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MORE RESULTS ON THE SMALLEST ONE-REALIZATION OF A GIVEN SET II

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Abstract

Let S be a finite set of positive integers. A mixed hypergraph $\mathcal{H}$ is a one-realization of S if its feasible set is S and each entry of its chromatic spectrum is either 0 or 1. The minimum number of vertices, denoted by $\delta_3(S)$, in a 3-uniform bi-hypergraph which is a one-realization of S was determined in [P. Zhao, K. Diao and F. Lu, *More result on the smallest one-realization of a given set*, Graphs Combin. 32 (2016) 835–850]. In this paper, we consider the minimum number of edges in a 3-uniform bi-hypergraph which already has the minimum number of vertices with respect of being a minimum bi-hypergraph that is one-realization of S. A tight lower bound on the number of edges in a 3-uniform bi-hypergraph which is a one-realization of S with $\delta_3(S)$ vertices is given.

Keywords: mixed hypergraph, feasible set, chromatic spectrum, gap, one-realization.

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1. Introduction

Coloring problems are among the most intensively studied combinatorial problems for both theoretical and practical reasons. The traditional coloring of graphs

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and hypergraphs requires that no edge is *monochromatic*; its dual problem, the so-called *co-coloring* of hypergraphs, requires that no edge is *polychromatic* (*rainbow*). Voloshin [19] combined both types of hypergraph colorings and introduced the concept of mixed hypergraph. A *mixed hypergraph* on a finite set X is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where $\mathcal{C}$ and $\mathcal{D}$ are families of subsets of X. The members of $\mathcal{C}$ and $\mathcal{D}$ are called $\mathcal{C}$*-edges* and $\mathcal{D}$*-edges*, respectively. The distinction between the two types of edges lies on the requirement in colorings. In a *proper* k*-coloring* $\varphi : X \to \{1, 2, \ldots, k\}$ of $\mathcal{H}$, each $\mathcal{C}$-edge has two vertices with a *Common* color, and each $\mathcal{D}$-edge has two vertices colored *Distinctly.* A *strict* k*-coloring* of $\mathcal{H}$ is a proper coloring using exactly k colors. The set

$$\Phi(\mathcal{H}) := \{k \mid \mathcal{H} \text{ has a strict } k\text{-coloring}\}$$

is termed the *feasible set* of $\mathcal{H}$. Each proper k-coloring φ of $\mathcal{H}$ induces a *feasible partition* $X_1 \cup X_2 \cup \cdots \cup X_k = X$, where the partition classes are the monochromatic subsets of X under φ, we denote by $X = \{X_1, X_2, \ldots, X_k\}$. For each $k \in \Phi(\mathcal{H})$, let r_k denote the number of feasible partitions of X into k nonempty classes. The vector $R(\mathcal{H}) = (r_1, r_2, \ldots, r_{\overline{\chi}})$ is called the *chromatic spectrum* of $\mathcal{H}$, where $\overline{\chi}$ is the maximum number in $\Phi(\mathcal{H})$. That is $r_i > 0$ if $i \in \Phi(\mathcal{H})$, otherwise $r_i = 0$.

The intersection of the two families of edges may not be empty. The members in $\mathcal{C} \cap \mathcal{D}$ are called *bi-edges*, and a mixed hypergraph with only bi-edges is called a *bi-hypergraph*, denoted by $\mathcal{H} = (X, \mathcal{B})$, where $\mathcal{B}$ is the set of bi-edges. A sub-hypergraph $\mathcal{H}' = (X', \mathcal{C}', \mathcal{D}')$ of $\mathcal{H}$ is called a *partial sub-hypergraph* (or *spanning sub-hypergraph*) of $\mathcal{H}$ if $X' = X$, and $\mathcal{H}'$ is called an *induced sub-hypergraph* of $\mathcal{H}$ on X', denoted by $\mathcal{H}[X']$, if $\mathcal{C}' = \{C \in \mathcal{C} | C \subseteq X'\}$ and $\mathcal{D}' = \{D \in \mathcal{D} | D \subseteq X'\}$. A mixed hypergraph is r*-uniform* if $|C| = |D| = r$ for every $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Two mixed hypergraphs $\mathcal{H}_1 = (X_1, \mathcal{C}_1, \mathcal{D}_1)$ and $\mathcal{H}_2 = (X_2, \mathcal{C}_2, \mathcal{D}_2)$ are *isomorphic* if there exists a bijection ϕ from X_1 to X_2 that preserve the incidence between vertices and edges and maps each $\mathcal{C}$-edge of $\mathcal{C}_1$ onto a $\mathcal{C}$-edge of $\mathcal{C}_2$ and maps each $\mathcal{D}$-edge of $\mathcal{D}_1$ onto a $\mathcal{D}$-edge of $\mathcal{D}_2$, and vice versa. The mixed hypergraph coloring has attracted a lot of attentions as witnessed by enormous number of papers on the subject, e.g., [2–4, 6–10, 15] and is widely applied in practice, e.g., [13, 16]. For more information, see [20] and the regularly updated website [21].

The concept of a mixed hypergraph coloring has led to the discovery of new principal properties of colorings that do not exist in classical graph and hypergraph colorings. Mixed hypergraphs may admit no proper colorings. A mixed hypergraph is *colorable* if it has proper colorings, otherwise, it is *uncolorable.* One of the surprising properties of mixed hypergraphs is that the chromatic spectrum may have gaps. Let $\mathcal{H}$ be a colorable mixed hypergraph. A *gap* in the chromatic spectrum of $\mathcal{H}$, or a gap of $\Phi(\mathcal{H})$, is an integer $k \notin \Phi(\mathcal{H})$ such that $\min(\Phi(\mathcal{H})) < k < \max(\Phi(\mathcal{H}))$. If $\Phi(\mathcal{H})$ has no gaps, then the spectrum or the feasible set is termed *continuous* or *gap-free*; otherwise it is said to be *broken.* Let S be a finite set of positive integers. We say that a mixed hypergraph $\mathcal{H}$ is a *realization* of S if $\Phi(\mathcal{H}) = S$. A mixed hypergraph $\mathcal{H}$ is a *one-realization* of S if it is a realization of S and each entry of the chromatic spectrum of $\mathcal{H}$ is either 0 or 1. Bacsó *et al.* [1] discussed the properties of uniform bi-hypergraphs $\mathcal{H}$ which are one-realizations of S when $|S| = 1$ and $\min(S) \geq 2$, in this case we also say that $\mathcal{H}$ is *uniquely colorable.* Jiang *et al.* [14] proved that S is the feasible set of some mixed hypergraph if and only if $1 \notin S$ or S is an interval containing 1, and Král [17] strengthened this result by showing that prescribing any positive integer r_k, there exists a mixed hypergraph which has precisely r_k strict k-colorings for all $k \in S$. Bujtás and Tuza [5] focused on the feasible set of uniform mixed hypergraphs and proved that for every integer $r \geq 3$, S is the feasible set of an r-uniform mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ with $|\mathcal{C}| + |\mathcal{D}| \geq 1$ if and only if (i) $\min(S) \geq r$, or (ii) $2 \leq \min(S) \leq r - 1$ and S contains all integers between $\min(S)$ and $r - 1$, or (iii) $\min(S) = 1$ and $S = \{1, \ldots, \overline{\chi}\}$ for some natural number $\overline{\chi} \geq 1$. Zhao *et al.* [24] strengthened this result by showing that prescribing any positive integer r_k, there exists a 3-uniform bi-hypergraph which has precisely r_k strict k-colorings for all $k \in S$.

