# THE CROSSING NUMBER OF THE HEXAGONAL GRAPH $\boldsymbol{H}_{3, n}$ 

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#### Abstract

In [C. Thomassen, Tilings of the torus and the Klein bottle and vertextransitive graphs on a fixed surface, Trans. Amer. Math. Soc. 323 (1991) 605-635], Thomassen described completely all (except finitely many) regular tilings of the torus $S_{1}$ and the Klein bottle $N_{2}$ into (3,6)-tilings, (4,4)-tilings and (6,3)-tilings. Many authors made great efforts to investigate the crossing number (in the plane) of the Cartesian product of an $m$-cycle and an $n$-cycle, which is a special $(4,4)$-tiling. For other tilings, there are quite rare results concerning on their crossing numbers. This motivates us in the paper to determine the crossing number of a hexagonal graph $H_{3, n}$, which is a special kind of (3,6)-tilings.


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## 1. Introduction

In [13], Thomassen described completely all (except finitely many) regular tilings of the torus $S_{1}$ and the Klein bottle $N_{2}$ into hexagons, quadrilaterals and triangles in which the vertices have degree 3,4 and 6 , respectively. To be more specific, let $G$ be a connected $d$-regular graph $(d \geq 3)$ and $\varphi$ a collection of $m$-cycles in $G$, assume that each edge of $G$ is contained in precisely two cycles in $\varphi$ and that, for each vertex $v$ in $G$, the edges incident with $v$ can be labelled $e_{1}, e_{2}, \ldots, e_{d}$ such that for each $i=1,2, \ldots, d$, there is a cycle in $\varphi$ containing $e_{i}$ and $e_{i+1}$ (where $e_{d+1}=e_{1}$ ). Then a surface $S$ can be obtained by letting the cycles of $\varphi$ be disjoint convex polygons in the Euclidean plane pasted together by the graph $G$, and $G$ is said to be a $(d, m)$-tiling of $S$. Using Euler's formula, Thomassen observed that a regular tiling of the torus or the Klein bottle fit into three categories: (3,6)-tilings, (4,4)-tilings and (6,3)-tilings.

Note that the Cartesian product of an $m$-cycle and an $n$-cycle, denoted by $C_{m} \square C_{n}$, is a special kind of (4,4)-tilings. It is well known that $C_{m} \square C_{n}$ can be embedded in the torus whose genus is 1, but cannot be embedded in the plane. Therefore, many authors made great efforts to determine the crossing number of $C_{m} \square C_{n}$ in the plane. However, determining the crossing number of graphs is a tedious problem [6], and only very few families of graphs whose crossing number are known $[3,4,8,9,14]$. According to its difficulty, it is not surprising that there are very few exact results concerning on the crossing number of $C_{m} \square C_{n}$ $[1,2,7,10,12]$.

For other regular tilings, to the best of our knowledge, there are quite rare results focus on determining their crossing numbers in the plane. Therefore, this arises our intensive interest in studying the problem, and this contribution is devoted to determine the crossing number of $H_{3, n}$, which is a special kind of (3,6)-tilings.


Figure 1. The embedding of the hexagonal graph $H_{3, n}$ of breadth three and length $n$ $(n \geq 3)$ in the torus.

## 2. Definitions

We shall introduce some basic definitions in this section.
All graphs considered here are finite, simple and connected. Let $G$ be a graph with vertex set $V$ and edge set $E$. The crossing number $\operatorname{cr}(G)$ of a graph $G$ is the minimum number of pairwise intersections of edges in a drawing of $G$ in the plane. It is well known that the crossing number of a graph is attained only in good drawings of the graph, which are the drawings where no edge crosses itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges have a common point. Let $D$ be a good drawing of the graph $G$, we denote the number of crossings in $D$ by $c r_{D}(G)$. If $D$ is a good drawing of $G$ satisfying $\operatorname{cr}_{D}(G)=c r(G)$, then $D$ is an optimal drawing of $G$. In a drawing $D$, if an edge is not crossed by any other edge, we say that it is clean in $D$, otherwise, we say it is crossed. For definitions not explained here, readers are referred to [5].

