Discussiones Mathematicae Graph Theory 39 (2019) 455–472 doi:10.7151/dmgt.2091

# ON THE CO-ROMAN DOMINATION IN GRAPHS

### Zehui Shao

Institute of Computing Science and Technology Guangzhou University, Guangzhou 510006, China

e-mail: zshao@gzhu.edu.cn

Seyed Mahmoud Sheikholeslami, Marzieh Soroudi

Department of Mathematics Azarbaijan Shahid Madani University Tabriz, I.R. Iran

e-mail: {s.m.sheikholeslami;m.soroudi}@@azaruniv.ac.ir

LUTZ VOLKMANN

Lehrstuhl II für Mathematik RWTH Aachen University 52056 Aachen, Germany

e-mail: volkm@math2.rwth-aachen.de

AND

## XINMIAO LIU

Beijing Foreign Studies University Beijing 100089, China

### Abstract

Let G = (V, E) be a graph and let  $f : V(G) \to \{0, 1, 2\}$  be a function. A vertex v is said to be protected with respect to f, if f(v) > 0 or f(v) = 0 and v is adjacent to a vertex of positive weight. The function f is a *co-Roman dominating function* if (i) every vertex in V is protected, and (ii) each  $v \in V$ with positive weight has a neighbor  $u \in V$  with f(u) = 0 such that the function  $f_{uv} : V \to \{0, 1, 2\}$ , defined by  $f_{uv}(u) = 1$ ,  $f_{uv}(v) = f(v) - 1$  and  $f_{uv}(x) = f(x)$  for  $x \in V \setminus \{v, u\}$ , has no unprotected vertex. The weight of f is  $\omega(f) = \sum_{v \in V} f(v)$ . The *co-Roman domination number* of a graph G, denoted by  $\gamma_{cr}(G)$ , is the minimum weight of a co-Roman dominating function on G. In this paper, we give a characterization of graphs of order n for which co-Roman domination number is  $\frac{2n}{3}$  or n - 2, which settles two open problem in [S. Arumugam, K. Ebadi and M. Manrique, *Co-Roman domination in graphs*, Proc. Indian Acad. Sci. Math. Sci. 125 (2015) 1–10]. Furthermore, we present some sharp bounds on the co-Roman domination number.

**Keywords:** co-Roman dominating function, co-Roman domination number, Roman domination.

2010 Mathematics Subject Classification: 05C69.

For terminology and notation on graph theory not given here, the reader is referred to [9, 10]. In this paper, G is a simple graph with vertex set V = V(G)and edge set E = E(G). The order |V| of G is denoted by n = n(G). For every vertex  $v \in V$ , the open neighborhood N(v) is the set  $\{u \in V \mid uv \in E\}$ and the closed neighborhood of v is the set  $N[v] = N(v) \cup \{v\}$ . The degree of a vertex  $v \in V$  is  $\deg_G(v) = \deg(v) = |N(v)|$ . A universal vertex is a vertex that is adjacent to all other vertices of G. The open neighborhood of a set  $S \subseteq V$  is the set  $N(S) = \bigcup_{v \in S} N(v)$ , and the closed neighborhood of S is the set  $N[S] = N(S) \cup S$ . For a set  $S \subseteq V(G)$  and a vertex  $v \in S$ , the private neighborhood of v with respect to S is the set  $pn(v; S) = \{u | u \in N(v), N(u) \cap S = \{v\}$ . A leaf is a vertex of degree one, and a *support vertex* is a vertex adjacent to a leaf. We also denote by  $L_v$  the set of all leaves adjacent to a support vertex v. For a vertex v in a rooted tree T, let D(v) denote the set of descendants of v and  $D[v] = D(v) \cup \{v\}$ . The maximal subtree at v is the subtree of T induced by D[v], and is denoted by  $T_v$ . A subdivision of an edge uv is obtained by replacing the edge uv with a path uwv, where w is a new vertex. The subdivision graph S(G) is the graph obtained from G by subdividing each edge of G. The subdivision star  $S(K_{1,t})$  for  $t \ge 2$ , is called a healthy spider. We write  $P_n$  for a path of length n-1 and  $K_{1,n}$  for a star. For integers  $r \geq s \geq 1$ , the double star DS(r, s) is the tree obtained by connecting the centers of two stars  $K_{1,r}$  and  $K_{1,s}$  with an edge. The *diameter* of G, denoted by  $\operatorname{diam}(G)$ , is the maximum value among minimum distances between all pairs of vertices of G. For a subset S of vertices of G, we denote by G[S] the subgraph induced by S. For a subset  $S \subseteq V(G)$  of vertices of a graph G and a function  $f: V(G) \to R$ , we define  $f(S) = \sum_{x \in S} f(x)$ . For a function  $f: V(G) \to \{0, 1, 2\}$ , let  $V_i = \{v \in V \mid f(v) = i\}$  for i = 0, 1, 2. Since these three sets determine f, we can equivalently write  $f = (V_0, V_1, V_2)$  (or  $f = (V_0^f, V_1^f, V_2^f)$  to refer f). We note that  $\omega(f) = |V_1| + 2|V_2|$ .