Naturally, it is required to determine the minimum number of vertices or edges in a realization or one-realization of S. It is readily seen that if $1 \in \Phi(\mathcal{H})$, then $\mathcal{H}$ has no $\mathcal{D}$-edges. Let $\min(S) \geq 2$. It has made a lot of progress in determining the minimum number of vertices of realizations or one-realizations of S. Kündgen *et al.* [18] initiated this problem and found a one-realization of $\{2, 4\}$ on 6 vertices for planar hypergraphs. Jiang *et al.* [14] determined the minimum number of vertices of realizations of S when $|S| = 2$ and $\max(S)-1 \notin S$. Král [17] gave an upper bound on the minimum number of vertices in a mixed hypergraph which is a one-realization of S, and Zhao *et al.* [25] determined this minimum number. Denoted by $\delta_3(S)$ the minimum number of vertices in a 3-uniform bi-hypergraph which is a one-realization of S. Zhao *et al.* [24] obtained an upper bound on $\delta_3(S)$, and Zhao *et al.* [22] proved the following result.

Theorem 1 (Zhao *et al.* [22])**.** *Suppose S is a finite set of positive integers with $|S| \geq 3$ and $\min(S) \geq 3$. Then*

$$\delta_3(S) = \begin{cases} 2\max(S), & \text{if } \max(S) - 1 \notin S, \\ 2\max(S) - 1, & \text{if } \max(S) - 1 \in S. \end{cases}$$

Furthermore, Diao *et al.* [11] generalized this result to r-uniform bi-hypergraphs for any $r \geq 3$ and determined the minimum number of vertices of r-uniform bi-hypergraphs which are one-realizations of S.

For the minimum number of edges of one-realizations of S, Diao *et al.* [12] determined the minimum number of $\mathcal{C}$-edges and the minimum number of $\mathcal{D}$-

edges of one-realizations of S; Zhao *et al.* [23] gave a tight lower bound on the number of edges of 3-uniform bi-hypergraphs which are one-realizations of S.

Theorem 2 (Zhao *et al.* [23]). *Suppose S is a finite set of positive integers with $|S| \geq 3$ and $\min(S) \geq 3$. Then we have*

$$|\mathcal{B}| \geq \begin{cases} \frac{\max(S)[\max(S)-1]}{2}, & \text{if } \max(S)-1 \notin S, \\ \frac{\max(S)[\max(S)-1]}{2} - 1, & \text{if } \max(S)-1 \in S, \end{cases}$$

holds for every 3-uniform bi-hypergraph $\mathcal{H} = (X, \mathcal{B})$ which is a one-realization of S.

Denote by $\mathcal{F}$ the set of all the 3-uniform bi-hypergraphs $\mathcal{H} = (X, \mathcal{B})$ which have $\delta_3(S)$ vertices and are one-realizations of S. In this paper, we focus on the minimum number of edges of all the 3-uniform bi-hypergraphs in $\mathcal{F}$ and get the following result.

Theorem 3. *Suppose S is a finite set of positive integers with $|S| \geq 3$ and $\min(S) \geq 3$. Then*

$$|\mathcal{B}| \geq \begin{cases} \frac{\max(S)[\max(S)-1]}{2} + 3, & \text{if } \max(S)-1 \notin S, \\ \frac{\max(S)[\max(S)-1]}{2} + 2, & \text{if } \max(S)-1 \in S, \end{cases}$$

holds for every 3-uniform bi-hypergraph $\mathcal{H} = (X, \mathcal{B}) \in \mathcal{F}$. Moreover, this lower bound is tight.

In the following, we always assume that $s \geq 3$ is an integer, $S = \{n_1, n_2, \ldots, n_s\}$ is a finite set of integers with $n_1 > n_2 > \cdots > n_s \geq 3$. $[n]$ is the set $\{1, 2, \ldots, n\}$. The lower bound in Theorem 3 will be gotten in Section 2. And we will show that the bound is tight in Section 3, by constructing two families of 3-uniform bi-hypergraphs which are one-realizations of S with $\delta_3(S)$ vertices and meet the lower bound in each case.

2. The Lower Bound

In this section we shall show that the number given in Theorem 3 is a lower bound on the number of edges of all the 3-uniform bi-hypergraphs in $\mathcal{F}$.

Theorem 4. *Let $\mathcal{H} = (X, \mathcal{B}) \in \mathcal{F}$. Then*

$$|\mathcal{B}| \geq \begin{cases} \frac{n_1(n_1-1)}{2} + 3, & \text{if } n_1 - 1 \notin S, \\ \frac{n_1(n_1-1)}{2} + 2, & \text{if } n_1 - 1 \in S. \end{cases}$$

Proof. Let

$$\delta = \begin{cases} \frac{n_1(n_1-1)}{2}, & \text{if } n_1 - 1 \notin S, \\ \frac{n_1(n-1-1)}{2} - 1, & \text{if } n_1 - 1 \in S. \end{cases}$$

By Theorem 1, we have $2n_1 - 1 \le |X| \le 2n_1$. Let $c = \{C_1, C_2, \ldots, C_{n_1}\}$ be a strict n_1-coloring of $\mathcal{H}$ with $|C_1| \ge |C_2| \ge \cdots \ge |C_{n_1}| \ge 1$. By Theorem 2, we have that $|\mathcal{B}| \ge \delta$. More exactly, there is at least one edge between each pair of the n_1 color classes if $n_1 - 1 \notin S$, or there is at most one pair of the color classes between which there are no edges otherwise.

In the following, we shall show that there exist three color classes, say C_p, C_q and C_t, such that each one of the three induced subhypergraphs $\mathcal{H}[C_1 \cup C_p]$, $\mathcal{H}[C_1 \cup C_p]$ and $\mathcal{H}[C_1 \cup C_p]$ has at least two edges, which implies that $|\mathcal{B}| \ge \delta + 3$.

We first focus on the orders of the n_1 color classes. Assume that $|C_{n_1-2}| = 1$. Then $\{C_1, C_2, \ldots, C_{n_1-2}, C_{n_1-1} \cup C_{n_1}\}$ and $\big\{C_1, C_2, \ldots, C_{n_1-3}, C_{n_1-2} \cup C_{n_1-1}, C_{n_1}\big\}$ are two distinct strict (n_1-1)-colorings, a contradiction to that $\mathcal{H}$ is a one-realization of S. Hence, $|C_i| \ge 2$, for every $i \in [n_1 - 2]$. Assume that $|C_1| = 2$. Pick $x_i \in C_i$, for every $i \in [n_1]$ and let $C_1' = \{x_i \mid i \in [n_1]\}$, $C_2' = X \setminus C_1'$. Then $\{C_1', C_2'\}$ is a strict 2-coloring of $\mathcal{H}$, a contradiction to that $n_s \ge 3$. Hence, $|C_1| > 2$. Then the fact that $2n_1 - 1 \le |X| \le 2n_1$ implies that $|C_1| = 3$, $|C_i| = 2$, $i \in [n_1 - 2] \setminus \{1\}$ and $|C_{n_1}| = 1$. Write $C_1 = \{x_1, y_1, z_1\}$, $C_i = \{x_i, y_i\}, i \in [n_1 - 2] \setminus \{1\}$ and $x_{n_1-1} \in C_{n_1-1}$, $C_{n_1} = \{x_{n_1}\}$. Then $\{x_1, y_1, z_1\} \notin \mathcal{B}$.