Figure 1 shows the embedding of the hexagonal graph $H_{3, n}$ of breadth three and length $n(n \geq 2)$ in the torus, it is seen that the number of 6 -cycles in the meridional (respectively, longitudinal) direction is three (respectively, $n$ ). To be more specific, $H_{3, n}$ is the graph with vertex set $V\left(H_{3, n}\right)=\left\{a_{i}, b_{i}, c_{i}: i=\right.$ $1,2, \ldots, 2 n\}$, and edge set $E\left(H_{3, n}\right)=\left\{a_{i} a_{i+1}, b_{i} b_{i+1}, c_{i} c_{i+1}: i=1,2, \ldots, 2 n\right\} \cup$ $\left\{a_{2 i-1} b_{2 i-1}, b_{2 i} c_{2 i}, c_{2 i-1} a_{2 i}: i=1,2, \ldots, n\right\}$. The indices are expressed modulo $2 n$. See Figure 1. Clearly, the hexagonal graph $H_{3, n}$ is 3 -regular, and can be viewed as by pasting together with 6 -cycles. Thus, $H_{3, n}$ is a special (3,6)-tiling.

Deleting all the edges $c_{2 i-1} a_{2 i}(i=1,2, \ldots, n)$ from Figure 1, the resulted graph is the hexagonal cylinder of breadth 2 and length $n$. In the hexagonal cylinder, two cycles $a_{1} a_{2} \cdots a_{2 n} a_{1}$ and $c_{1} c_{2} \cdots c_{2 n} c_{1}$ are called peripheral cycles.

The hexagonal graph $H_{m, n}$ of breadth $m$ and length $n$ can be defined as: let $a_{1} a_{2} \cdots a_{2 n} a_{1}$ and $c_{1} c_{2} \cdots c_{2 n} c_{1}$ be two peripheral cycles of the hexagonal cylinder of breadth $m-1$ and length $n, H_{m, n}$ is obtained from the hexagonal cylinder of breadth $m-1$ and length $n$ by adding all the edges $a_{2 i} c_{2 i-1}(i=1,2, \ldots, n)$ when $m$ is odd, and by adding all the edges $a_{2 i} c_{2 i}(i=1,2, \ldots, n)$ when $m$ is even. Figure 2 is an embedding of $H_{m, n}$ in the torus when $m$ is even and $m \geq 4$.

It is easy to see that $H_{2, n}$ is planar, therefore, we begin to investigate the crossing number of $H_{m, n}$ for $m=3$, and get the main result.
Theorem 1. For $n \geq 2, \operatorname{cr}\left(H_{3, n}\right)=n$.

## 3. The Proof of Theorem 1

We shall proceed our proof of Theorem 1 by induction on $n$. The base case is $n=2$, which needs to be discussed firstly.
Lemma 2. $\operatorname{cr}\left(H_{3,2}\right)=2$.


Figure 2. The embedding of $H_{m, n}$ in the torus for $m$ is even and $m \geq 4$.

Proof. Figure 3 shows a good drawing of $H_{3,2}$ in the plane, which indicates that $\operatorname{cr}\left(H_{3,2}\right) \leq 2$. We prove the reverse inequality by assuming to the contrary that there is a good drawing $D$ of $H_{3,2}$ with fewer than 2 crossings, then $\operatorname{cr}_{D}\left(H_{3,2}\right)=1$ since $H_{3,2}$ contains a subdivision of $K_{3,3}$ whose crossing number is 1 [11], see Figure 4. Thus, a planar graph can be obtained from $D$ by removing one of the crossed edge. Nevertheless, one can testify that, for any $e \in E\left(H_{3,2}\right), H_{3,2}-e$ contains a subdivision of $K_{3,3}$. This contradiction completes the proof.


Figure 3. A good drawing of $H_{3,2}$ in the plane.


Figure 4. A subdivision of $K_{3,3}$.

For $1 \leq i \leq n$, let $F_{i}=\left\{a_{2 i-2} a_{2 i-1}, a_{2 i-1} a_{2 i}, b_{2 i-2} b_{2 i-1}, b_{2 i-1} b_{2 i}, c_{2 i-2} c_{2 i-1}\right.$, $\left.c_{2 i-1} c_{2 i}, a_{2 i-1} b_{2 i-1}, b_{2 i} c_{2 i}, c_{2 i-1} a_{2 i}\right\}$, the indices are read modulo $2 n$. Then $F_{1}, F_{2}$, $\ldots, F_{n}$ is a partition of $E\left(H_{3, n}\right)$, that is to say, $E\left(H_{3, n}\right)=\bigcup_{i=1}^{n} F_{i}$, and $F_{i} \cap F_{j}=$ $\emptyset$ for $i \neq j$.