A Roman dominating function on a graph G, abbreviated RD-function, is a function  $f: V(G) \longrightarrow \{0, 1, 2\}$  satisfying the condition that every vertex u for which f(u) = 0 is adjacent to at least one vertex v for which f(v) = 2. The weight,  $\omega(f)$ , of f is defined as  $\omega(f) = \sum_{v \in V} f(v)$ . The Roman domination number, denoted by  $\gamma_R(G)$ . An RD-function with minimum weight  $\gamma_R(G)$  in Gis called a  $\gamma_R(G)$ -function. The definition of the Roman dominating function was given multiplicity by Steward [14] and ReVelle and Rosing [13]. Roman domination is now well studied in graph theory [1,3-7,15].

Let  $f: V(G) \to \{0, 1, 2\}$  be a function. A vertex v is said to be protected with respect to f, if f(v) > 0 or f(v) = 0 and v is adjacent to a vertex of positive weight. The function f is a weak Roman dominating function if for every vertex u with f(u) = 0 there exists a vertex v adjacent to u such that  $f(v) \in \{1, 2\}$ and the function  $f_{uv}: V \to \{0, 1, 2\}$ , defined by  $f_{uv}(u) = 1$ ,  $f_{uv}(v) = f(v) - 1$ and  $f_{uv}(x) = f(x)$  for  $x \in V \setminus \{v, u\}$ , has no unprotected vertex. The weak Roman domination number of a graph G, denoted by  $\gamma_r(G)$ , is the minimum weight among all weak Roman dominating functions on G. The weak Roman domination number was introduced by Henning and Hedetniemi in [11].

The function  $f: V(G) \to \{0, 1, 2\}$  is a co-Roman dominating function, abbreviated CRDF if (i) every vertex in V is protected, and (ii) each  $v \in V$  with positive weight has a neighbor  $u \in V$  with f(u) = 0 such that the function  $f_{uv}: V \to \{0, 1, 2\}$ , defined by  $f_{uv}(u) = 1$ ,  $f_{uv}(v) = f(v) - 1$  and  $f_{uv}(x) = f(x)$  for  $x \in V \setminus \{v, u\}$ , has no unprotected vertex. The weight of f is  $\omega(f) = \sum_{v \in V} f(v)$ . The co-Roman domination number of a graph G, denoted by  $\gamma_{cr}(G)$ , is the minimum weight of a co-Roman dominating function on G. It follows from the definitions that for any connected graph G of order  $n \geq 2$ ,

(1) 
$$\gamma_{cr}(G) \le n-1.$$

The co-Roman domination in graphs was investigated by Arumugam *et al.* in [2]. The proof of the next results can be found in [2].

**Proposition 1.** If H is a spanning subgraph of a graph H, then  $\gamma_{cr}(G) \leq \gamma_{cr}(H)$ .

**Proposition 2.** For  $n \ge 2$ ,  $\gamma_{cr}(K_{1,n}) = 2$ .

**Proposition 3.** For  $n \ge 4$ ,  $\gamma_{cr}(P_n) = \gamma_{cr}(C_n) = \left\lceil \frac{2n}{5} \right\rceil$ .

**Proposition 4.** For every tree T of order  $n \ge 2$ ,  $\gamma_{cr}(T) \le \frac{2n}{3}$ .

The next result is an immediate consequence of Propositions 1 and 4.

**Corollary 5.** For every connected graph G of order  $n \ge 2$ ,  $\gamma_{cr}(G) \le \frac{2n}{3}$ .

**Observation 6.** Let G be a graph of order  $n \ge 2$ . Then  $\gamma_{cr}(G) = 1$  if and only if G has two vertices of degree n - 1.

**Theorem 7.** For every graph G,  $\gamma_{cr}(G) \leq \gamma_r(G)$ .

In [2], the authors posed the following open problems.

**Problem 1.** Characterize graphs G of order n such that  $\gamma_{cr}(G) = n - 2$ .

**Problem 2.** Characterize trees T of order n such that  $\gamma_{cr}(T) = \frac{2n(T)}{3}$ .

**Problem 3.** Characterize graphs G such that  $\gamma_{cr}(T) = \gamma(G)$ .

In this paper, we settle the above open problems. Furthermore, we establish some sharp bounds on the co-Roman domination number.

1. Graphs G with  $\gamma_{cr}(G) = \gamma_r(G)$  or  $\gamma_{cr}(G) = \gamma(G)$ 

In this section, we study the properties of graphs G for which  $\gamma_{cr}(G) = \gamma_r(G)$  or  $\gamma_{cr}(G) = \gamma(G)$ .