We then focus on the edges between C_1 and the other color classes. We first claim that there exists an edge in $\mathcal{B}$, say B_1, such that $C_1 \cap B_1 = \{x_1, y_1\}$. Suppose for a contradiction that $|\{x_1, y_1\} \cap B| \le 1$, for every $B \in \mathcal{B}$. Let $C_1' = \{x_i, y_1 \mid i \in [n_1]\}$ and $C_2' = X \setminus C_1'$. Then $\{C_1', C_2'\}$ is a strict 2-coloring of $\mathcal{H}$, a contradiction. Hence, our claim is valid. Similarly, there exist two edges in $\mathcal{B}$, say B_2 and B_3, such that $C_1 \cap B_2 = \{x_1, z_1\}$, $C_1 \cap B_3 = \{y_1, z_1\}$. We then claim that there exists an integer in $[n_1 - 1] \setminus \{1\}$, say p, such that $\{x_p, x_1, y_1\}, \{y_p, x_1, y_1\} \in \mathcal{B}$. Suppose for a contradiction that $\{x_k, x_1, y_1\} \notin \mathcal{B}$ or $\{y_k, x_1, y_1\} \notin \mathcal{B}$, for every $k \in [n_1 - 1] \setminus \{1\}$. Write $C_1'' = \{x_1, y_1\} \cup \{x_i, y_j \mid 2 \le i, j \le n_1 - 1 \wedge \{x_i, x_1, y_1\} \notin \mathcal{B}, \{x_j, x_1, y_1\} \in \mathcal{B}\}$, $C_2'' = X \setminus C_1''$. Then $\{C_1'', C_2''\}$ is a strict 2-coloring of $\mathcal{H}$, a contradiction. Hence, our claim is valid. Similarly, there exist two integers in $[n_1 - 1] \setminus \{1\}$, say q and t, such that $\{x_q, x_1, z_1\}, \{y_q, x_1, z_1\} \in \mathcal{B}$ and $\{x_t, y_1, z_1\}, \{y_t, y_1, z_1\} \in \mathcal{B}$. The desired result follows. ■

3. The Bi-Hypergraphs Which Meet the Lower Bound

In this section we shall construct the 3-uniform bi-hypergraphs which are one-realizations of S with $\delta_3(S)$ vertices and meet the lower bound.

3.1. The case of $n_1 - 1 \notin S$

Construction I. Suppose $s \geq 3$ and $n_s, n_i - n_{i+1} \geq 56$, $i \in [s-1]$. Set

$$
\begin{aligned}
l_1 &= n_1 - n_2 - 4,\ l_s = n_s - 2,\ l_i = n_i - n_{i+1} - 1,\ 2 \leq i \leq s-1,\\
M_s &= \{1, 2, \ldots, l_s\},\ M_i = \{0, 1, \ldots, l_i\},\ i \in [s-1],\\
X_s &= \{x_s^t, y_s^t \mid t \in M_s\},\ \text{where } x_s^t = (\underbrace{t, \ldots, t}_{s}, 1),\ y_s^t = (t, \ldots, t, 0),\\
Y_s &= \{x_{n_s-1}, y_{n_s-1}, z_{n_s-1}\} \text{ where } x_{n_s-1} = (\underbrace{n_s - 1, \ldots, n_s - 1}_{s}, 1),\\
y_{n_s-1} &= (n_s - 1, \ldots, n_s - 1, 0),\quad z_{n_s-1} = (n_s - 1, \ldots, n_s - 1, 2),\\
X_i &= \{x_i^t, y_i^t \mid t \in M_i\},\ i \in [s-1],\ \text{where}\\
x_i^t &= (n_{i+1} + t, \ldots, n_{i+1} + t, n_{i+1}, \ldots, n_s, 1),\\
y_i^t &= (\underbrace{n_{i+1} + t, \ldots, n_{i+1} + t}_{i}, \underbrace{1, \ldots, 1}_{s-i}, 0),\\
X_0 &= \{x_{n_1}, x_{n_1-h}, y_{n_1-h} \mid h = 1, 2, 3\},\ \text{where }\ x_{n_1} = (n_1, n_2, \ldots, n_s, 1),\\
x_{n_1-h} &= (n_1 - h, n_2, \ldots, n_s, 1),\ y_{n_1-h} = (n_1 - h, 4 - h, \ldots, 4 - h, 0),\\
X_{n_1, \ldots, n_s} &= \left(\bigcup_{i=0}^{s} X_i \right) \cup Y_s \text{ and}\\
\mathcal{B}_i &= \{\{x_i^p, x_i^t, y_i^t\} \mid p, t \in M_i,\ p < t,\ p + t \equiv 0 \pmod 2\}\\
&\cup \{\{y_i^q, x_i^t, y_i^t\} \mid q, t \in M_i,\ q > t,\ q + t \equiv 1 \pmod 2\},\ i \in [s],\\
\mathcal{B}_{kj} &= \{\{x_k^t, x_j^p, y_j^p\}, \{y_k^t, x_j^q, y_j^q\} \mid t \in M_k,\ 0 \leq p \leq 3,\ 4 \leq q \leq l_j\},\\
&\quad 1 \leq k < j \leq s-1,\\
\mathcal{B}_{ks} &= \{\{x_k^t, x_s^p, y_s^p\}, \{y_k^t, x_s^q, y_s^q\} \mid t \in M_k,\ p \in [4],\ 5 \leq q \leq l_s\},\ k \in [s-1],\\
\mathcal{E}_s &= \{\{x_{n_s-1}, x_s^t, y_s^t\} \mid t = 1, 2, 3, l_s - 2, l_s - 1, l_s\}\\
&\cup \{\{y_{n_s-1}, x_s^t, y_s^t\} \mid t = 4, 5, 6, l_s - 5, l_s - 4, l_s - 3\},\\
&\cup \{\{x_s^7, x_{n_s-1}, y_{n_s-1}\}, \{y_s^7, x_{n_s-1}, y_{n_s-1}\}, \{y_s^8, x_{n_s-1}, y_{n_s-1}\}\}\\
&\cup \{\{x_s^9, y_{n_s-1}, z_{n_s-1}\}, \{y_s^9, y_{n_s-1}, z_{n_s-1}\}, \{y_s^{10}, y_{n_s-1}, z_{n_s-1}\}\}\\
&\cup \{\{x_s^{11}, x_{n_s-1}, z_{n_s-1}\}, \{y_s^{11}, x_{n_s-1}, z_{n_s-1}\}, \{y_s^{12}, x_{n_s-1}, z_{n_s-1}\}\}\\
&\cup \{\{z_{n_s-1}, x_s^t, y_s^t\} \mid 13 \leq t \leq l_s - 6\},\\
\mathcal{B}_0 &= \{\{x_{n_1-3}, x_i^p, y_i^p\}, \{y_{n_1-3}, x_i^q, y_i^q\} \mid i \in [s-1],\ 0 \leq p \leq 3,\ 4 \leq q \leq l_i\}\\
&\cup \{\{x_{n_1-3}, x_s^p, y_s^p\}, \{y_{n_1-3}, x_s^q, y_s^q\} \mid p \in [4],\ 5 \leq q \leq l_s\}\\
&\cup \{\{x_{n_1-h}, x_i^t, y_i^t\} \mid h = 0, 1, 2, i \in [s], t \in M_i\}\\
&\cup \{\{x_{n_1}, x_{n_1-h}, y_{n_1-h}\} \mid h \in [3]\}\\
&\cup \{\{x_{n_1-h}, x_{n_s-1}, y_{n_s-1}\}, \{x_k^t, x_{n_s-1}, y_{n_s-1}\} \mid 0 \leq h \leq 3,\\
&\quad k \in [s-1], t \in M_k\}
\end{aligned}
$$

