2r} \left(n^2 + 2 \left\lfloor \frac{r\delta}{r-1} \right\rfloor^2 - 2n \left\lfloor \frac{r\delta}{r-1} \right\rfloor \right) \right\rfloor + \delta - 1$$

does not guarantee $\lambda = \delta$.

Example 5. Let $r \ge 2$ and $q \ge 1$ be integers. Let H_1 and H_2 be two disjoint copies of the complete *r*-partite graph with q vertices in each partite set. Define H as the union of H_1 and H_2 by adding $\delta - 1 = q(r-1) - 1$ edges between H_1 and H_2 such that $\omega(H) \le r$. Then H has order n = 2qr, minimum degree $\delta = q(r-1)$ such that

$$m(H) = q^2 r(r-1) + q(r-1) - 1$$

= $\left\lfloor \frac{r-1}{2r} \left(n^2 + 2 \left\lfloor \frac{r\delta}{r-1} \right\rfloor^2 - 2n \left\lfloor \frac{r\delta}{r-1} \right\rfloor \right) \right\rfloor + \delta - 1,$

but obviously, $\lambda(H) = \delta(H) - 1$.

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3. Super Edge-Connected Graphs

Theorem 6. Let $r \ge 2$ be an integer, and let G be a connected graph of order n, size m, minimum degree $\delta \ge 2$, edge-connectivity λ and $\omega(G) \le r$. If $\delta \ge 3$ or $r \ge 3$ and

$$m > \left\lfloor \frac{r-1}{2r} \left(n^2 + 2\left(\left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1 \right)^2 - 2n\left(\left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1 \right) \right) \right\rfloor + \delta,$$

then G is super- λ .

Proof. Suppose to the contrary that G is not super- λ . Then there exist two disjoint sets $X, Y \subset V(G)$ such that $X \cup Y = V(G), |X|, |Y| \ge 2$ and $|[X, Y]| = \lambda$. Assume, without loss of generality, that $2 \le |X| \le |Y|$.

We first show that X contains at least δ vertices. Otherwise, suppose that X contains at most $\delta - 1$ vertices. Then we obtain

$$\delta|X| \le \sum_{x \in X} d_G(x) \le |X| (|X| - 1) + \lambda \le (\delta - 1) (|X| - 1) + \delta,$$

which implies that $|X| \leq 1$, contradicting that $|X| \geq 2$. Thus $|X| \geq \delta$.

Next we show that $|X| \ge |(r\delta)/(r-1)| - 1$. If $\delta = 2$ and $r \ge 3$, then

$$|X| \ge \delta = 2 \ge \lfloor (2r)/(r-1) \rfloor - 1 = \lfloor (r\delta)/(r-1) \rfloor - 1$$

Let now $\delta \geq 3$. Suppose to the contrary that X contains at most $\lfloor (r\delta)/(r-1) \rfloor - 2$ vertices. Since $2|E(G[X])| = \sum_{x \in X} d_G(x) - \lambda$, we conclude from (1) that

$$\begin{split} |X|\delta &\leq \sum_{x \in X} d_G(x) \leq 2\left\lfloor \frac{(r-1)|X|^2}{2r} \right\rfloor + \lambda \leq \frac{(r-1)|X|^2}{r} + \delta \\ &\leq |X|\frac{r-1}{r} \left(\left\lfloor \frac{r\delta}{r-1} \right\rfloor - 2 \right) + \delta \leq |X|\frac{r-1}{r} \left(\frac{r\delta}{r-1} - 2 \right) + \delta \\ &= |X|\delta - \frac{2(r-1)}{r}|X| + \delta \end{split}$$

and thus $|X| \leq \frac{r\delta}{2(r-1)}$. Using this argument once more, we arrive at

$$|X|\delta \leq \frac{(r-1)|X|^2}{r} + \delta \leq |X|\frac{r-1}{r} \cdot \frac{r\delta}{2(r-1)} + \delta = \frac{|X|\delta}{2} + \delta$$

and thus $|X| \leq 2$, which contradicts the fact that $3 \leq \delta \leq |X|$. Hence we have shown that $|X| \geq \lfloor (r\delta)/(r-1) \rfloor - 1$ when $\delta \geq 3$ or $r \geq 3$. Since |X| + |Y| = n