Let $D$ be a good drawing of $H_{3, n}$, we define $f_{D}\left(F_{i}\right)(1 \leq i \leq n)$ to be the
function counting the number of crossings related to $F_{i}$ in $D$ as follows:

$$
f_{D}\left(F_{i}\right)=c r_{D}\left(F_{i}, F_{i}\right)+\frac{1}{2} \sum_{1 \leq j \leq n, j \neq i} c r_{D}\left(F_{i}, F_{j}\right)
$$

By counting the number of crossings in $D$, we can get
Lemma 3. $\operatorname{cr}_{D}\left(H_{3, n}\right)=\sum_{i=1}^{n} f_{D}\left(F_{i}\right)$.
Lemma 4. $\operatorname{cr}\left(H_{3, n}\right) \geq n$ for $n \geq 2$.
Proof. We prove the lemma by induction on $n$. Lemma 2 enforces the inequality holds for $n=2$. Suppose that $\operatorname{cr}\left(H_{3, k}\right) \geq k$ for $k<n$, and that there exists a good drawing $D$ of $H_{3, n}$ satisfying $\operatorname{cr}_{D}\left(H_{3, n}\right)<n$. Together with our assumption, it has $\operatorname{cr}_{D}\left(H_{3, n}\right)=n-1$ since $H_{3, n}$ contains a subdivision of $H_{3, n-1}$.

Let $E_{0}=\left\{a_{2 i-1} b_{2 i-1}, b_{2 i} c_{2 i}, c_{2 i-1} a_{2 i}: i=1,2, \ldots, n\right\}$. For any $e \in E_{0}$, it is not difficult to see that $H_{3, n}-e$ contains a subgraph homeomorphic to $H_{3, n-1}$, therefore, $e$ must be clean in $D$, otherwise, a good drawing of $H_{3, n-1}$ with less than $n-1$ crossings can be constructed from $D$ by removing $e$.

By combining Lemma 3 with the fact that $\operatorname{cr}_{D}\left(H_{3, n}\right)=n-1$, there must exist an $i(1 \leq i \leq n)$ such that $f_{D}\left(F_{i}\right)<1$. Without loss of generality, let $f_{D}\left(F_{2}\right)<1$.

The following two cases are considered.
Case 1. $f_{D}\left(F_{2}\right)=0$. That is to say, all the edges of $F_{2}$ are clean in $D$. Note that the subgraph induced on six edges, $\left\{a_{3} a_{4}, a_{4} c_{3}, c_{3} c_{4}, c_{4} b_{4}, b_{4} b_{3}, b_{3} a_{3}\right\}$, of $F_{2}$ is a 6 -cycle, thus, the subdrawing of the 6 -cycle partite the plane into two faces.

We conclude that vertices $b_{2}$ and $c_{2}$ must lie in the same face since $b_{2} c_{2} \in E_{0}$. Without loss of generality, assume that both $b_{2}$ and $c_{2}$ lie in the interior face Int $C$. Moreover, the vertex $a_{2}$ should also lie in $\operatorname{Int} C$, otherwise, the path $a_{2} c_{1} c_{2}$ will cross the boundary of the 6 -cycle.

Consider now the edge $b_{2} c_{2}$, it is clean in $D$ since $b_{2} c_{2} \in E_{0}$. Therefore, the subdrawing of $F_{2} \cup\left\{b_{2} c_{2}\right\}$ must be as shown in Figure 1. The face Int $C$ has been divided into two regions, with vertices $a_{2}$ and $c_{4}$ do not lie on the boundary of the same region. Hence, the path $a_{2} c_{1} c_{2 n} c_{2 n-1} \cdots c_{5} c_{4}$ will cross $F_{2}$ at least once, which is contradicts with $f_{D}\left(F_{2}\right)=0$.