$$\begin{aligned}
& \cup \ \big\{\{x_{n_1-1}, x_{n_1-2}, y_{n_1-2}\}, \{x_{n_1-2}, x_{n_1-3}, y_{n_1-3}\}, \{x_{n_1-3}, x_{n_1-1}, y_{n_1-1}\}\big\} \\
& \cup \ \big\{\{y_s^1, x_{n_1-3}, y_{n_1-3}\}, \{y_s^2, x_{n_1-2}, y_{n_1-2}\}\big\} \\
& \cup \ \big\{\{y_s^3, x_{n_1-1}, y_{n_1-1}\}, \{y_1^4, x_{n_1-3}, y_{n_1-3}\}\big\}, \\
\mathcal{E} &= \big\{\{y_{n_1-3}, x_1^4, y_1^4\}, \{x_{n_1-3}, x_s^1, y_s^1\}, \{x_{n_1-2}, x_s^2, y_s^2\}, \{x_{n_1-1}, x_s^3, y_s^3\}\big\}, \\
\mathcal{B}_{n_1,\ldots,n_s} &= \left(\left(\bigcup_{i=0}^{s} \mathcal{B}_i\right) \cup \left(\bigcup_{k=1}^{s-1} \bigcup_{j=k+1}^{s} \mathcal{B}_{kj}\right) \cup \mathcal{E}_s\right) \setminus \mathcal{E}.
\end{aligned}$$

Then $\mathcal{H}_{n_1,\ldots,n_s} = (X_{n_1,\ldots,n_s}, \mathcal{B}_{n_1,\ldots,n_s})$ is a 3-uniform bi-hypergraph with $2n_1$ vertices and $\frac{n_1(n_1-1)}{2} + 3$ edges.

Clearly, for each $i \in [s]$,

$$c_i = \big\{X_{i1}, X_{i2}, \ldots, X_{in_i}\big\}$$

is a strict n_i-coloring of $\mathcal{H}_{n_1,\ldots,n_s}$, where X_{ij} is the subset of $X_{n_1,\ldots,n_s}$ that consists of the vertices whose i-th entry is j. Then we shall prove that $\mathcal{H}_{n_1,\ldots,n_s}$ has no other strict colorings except $c_1, c_2, \ldots, c_s$. That is to say, $\mathcal{H}_{n_1,\ldots,n_s}$ is a one-realization of S.

For a strict coloring c of a mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, we denote by $c(x)$ the color of the vertex $x \in X$ under c, and denote by $c(Y)$ the set of the colors on $Y \subseteq X$ under c. In the rest of this subsection, we always assume that $c = \{C_1, C_2, \ldots, C_m\}$ is a strict coloring of $\mathcal{H}_{n_1,\ldots,n_s}$ with $c(x_{n_1}) = d$. Note that for each $i \in [s]$ and $t \in M_i$, since $\{x_{n_1}, x_i^t, y_i^t\}$ is an edge, we have that $d \in \{c(x_i^t), c(y_i^t)\}$ if $c(x_i^t) \neq c(y_i^t)$, or $d \neq c(x_i^t)(= c(y_i^t))$ otherwise.

Note that the construction of $\mathcal{B}_i$, for every $i \in [l_s] \cup \{0\}$, $\mathcal{B}_{kj}$, for every $1 \le k < j \le s-1$, or $\mathcal{B}_{ks}$, for every $k \in [s-2]$ here is the same as the construction of $\mathcal{B}_i$, for every $i \in [l_s] \cup \{0\}$, $\mathcal{B}_{kj}$, for every $1 \le k < j \le s-1$, or $\mathcal{B}_{ks}$, for every $k \in [s-2]$ in Construction I in [23], respectively. Hence, we have the following results.

Lemma 5 (Zhao *et al.* [23])**.** *Suppose $i \in [s]$ and $c(x_i^v) \neq c(y_i^v)$ for some $v \in M_i$. Then there exists at most one integer in M_i, say p, such that $c(x_i^p) = c(y_i^p)$.*

Lemma 6 (Zhao *et al.* [23])**.** *Suppose $c(x_s^k) = c(y_s^k) = k$, for every $k \in [l_s]$. Then one of the following conditions holds.*

(1) *$c(x_1^t) = c(y_1^t)$, for every $t \in M_1$, $c(x_{n_1-h}) = c(y_{n_1-h})$, for every $h \in [3]$, or*

(2) *$c(x_1^t) = c(x_{n_1-h}) = d$, $c(y_{n_1-3}) = c(y_1^t) = 1$, for every $h \in [3] \wedge t \in M_1$; $c(y_{n_1-2}) = 2$, $c(y_{n_1-1}) = 3$.*

Lemma 7 (Zhao *et al.* [23])**.** *Suppose $c(x_s^k) = c(y_s^k) = k$, $k \in [l_s]$. If $c(x_i^v) \neq c(y_i^v)$ for some $i \in [s-1] \setminus \{1\} \wedge v \in M_i$, then $c(x_j^t) = d, c(y_j^t) = 1$, for every $j \in [i] \wedge t \in M_j$.*

Lemma 8. *We may reorder the color classes such that* $c(x_s^k) = c(y_s^k) = k$*,* $k \in [l_s]$*, and* $c(x_{n_s-1}) = c(y_{n_s-1}) = c(z_{n_s-1}) = n_s - 1$*.*

Proof. By Lemma 5 we have the following three possible cases.

Case 1. $c(x_s^k) = c(y_s^k)$, $k \in [l_s]$. Pick integers $p, q \in [l_s]$ such that $p < q$. Then $\{x_s^p, x_s^q, y_s^q\}$ is an edge if $p + q \equiv 0 \pmod 2$, or $\{y_s^q, x_s^p, y_s^p\}$ is an edge if $p + q \equiv 1 \pmod 2$, which implies that $c(x_s^p) \neq c(x_s^q)$. Hence, we may reorder the color classes such that $c(x_s^k) = c(y_s^k) = k$, $k \in [l_s]$. Then $d \notin [l_s]$. We focus on the colors of the vertices x_{n_s-1}, y_{n_s-1} and z_{n_s-1}. If $c(x_{n_s-1}) \neq c(y_{n_s-1})$, then $\{c(x_{n_s-1}), c(y_{n_s-1})\} = \{7, d\}$ since $\{y_s^7, x_{n_s-1}, y_{n_s-1}\}$ and $\{x_{n_1}, x_{n_s-1}, y_{n_s-1}\}$ are edges, which implies that the edge $\{y_s^8, x_{n_s-1}, y_{n_s-1}\}$ is polychromatic, a contradiction. Hence, $c(x_{n_s-1}) = c(y_{n_s-1})$. If $c(x_{n_s-1}) \neq c(z_{n_s-1})$, then $\{c(x_{n_s-1}), c(z_{n_s-1})\} = \{11, 12\}$ since $\{y_s^{11}, x_{n_s-1}, z_{n_s-1}\}$ and $\{y_s^{12}, x_{n_s-1}, z_{n_s-1}\}$ are edges, which implies that the edge $\{y_s^9, y_{n_s-1}, z_{n_s-1}\}$ is polychromatic, a contradiction. Therefore, $c(x_{n_s-1}) = c(z_{n_s-1})$. Then from the edges in $\mathcal{E}_s$, one gets that $c(x_{n_s-1}), c(y_{n_s-1}), c(z_{n_s-1}) \notin [l_s]$. Hence, we may reorder the color classes such that $c(x_{n_s-1}) = c(y_{n_s-1}) = c(z_{n_s-1}) = n_s - 1$, as desired.