and $|X| \leq n/2$, the inequalities (1) and (2) lead to

$$\begin{split} m \ &= \ \left| E\left(G[X]\right) \right| + \left| E\left(G[Y]\right) \right| + \lambda \leq \left\lfloor \frac{(r-1)|X|^2}{2r} \right\rfloor + \left\lfloor \frac{(r-1)|Y|^2}{2r} \right\rfloor + \delta \\ &= \left\lfloor \frac{(r-1)|X|^2}{2r} \right\rfloor + \left\lfloor \frac{(r-1)(n-|X|)^2}{2r} \right\rfloor + \delta \\ &\leq \left\lfloor \frac{(r-1)}{2r} \left(|X|^2 + (n-|X|)^2 \right) \right\rfloor + \delta \\ &= \left\lfloor \frac{(r-1)}{2r} \left(n^2 + 2(|X|^2 - n|X|) \right) \right\rfloor + \delta \\ &\leq \left\lfloor \frac{(r-1)}{2r} \left(n^2 + 2\left(\left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1 \right)^2 - 2n\left(\left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1 \right) \right) \right\rfloor + \delta, \end{split}$$

a contradiction to the hypothesis. Thus G is super- λ .

Theorem 2 is the special case
$$r = 2$$
 of Theorem 6. The next family of graphs
shows that Theorem 6 is best possible in the sense that

$$m = \left\lfloor \frac{r-1}{2r} \left(n^2 + 2\left(\left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1 \right)^2 - 2n\left(\left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1 \right) \right) \right\rfloor + \delta$$

does not guarantee that the graph is super- λ .

Example 7. Let $r \ge 2$ and $q \ge 3$ be integers. Let H_1 be the complete *r*-partite graph with q-1 vertices in one partite set and q vertices in r-1 partite sets, and let H_2 be the complete *r*-partite graph with q vertices in each partite set. Define H as the union of H_1 and H_2 by adding $\delta = q(r-1)$ edges between H_1 and H_2 such that $\omega(H) \le r$ and $\delta(H) = \delta = q(r-1)$. Then H has order n = 2qr - 1, minimum degree $\delta = q(r-1)$ such that

$$m(H) = q^{2}r^{2} - q^{2}r$$
$$= \left\lfloor \frac{r-1}{2r} \left(n^{2} + 2\left(\left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1 \right)^{2} - 2n\left(\left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1 \right) \right) \right\rfloor + \delta$$

but obviously, H is not super- λ .

Our last example demonstrates that Theorem 6 is not valid for $\delta = 2$ and r = 2 in general.

Example 8. Let $q \ge 2$ be an integer, and let $K_{q,q}$ be the complete bipartite graph with the partite sets $X = \{x_1, x_2, \ldots, x_q\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$, and let u and v be two further vertices. Define the graph H as the union of $K_{q,q}$, u and

v together with the edges uv, ux_1 and vx_2 . Then H has order n(H) = 2q + 2, minimum degree $\delta(H) = 2$ and $\omega(H) \leq 2$. Furthermore,

$$m(H) = q^{2} + 3 > q^{2} - q + 4$$
$$= \left\lfloor \frac{r - 1}{2r} \left(n^{2} + 2\left(\left\lfloor \frac{r\delta}{r - 1} \right\rfloor - 1 \right)^{2} - 2n\left(\left\lfloor \frac{r\delta}{r - 1} \right\rfloor - 1 \right) \right) \right\rfloor + \delta$$

but H is not super- λ .

References

- D. Bauer, C. Suffel, F. Boesch and R. Tindell, Connectivity extremal problems and the design of reliable probabilistic networks, in: The Theory and Applications of Graphs, Kalamazoo MI (Wiley, New York, 1981) 45–54.
- G. Chartrand, A graph-theoretic approach to a communications problem, SIAM J. Appl. Math. 14 (1966) 778–781. doi:10.1137/0114065
- [3] A. Hellwig and L. Volkmann, Maximally edge-connected and vertex-connected graphs and digraphs: A survey, Discrete Math. 308 (2008) 3265–3296. doi:10.1016/j.disc.2007.06.035
- [4] A.K. Kelmans Asymptotic formulas for the probability of k-connectedness of random graphs, Theory Probab. Appl. 17 (1972) 243-254. doi:10.1137/1117029
- [5] P. Turán, On an extremal problem in graph theory, Mat. Fiz. Lapok 48 (1941) 436–452, in Hungarian.
- [6] L. Volkmann and Z.-M. Hong, Sufficient conditions for maximally edge-connected and super-edge-connected graphs, Commun. Comb. Optim. 2 (2017) 35–41.

Received 8 June 2017 Revised 19 October 2017 Accepted 23 October 2017