Case 2. $\quad f_{D}\left(F_{2}\right)>0$. From the definition of $f_{D}$, it has $f_{D}\left(F_{2}\right)=\frac{1}{2}$ and $\operatorname{cr}_{D}\left(F_{2}, F_{2}\right)=0$, which means that exactly one edge of $F_{2}$ is crossed in $D$, and that $F_{2}$ does not have internal crossing in $D$.

By the analogous arguments to those of Case 1 , the subdrawing of the 6 -cycle $a_{3} a_{4} c_{3} c_{4} b_{4} b_{3} a_{3}$ partite the plane into two faces, moreover, the vertices $b_{2}$ and $c_{2}$ should lie in the same face since the edge $b_{2} c_{2}$ is clean in $D$. Without loss of


Figure 5. The subdrawing of $F_{2} \cup\left\{b_{2} c_{2}\right\}$.
generality, assume that both $b_{2}$ and $c_{2}$ lie in the interior face $\operatorname{Int} C$. By adding two edges $b_{2} b_{3}$ and $c_{2} c_{3}$ without any internal crossing occurred in $F_{2}$, and by adding the clean edge $b_{2} c_{2}$, one can see that the face $\operatorname{Int} C$ has been divided into two regions, denoted as $x$ and $y$. See Figure 5 .

Consider the vertex $a_{2}$. If $a_{2}$ lies in the exterior face $\operatorname{Ext} C$, then the two edge-disjoint paths $a_{2} a_{1} b_{1} b_{2}$ and $a_{2} c_{1} c_{2}$ will cross the boundary of the 6 -cycle at least once respectively, contradicts with $f_{D}\left(F_{2}\right)=\frac{1}{2}$. If $a_{2}$ lies in the region $y$, then there will be at least one internal crossing in $F_{2}$ made by the edge $a_{2} a_{3}$, which is impossible. Hence, $a_{2}$ must lie in the region $x$.

Now consider the vertices $a_{5}$ and $b_{5}$, they must lie in the same region since $a_{5} b_{5} \in E_{0}$. The following three subcases are discussed according to in which region do $a_{5}$ and $b_{5}$ lie.

Subcase 2.1. Both $a_{5}$ and $b_{5}$ lie in ExtC. Notice that vertices $a_{2}$ and $a_{5}$ do not lie on the boundary of a same region, thus the two edge-disjoint paths $a_{2} a_{1} a_{2 n} a_{2 n-1} \cdots a_{5}$ and $a_{2} c_{1} c_{2 n} c_{2 n-1} \cdots c_{6} b_{6} b_{5} a_{5}$ will cross at least once with the edges of $F_{2}$, respectively, which implies $f_{D}\left(F_{2}\right) \geq 1$, contradicts with $f_{D}\left(F_{2}\right)=\frac{1}{2}$.

Subcase 2.2. Both $a_{5}$ and $b_{5}$ lie in the region $x$. Remind that the edge $b_{2} c_{2}$ is clean in $D$, therefore, the edge $b_{4} b_{5}$ and the path $c_{4} c_{5} c_{6} b_{6} b_{5}$ will cross at least once with the edges of $F_{2}$, respectively, which is impossible.

Subcase 2.3. Both $a_{5}$ and $b_{5}$ lie in the region $y$. By the analogous arguments to that of Subcase 2.2, the edge $a_{4} a_{5}$ and the path $a_{2} a_{1} a_{2 n} a_{2 n-1} \cdots a_{5}$ will cross at least once with the edges of $F_{2}$, respectively, which is absurd.

All the above contradictions confirm that $\operatorname{cr}\left(H_{3, n}\right) \geq n$.

Lemma 5. $\operatorname{cr}\left(H_{m, n}\right) \leq(m-2) n$ for $n \geq 2$.


Figure 6. A good drawing of $H_{m, n}$ when $m$ is odd.


Figure 7. A good drawing of $H_{m, n}$ when $m$ is even.

Proof. For $m$ is odd (respectively, even), Figure 6 (respectively, Figure 7) demonstrates a good drawing of $H_{m, n}$ in the plane with exactly $(m-2) n$ crossings. Thus, $c r\left(H_{m, n}\right) \leq(m-2) n$.

According to Lemmas 4 and 5, Theorem 1 is easily followed.

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