Case 2. $c(x_s^k) \neq c(y_s^k)$, for every $k \in [l_s]$. Suppose $\{c(x_s^1), c(y_s^1)\} = \{1, d\}$. There are the following two possible subcases.

Subcase 2.1. $\{c(x_s^k), c(y_s^k)\} = \{1, d\}$, for every $k \in [l_s]$. From the edges $\{x_{n_s-1}, x_s^1, y_s^1\}$, $\{y_{n_s-1}, x_s^4, y_s^4\}$ and $\{z_{n_s-1}, x_s^{13}, y_s^{13}\}$, we have $x_{n_s-1}, y_{n_s-1}, z_{n_s-1} \in C_1 \cup C_d$. Then at least two of the three vertices x_{n_s-1}, y_{n_s-1} and z_{n_s-1} have the same color. If $c(x_{n_s-1}) = c(y_{n_s-1})$, then the edge $\{x_s^7, x_{n_s-1}, y_{n_s-1}\}$ or $\{y_s^7, x_{n_s-1}, y_{n_s-1}\}$ is monochromatic, a contradiction; if $c(x_{n_s-1}) = c(z_{n_s-1})$, then the edge $\{x_s^{11}, x_{n_s-1}, z_{n_s-1}\}$ or $\{y_s^{11}, x_{n_s-1}, z_{n_s-1}\}$ is monochromatic, a contradiction; if $c(y_{n_s-1}) = c(z_{n_s-1})$, then the edge $\{x_s^9, y_{n_s-1}, z_{n_s-1}\}$ or $\{y_s^9, y_{n_s-1}, z_{n_s-1}\}$ is monochromatic, also a contradiction.

Subcase 2.2. $\{c(x_s^v), c(y_s^v)\} \neq \{1, d\}$ for some $v \in [l_s] \setminus \{1\}$. Write $T = \{t \in [l_s] \mid \{c(x_s^t), c(y_s^t)\} \neq \{1, d\}\}$, $p = \min(T)$ and $\{c(x_s^p), c(y_s^p)\} = \{b, d\}$, where $b \neq 1$. Then $\{c(x_s^k), c(y_s^k)\} = \{1, d\}$, for every $k \in [p-1]$. The edge $\{y_s^p, x_s^{p-1}, y_s^{p-1}\}$ implies that $c(x_s^p) = b$, $c(y_s^p) = d$. For each integer $k \in \{p + 2, \ldots, l_s\}$ such that $k + p \equiv 0 \pmod 2$, $c(x_s^k) = b$, $c(y_s^k) = d$ since $\{x_s^p, x_s^k, y_s^k\}$ and $\{y_s^k, x_s^{p-1}, y_s^{p-1}\}$ are edges. For each integer $k \in [p-1]$ such that $k + p \equiv 0 \pmod 2$, the edge $\{x_s^k, x_s^p, y_s^p\}$ implies that $c(x_s^k) = d$, $c(y_s^k) = 1$. The edges $\{x_{n_s-1}, x_s^1, y_s^1\}$, $\{x_{n_s-1}, x_s^{l_s}, y_s^{l_s}\}$ and $\{x_{n_s-1}, x_s^{l_s-1}, y_s^{l_s-1}\}$ imply that $c(x_{n_s-1}) = d$. Assume that $p \geq l_s - 2$. The edges $\{y_{n_s-1}, x_s^4, y_s^4\}$ and $\{x_{n_1}, x_{n_s-1}, y_{n_s-1}\}$ imply that $c(y_{n_s-1}) = 1$. From the edge $\{z_{n_s-1}, x_s^{13}, y_s^{13}\}$, one gets $z_{n_s-1} \in C_1 \cup C_d$. If $c(z_{n_s-1}) = 1$, then the edge $\{x_s^9, y_{n_s-1}, z_{n_s-1}\}$ or $\{y_s^9, y_{n_s-1}, z_{n_s-1}\}$ is monochromatic, a contradiction; if $c(z_{n_s-1}) = d$, then the edge $\{x_s^{11}, x_{n_s-1}, z_{n_s-1}\}$ or $\{y_s^{11}, x_{n_s-1}, z_{n_s-1}\}$ is monochromatic, also a contradiction. Assume that $5 \leq$

$p \leq l_s - 3$. Then $c(y_{n_s-1}) = d$ since $\{y_{n_s-1}, x_s^4, y_s^4\}$, $\{y_{n_s-1}, x_s^{l_s-3}, y_s^{l_s-3}\}$ and $\{y_{n_s-1}, x_s^{l_s-4}, y_s^{l_s-4}\}$ are edges, which implies that the edge $\{x_{n_1}, x_{n_s-1}, y_{n_s-1}\}$ is monochromatic, a contradiction. Assume that $2 \leq p \leq 4$. From the edges $\{x_1^1, x_s^1, y_s^1\}$, $\{x_1^1, x_s^p, y_s^p\}$ and $\{y_1^1, x_s^{p+4}, y_s^{p+4}\}$, we have $c(x_1^1) = d, c(y_1^1) = b$. For each integer $k \in \{5, \ldots, l_s\}$, the edge $\{y_1^1, x_s^k, y_s^k\}$ implies that $\{c(x_s^k), c(y_s^k)\} = \{b, d\}$. Since $\{y_{n_s-1}, x_s^5, y_s^5\}$ and $\{x_{n_1}, x_{n_s-1}, y_{n_s-1}\}$ are edges, $c(y_{n_s-1}) = b$. From the edges $\{x_s^9, y_{n_s-1}, z_{n_s-1}\}$, $\{y_s^9, y_{n_s-1}, z_{n_s-1}\}$, we have $c(z_{n_s-1}) = d$, which implies that the edge $\{x_s^{11}, x_{n_s-1}, z_{n_s-1}\}$ or $\{y_s^{11}, x_{n_s-1}, z_{n_s-1}\}$ is monochromatic, a contradiction.

Case 3. There exists an integer in $[l_s]$, say p, such that $c(x_s^p) = c(y_s^p)$ and $c(x_s^t) \neq c(y_s^t)$, for every $t \in [l_s] \setminus \{p\}$.

Suppose $a = c(x_s^p) = c(y_s^p)$. If $p > 1$, then the edge $\{y_s^p, x_s^k, y_s^k\}$ implies that $\{c(x_s^k), c(y_s^k)\} = \{a, d\}$ for each integer $k \in [p-1]$ such that $k + p \equiv 1 \pmod 2$. Moreover, if $p \leq l_s - 2$, then the edge $\{x_s^p, x_s^k, y_s^k\}$ implies that $\{c(x_s^k), c(y_s^k)\} = \{a, d\}$ for each integer $k \in \{p+1, \ldots, l_s\}$ such that $k + p \equiv 0 \pmod 2$. Suppose $|c(X_s)| = 2$. Then from the edges in $\mathcal{E}_s$, one gets that $c(x_{n_s-1}), c(y_{n_s-1}), c(z_{n_s-1}) \in \{a, d\}$. Similarly to Subcase 2.1, we may have contradictions. Suppose $|c(X_s)| \geq 3$. Then $\{c(x_s^v), c(y_s^v)\} \neq \{a, d\}$ for some integer $v \in [l_s] \setminus \{p\}$. Write $T = \{t \in [l_s] \mid \{c(x_s^t), c(y_s^t)\} \neq \{a, d\}\}$, $q = \min(T)$ and $\{b, d\} = \{c(x_s^q), c(y_s^q)\}$, where $b \neq a$. There are the following two possible subcases.

Case 3.1. $q \in [p-1]$. Then $q + p \equiv 0 \pmod 2$ and $\{c(x_s^{q+1}), c(y_s^{q+1})\} = \{a, d\}$. From the edges $\{y_s^{q+1}, x_s^q, y_s^q\}$ and $\{y_s^p, x_s^{q+1}, y_s^{q+1}\}$, we have $c(x_s^{q+1}) = a$, $c(y_s^{q+1}) = d$. For each integer $k \in \{q+3, \ldots, l_s\}$ such that $k + q \equiv 1 \pmod 2$, $c(x_s^k) = a$, $c(y_s^k) = d$ since $\{x_s^{q+1}, x_s^k, y_s^k\}$ and $\{y_s^k, x_s^q, y_s^q\}$ are edges. Suppose $q = 1$. Then $c(x_s^k) = a, c(y_s^k) = d$ for each even integer $k \in [l_s]$. Assume that $p \geq 5$. From the edges $\{x_1^1, x_s^1, y_s^1\}$, $\{x_1^1, x_s^2, y_s^2\}$ and $\{y_1^1, x_s^p, y_s^p\}$, $\{y_1^1, x_s^6, y_s^6\}$, we have $c(x_1^1) = c(y_1^1) = d$, which implies that the edge $\{x_{n_1}, x_1^1, y_1^1\}$ is monochromatic, a contradiction. Assume that $p = 3$. Then $\{c(x_s^k), c(y_s^k)\} = \{a, d\}$ for each odd integer $k \in \{5, \ldots, l_s\}$. Since $\{x_{n_s-1}, x_s^2, y_s^2\}$, $\{x_{n_s-1}, x_s^3, y_s^3\}$ and $\{y_{n_s-1}, x_s^5, y_s^5\}$, $\{x_{n_1}, x_{n_s-1}, y_{n_s-1}\}$ are edges, $c(x_{n_s-1}) = d$, $c(y_{n_s-1}) = a$. From the edge $\{z_{n_s-1}, x_s^{13}, y_s^{13}\}$, we have $c(z_{n_s-1}) \in \{a, d\}$. Then the edge $\{x_s^9, y_{n_s-1}, z_{n_s-1}\}$ or $\{y_s^9, y_{n_s-1}, z_{n_s-1}\}$ is monochromatic if $c(z_{n_s-1}) = a$, a contradiction; the edge $\{x_s^{11}, x_{n_s-1}, z_{n_s-1}\}$ or $\{y_s^{11}, x_{n_s-1}, z_{n_s-1}\}$ is monochromatic if $c(z_{n_s-1}) = d$, also a contradiction. Suppose $q > 1$. From the edge $\{y_s^q, x_s^{q-1}, y_s^{q-1}\}$, we have $c(x_s^q) = b$, $c(y_s^q) = d$. For each integer $k \in \{q+2, \ldots, l_s\}$ such that $k + q \equiv 0 \pmod 2$, from the edges $\{x_s^q, x_s^k, y_s^k\}$ and $\{y_s^k, x_s^{k-1}, y_s^{k-1}\}$, we have $c(x_s^k) = b, c(y_s^k) = d$. Assume that $p \leq l_s - 2$. Then the edge $\{x_s^p, x_s^{p+2}, y_s^{p+2}\}$ is polychromatic, a contradiction. Assume that $p \geq l_s - 1$. If $q \leq 4$, then $c(x_1^1) = c(y_1^1) = d$ since $\{x_1^1, x_s^q, y_s^q\}$, $\{x_1^1, x_s^{q-1}, y_s^{q-1}\}$ and $\{y_1^1, x_s^p, y_s^p\}$, $\{y_1^1, x_s^{p-1}, y_s^{p-1}\}$ are edges, which implies that the edge $\{x_{n_1}, x_1^1, y_1^1\}$ is monochromatic, a contradiction. If

$5 \le q \le l_s - 3$, then $c(x_{n_s-1}) = c(y_{n_s-1}) = d$ since $\{x_{n_s-1}, x_s^{l_s-h}, y_s^{l_s-h}\}, h = 0,1,2$ and $\{y_{n_s-1}, x_s^{l_s-h}, y_s^{l_s-h}\}, h = 3,4,5$ are edges, which implies that the edge $\{x_{n_1}, x_{n_s-1}, y_{n_s-1}\}$ is monochromatic, a contradiction. Let $q = l_s - 2$. Then $p = l_s$. From the edges $\{x_{n_s-1}, x_s^{l_s}, y_s^{l_s}\}$, $\{x_{n_s-1}, x_s^{l_s-1}, y_s^{l_s-1}\}$ and $\{y_{n_s-1}, x_s^{l_s-3}, y_s^{l_s-3}\}$, we have $c(x_{n_s-1}) = d$, $c(y_{n_s-1}) = a$. Since $\{x_s^9, y_{n_s-1}, z_{n_s-1}\}$ and $\{y_s^9, y_{n_s-1}, z_{n_s-1}\}$ are edges, one gets $c(z_{n_s-1}) = d$, which implies that the edge $\{x_s^{11}, x_{n_s-1}, z_{n_s-1}\}$ or $\{y_s^{11}, x_{n_s-1}, z_{n_s-1}\}$ is monochromatic, a contradiction.

Case 3.2. $q > p$. Then $q + p \equiv 1 \pmod 2$. Moreover, $\{c(x_s^k), c(y_s^k)\} = \{a, d\}$ for each integer $k \in [q-1] \setminus \{p\}$. If $q > p+1$, then $c(x_s^{p+1}) = a$, $c(y_s^{p+1}) = d$ since $\{y_s^{p+1}, x_s^p, y_s^p\}$ is an edge, which implies that the edge $\{x_s^{p+1}, x_s^q, y_s^q\}$ is polychromatic, a contradiction. Hence, $q = p+1$. For each integer $k \in \{p+2, \dots, l_s\}$ such that $k+p \equiv 0 \pmod 2$, the edges $\{x_s^p, x_s^k, y_s^k\}$ and $\{y_s^k, x_s^q, y_s^q\}$ imply that $c(x_s^k) = a$, $c(y_s^k) = d$. Suppose $p = 1$. The edges $\{x_{n_s-1}, x_s^1, y_s^1\}$ and $\{x_{n_s-1}, x_s^3, y_s^3\}$ imply that $c(x_{n_s-1}) = d$. From the edges $\{y_{n_s-1}, x_s^5, y_s^5\}$ and $\{x_{n_1}, x_{n_s-1}, y_{n_s-1}\}$, we have $c(y_{n_s-1}) = a$. Since $\{x_s^9, y_{n_s-1}, z_{n_s-1}\}$ and $\{y_s^9, y_{n_s-1}, z_{n_s-1}\}$ are edges, $c(z_{n_s-1}) = d$, which implies that the edge $\{y_s^{11}, x_{n_s-1}, z_{n_s-1}\}$ is monochromatic, a contradiction. Suppose $p > 1$. The edge $\{x_s^k, x_s^q, y_s^q\}$ implies that $c(x_s^k) = d$, $c(y_s^k) = a$ for each integer $k \in [p-1]$ such that $k + p \equiv 1 \pmod 2$. For each integer $k \in [p-1]$ such that $k + p \equiv 0 \pmod 2$, from the edges $\{x_s^k, x_s^p, y_s^p\}$ and $\{y_s^{p-1}, x_s^k, y_s^k\}$, we have $c(x_s^k) = d, c(y_s^k) = a$. Assume that $c(x_s^q) = b$ and $c(y_s^q) = d$. Then for each integer $k \in \{q+1, \dots, l_s\}$ such that $k + q \equiv 0 \pmod 2$, $c(x_s^k) = b$, $c(y_s^k) = d$ since $\{x_s^q, x_s^k, y_s^k\}$ and $\{y_s^k, x_s^{k-1}, y_s^{k-1}\}$ are edges. If $p \le 4$, then $c(x_1^1) = c(y_1^1) = d$ since $\{x_1^1, x_s^1, y_s^1\}$, $\{x_1^1, x_s^p, y_s^p\}$ and $\{y_1^1, x_s^{l_s}, y_s^{l_s}\}$, $\{y_1^1, x_s^{l_s-1}, y_s^{l_s-1}\}$ are edges, a contradiction. If $5 \le p \le l_s - 3$, then $c(x_{n_s-1}) = c(y_{n_s-1}) = d$ since $\{x_{n_s-1}, x_s^{l_s}, y_s^{l_s}\}$, $\{x_{n_s-1}, x_s^{l_s-1}, y_s^{l_s-1}\}$ and $\{y_{n_s-1}, x_s^{l_s-3}, y_s^{l_s-3}\}$, $\{y_{n_s-1}, x_s^{l_s-4}, y_s^{l_s-4}\}$ are edges, a contradiction. Let $p \ge l_s - 2$. From the edges $\{x_{n_s-1}, x_s^1, y_s^1\}$ and $\{x_{n_s-1}, x_s^p, y_s^p\}$, we have $c(x_{n_s-1}) = d$. Since $\{y_{n_s-1}, x_s^4, y_s^4\}$, $\{x_{n_1}, x_{n_s-1}, y_{n_s-1}\}$ and $\{x_s^{11}, x_{n_s-1}, z_{n_s-1}\}$, $\{z_{n_s-1}, x_s^{13}, y_s^{13}\}$ are edges, $c(y_{n_s-1}) = c(z_{n_s-1}) = a$, which implies that the edge $\{y_s^9, y_{n_s-1}, z_{n_s-1}\}$ is monochromatic, a contradiction. Assume that $c(x_s^q) = d$, $c(y_s^q) = b$. If $p > 2$, then the edge $\{y_s^q, x_s^{p-2}, y_s^{p-2}\}$ is polychromatic, a contradiction. Let $p = 2$. From the edges $\{x_{n_s-1}, x_s^1, y_s^1\}$, $\{x_{n_s-1}, x_s^2, y_s^2\}$ and $\{y_{n_s-1}, x_s^4, y_s^4\}$, we have $c(x_{n_s-1}) = d, c(y_{n_s-1}) = a$. Since $\{y_{n_s-1}, x_s^5, y_s^5\}$ is an edge, $\{c(x_s^5), c(y_s^5)\} = \{a, d\}$. The edge $\{y_s^5, x_s^2, y_s^2\}$ implies that $c(x_s^5) = a, c(y_s^5) = d$. For each odd integer $k \in \{7, \dots, l_s\}$, from the edges $\{x_s^5, x_s^k, y_s^k\}$ and $\{y_s^k, x_s^2, y_s^2\}$, we have $c(x_s^k) = a$, $c(y_s^k) = d$. Then $c(z_{n_s-1}) = a$ since $\{y_s^{11}, x_{n_s-1}, z_{n_s-1}\}$, $\{z_{n_s-1}, x_s^{13}, y_s^{13}\}$ are edges, which implies that the edge $\{x_s^9, y_{n_s-1}, z_{n_s-1}\}$ is monochromatic, a contradiction. The proof is completed. ■

Theorem 9. *$\mathcal{H}_{n_1,\dots,n_s}$ is a one-realization of S.*

Proof. Let $c = \{C_1, C_2, \dots, C_m\}$ be a strict coloring of $\mathcal{H}_{n_1,\dots,n_s}$ with $c(x_{n_1}) = d$.

By Lemma 8 we may reorder the color classes such that $c(x_s^k) = c(y_s^k) = k$, for every $k \in [l_s]$, $c(x_{n_s-1}) = c(y_{n_s-1}) = c(z_{n_s-1}) = n_s - 1$. Moreover, $d \notin [n_s - 1]$. We focus on the colors of the other vertices and have the following two possible cases.

Case 1. $c(x_j^v) \neq c(y_j^v)$ for some $j \in [s-1] \setminus \{1\} \wedge v \in M_j$. Set $M = \{j \in [s-1] \setminus \{1\} \mid c(x_j^v) \neq c(y_j^v)$ for some $v \in M_j\}$ and $i = \max(M)$. That is to say, $c(x_h^t) = c(y_h^t)$, for every $h \in \{i+1, \ldots, s\} \wedge t \in M_h$. Pick $h \in \{i+1, \ldots, s-1\}$. For any integers $p, q \in M_h$ such that $p < q$, $c(x_h^p) \neq c(x_h^q)$ since $\{x_h^p, x_h^q, y_h^q\}$ or $\{y_h^q, x_h^p, y_h^p\}$ is an edge. For each $j \in \{h+1, \ldots, s\}$, $t \in M_j$ and $p \in M_h$, since $\{x_h^p, x_j^t, y_j^t\}$ or $\{y_h^p, x_j^t, y_j^t\}$ is an edge, $c(x_h^p) \neq c(x_j^t)$. The edge $\{x_h^p, x_{n_s-1}, y_{n_s-1}\}$ implies that $c(x_h^p) \neq n_s - 1$ for each $p \in M_h$. Hence, we may reorder the color classes such that $c(x_h^t) = c(y_h^t) = n_h + t$, for every $h \in \{i+1, \ldots, s-1\} \wedge t \in M_h$. By Lemma 7 we have that $c(x_k^t) = d, c(y_k^t) = 1$, for every $k \in [i] \wedge t \in M_k$. By Lemma 6 we have $c(x_{n_1-h}) = d$, for every $h \in [3]$, and $c(y_{n_1-3}) = 1$, $c(y_{n_1-2}) = 2$, $c(y_{n_1-1}) = 3$. Hence, $c = c_{i+1}$.

Case 2. $c(x_i^t) = c(y_i^t)$, for every $i \in [s-1] \setminus \{1\} \wedge t \in M_i$. Then we may reorder the color classes such that $c(x_i^t) = c(y_i^t) = n_i + t$, for every $i \in \{2, \ldots, s-1\} \wedge t \in M_i$. Suppose $c(x_1^0) = c(y_1^0)$. By Lemma 6 we have $c(x_1^t) = c(y_1^t)$, for every $t \in M_1$, and $c(x_{n_1-h}) = c(y_{n_1-h}), h = 1, 2, 3$, which implies that $c = c_1$. Suppose $c(x_1^0) \neq c(y_1^0)$. By Lemma 6 we get that $c(x_{n_1-1}) = c(x_{n_1-2}) = c(x_{n_1-3}) = c(x_1^t) = d$, $c(y_{n_1-3}) = c(y_1^t) = 1$, for every $t \in M_1$, and $c(y_{n_1-2}) = 2, c(y_{n_1-1}) = 3$, which implies that $c = c_2$. The proof is completed. ■

3.2. The case of $n_1 - 1 \in S$

Construction II. Let $s \geq 4$, $n_2 = n_1 - 1$, $n_s, n_i - n_{i+1} \geq 56$, $i \in [s-1] \setminus \{1, 2\}$. Set

$$\begin{aligned}
M_i' &= M_i,\ l_i' = l_i,\ i \in [s] \setminus \{1,2\},\ l_2' = n_2 - n_3 - 4, M_2' = \{0, 1, \ldots, l_2'\},\\
X_i' &= X_i,\ i \in [s] \setminus \{1,2\},\ Y_s' = Y_s,\\
X_2' &= \{x_2^t, y_2^t \mid t \in M_2'\},\ \text{where}\\
x_2^t &= (n_3 + t, n_3 + t, n_3, \ldots, n_s, 1),\ y_2^t = (n_3 + t, n_3 + t, \underbrace{1, \ldots, 1}_{s-2}, 0),\\
X_0' &= \{x_{n_2-h}, y_{n_2-h} \mid h = 1, 2, 3\} \cup \{x_{n_2}, x_{n_1}\},\ \text{where}\\
x_{n_2-h} &= (n_2 - h, n_2 - h, n_3, \ldots, n_s, 1),\\
y_{n_2-h} &= (n_2 - h, n_2 - h, 4 - h, \ldots, 4 - h, 0),\\
x_{n_2} &= (n_2, n_2, n_3, \ldots, n_s, 1),\quad x_{n_1} = (n_1, n_2, \ldots, n_s, 1),\\
X' &= \left(\bigcup_{i=2}^{s} X_i'\right) \cup X_0' \cup Y_s',
\end{aligned}$$

$\mathcal{B}'_i = \mathcal{B}_i, i \in [s] \setminus \{1,2\},\ \mathcal{B}'_{kj} = \mathcal{B}_{kj}, 2 \le k < j \le s,\ \mathcal{E}'_s = \mathcal{E}_s,$

$$\begin{aligned}
\mathcal{B}'_0 = &\{\{x_{n_2-3}, x_i^p, y_i^p\}, \{y_{n_2-3}, x_i^q, y_i^q\} \mid i \in [s-1]\setminus\{1\},\ 0 \le p \le 3,\ 4 \le q \le l'_i\}\\
&\cup \{\{x_{n_2-3}, x_s^p, y_s^p\}, \{y_{n_2-3}, x_s^q, y_s^q\} \mid p \in [4],\ 5 \le q \le l'_s\}\\
&\cup \{\{x_{n_2-h}, x_i^t, y_i^t\}, \{x_{n_1}, x_i^t, y_i^t\}, \mid 0 \le h \le 2, i \in [s] \setminus \{1\}, t \in M'_i\}\\
&\cup \{\{x_k^t, x_{n_s-1}, y_{n_s-1}\} \mid k \in [s-1], t \in M'_k\}\\
&\cup \{\{x_{n_2-h}, x_{n_s-1}, y_{n_s-1}\}, \{x_{n_1}, x_{n_s-1}, y_{n_s-1}\} \mid h = 0,1,2\}\\
&\cup \{\{x_{n_2}, x_{n_2-h}, y_{n_2-h}\}, \{x_{n_1}, x_{n_2-h}, y_{n_2-h}\} \mid h = 1,2,3\}\\
&\cup \{\{x_{n_2-1}, x_{n_2-2}, y_{n_2-2}\}, \{x_{n_2-2}, x_{n_2-3}, y_{n_2-3}\}, \{x_{n_2-3}, x_{n_2-1}, y_{n_2-1}\}\}\\
&\cup \{\{y_s^1, x_{n_2-3}, y_{n_2-3}\}, \{y_s^2, x_{n_2-2}, y_{n_2-2}\}, \{y_s^3, x_{n_2-1}, y_{n_2-1}\}, \{y_2^4, x_{n_2-3}, y_{n_2-3}\}\},
\end{aligned}$$

$\mathcal{E}' = \{\{y_{n_2-3}, x_2^4, y_2^4\}, \{x_{n_2-3}, x_s^1, y_s^1\}, \{x_{n_2-2}, x_s^2, y_s^2\}, \{x_{n_2-1}, x_s^3, y_s^3\}\},$

$$\mathcal{B}' = \left(\left(\bigcup_{i=2}^{s} \mathcal{B}'_i\right) \cup \left(\bigcup_{k=2}^{s-1} \bigcup_{j=k+1}^{s} \mathcal{B}'_{kj}\right) \cup \mathcal{B}'_0 \cup \mathcal{E}'_s\right) \setminus \mathcal{E}'.$$

Then $\mathcal{H}' = (X', \mathcal{B}')$ is a 3-uniform bi-hypergraph with $2n_1 - 1$ vertices and $\frac{n_1(n_1-1)}{2} + 2$ edges. Moreover, for each $i \in [s]$,

$$c'_i = \left\{X'_{i1}, X'_{i2}, \ldots, X'_{in_i}\right\} \tag{1}$$

is a strict n_i-coloring of $\mathcal{H}'$, where X'_{ij} is the set of the vertices in X' whose i-th entry is j. Furthermore, we have the following result.

Theorem 10. *$\mathcal{H}' = (X', \mathcal{B}')$ is a one-realization of S.*

Proof. It suffices to prove that $\mathcal{H}'$ has no other strict colorings except $c'_1, c'_2, \ldots, c'_s$. Let $Y = X' \setminus \{x_{n_1}\}$. Note that $\mathcal{H}'[Y]$ is isomorphic to $\mathcal{H}_{n_2,n_3,\ldots,n_s}$ under the following bijection

$$\begin{aligned}
\varphi: \quad\quad\quad\quad Y &\to X_{n_2,n_3,\ldots,n_s}\\
(x_2, x_2, x_3, \ldots, x_s) &\to (x_2, x_3, \ldots, x_s).
\end{aligned}$$

Then all of the strict colorings of $\mathcal{H}'[Y]$ are as follows

$$c''_i = \left\{X''_{i1}, X''_{i2}, \ldots, X''_{in_i}\right\}, \quad i \in [s] \setminus \{1\}, \tag{2}$$

where $X''_{ij} = X'_{ij} \cap Y, j \in [n_i]$. Assume that $c' = \{C'_1, C'_2, \ldots, C'_m\}$ is a strict coloring of $\mathcal{H}'$ and c'' is the restriction of c' on Y. There are the following two possible cases.

Case 1. $c'|_Y = c''_2$. That is to say, $c'(x_s^k) = c'(y_s^k) = k$, for every $k \in [l_s]$; $c'(x_{n_s-1}) = c'(y_{n_s-1}) = c'(y_{n_s-1}) = n_s - 1$; $c'(x_i^t) = c'(y_i^t) = n_i + t$, for every $i \in [s-1] \setminus \{1\} \wedge t \in M'_i$; $c'(x_{n_2-h}) = c'(y_{n_2-h}) = n_2 - h, h = 1,2,3$; and

$c'(x_{n_2}) = n_2$. Then $c(x_{n_1}) \neq k$, for every $k \in [n_2 - 1]$. Hence, $c' = c'_2$ if $c'(x_{n_1}) = c'(x_{n_2})$, or $c' = c'_1$ otherwise.

Case 2. $c'|_Y = c''_i$ for some $i \in [s] \setminus \{1, 2\}$. From the construction of $\mathcal{H}'$, we have that the edges that contain the vertex x_{n_1} is 1-1 correspondence to the edges that contain the vertex x_{n_2}. Hence, $c'(x_{n_1}) = c'(x_{n_2})$, which implies that $c' = c'_i$, as desired. ∎

Note that $|X_{n_1,\ldots,n_s}| = 2n_1$, $|X'| = 2n_1 - 1$ and $|\mathcal{B}_{n_1,\ldots,n_s}| = \frac{n_1(n_1-1)}{2} + 3$, $|\mathcal{B}'| = \frac{n_1(n_1-1)}{2} + 2$. Combining Theorems 4, 9 and 10, we get the desired result of Theorem 1.1.

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