**Proposition 8.** Let G be a connected graph of order at least two. Then  $\gamma_{cr}(G) = \gamma_r(G)$  if and only if there exists a  $\gamma_{cr}(G)$ -function  $f = (V_0, V_1, V_2)$  such that each vertex  $x \in V_0$ , either has a neighbor x' in  $V_2$  or has a neighbor x' in  $V_1$  for which  $pn(x', V_1 \cup V_2) \subseteq N[x]$ .

**Proof.** If there exists a  $\gamma_{cr}(G)$ -function  $f = (V_0, V_1, V_2)$  such that each vertex  $x \in V_0$ , has a neighbor  $x' \in V_1 \cup V_2$  with  $pn(x', V_1 \cup V_2) \subseteq N[x]$ , then clearly f is a weak Roman dominating function of G and so  $\gamma_r(G) \leq \gamma_{cr}(G)$ . It follows from Theorem 7 that  $\gamma_r(G) = \gamma_{cr}(G)$ .

Conversely, let  $\gamma_r(G) = \gamma_{cr}(G)$ . There exists a  $\gamma_r(G)$ -function  $f = (V_0, V_1, V_2)$ such that f is a co-Roman dominating function of G (see Theorem 3.3 of [2]). By assumption, f is a  $\gamma_{cr}(G)$ -function. Assume  $x \in V_0$  is an arbitrary vertex. Since f is a weak Roman dominating function, x has a neighbor x' in  $V_1 \cup V_2$ such that the function  $g: V(G) \to \{0, 1, 2\}$  defined by g(x) = 1, g(x') = f(x') - 1and g(u) = f(u) otherwise, is safe. If x has a neighbor in  $V_2$ , then we are done. Assume x has no neighbor in  $V_2$ . It follows that  $x' \in V_1$ . Since f is safe, we must have  $pn(x', V_1 \cup V_2) \subseteq N[x]$  and the proof is complete.

**Proposition 9.** Let G be a connected graph of order at least two. Then  $\gamma(G) = \gamma_{cr}(G)$  if and only if there exists a  $\gamma(G)$ -set S such that each vertex  $x \in S$  has a neighbor  $x' \in V \setminus S$  with  $pn(x, S) \subseteq N[x']$ .

**Proof.** Let  $\gamma(G) = \gamma_{cr}(G)$ . Assume  $f = (V_0, V_1, V_2)$  is a  $\gamma_{cr}(G)$ -function. Since  $V_1 \cup V_2$  is a dominating set, we deduce from  $\gamma(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_{cr}(G)$  that  $V_2 = \emptyset$  and  $V_1$  is a  $\gamma(G)$ -set. Let  $x \in V_1$  be an arbitrary vertex. Since f is a co-Roman dominating function, there is a vertex  $x' \in V_0 \cap N(x)$  such that  $((V_0 \setminus \{x'\}) \cup \{x\}, (V_1 \setminus \{x\}) \cup \{x'\}, \emptyset)$  is a  $\gamma_{cr}(G)$ -function. It follows that  $(V_1 \setminus \{x\}) \cup \{x'\}$  is a  $\gamma(G)$ -set and this implies that  $pn(x, V_1) \subseteq N[x']$ .

Conversely, let S be a  $\gamma(G)$ -set such that each vertex  $x \in S$  has a neighbor  $x' \in V \setminus S$  with  $pn(x, S) \subseteq N[x']$ . Then the function  $f = (V(G) \setminus S, S, \emptyset)$  is clearly a co-Roman dominating function of weight  $\gamma(G)$  and so  $\gamma_{cr}(G) \leq \gamma(G)$ . It follows that  $\gamma_{cr}(G) = \gamma(G)$ .

**Corollary 10.** Let G be a connected graph of order at least two with  $\gamma(G) = \gamma_{cr}(G)$ . Then for any  $\gamma_{cr}(G)$ -function  $f = (V_0, V_1, V_2), V_2 = \emptyset$ .

**Corollary 11.** Let G be a connected graph of order at least two. If  $\gamma(G) = \gamma_{cr}(G)$ , then G has no strong support vertex.

For a tree T, let  $M(T) = \{v \mid \text{ there exists a } \gamma_{cr}(T)\text{-function } f \text{ such that } f(v) = 1\}$ . In what follows, we present a constructive characterization of trees T with  $\gamma(T) = \gamma_{cr}(T)$ . In order to do this, we define a family of trees as follows. Let  $\mathcal{T}$  be the collection of trees T that can be obtained from a sequence  $T_1, T_2, \ldots, T_k$  of trees for some  $k \geq 1$ , where  $T_1$  is a  $P_2$  and  $T = T_k$ . If  $k \geq 2$ ,  $T_{i+1}$  can be obtained from  $T_i$  by one of the following three operations. Let one vertex of  $P_2$  be considered as a support vertex.

**Operation**  $\mathcal{O}_1$ . If  $v \in M(T_i)$ , then the tree  $T_{i+1}$  is obtained from  $T_i$  by adding a pendant  $P_3 = xyz$  and adding the edge vx (see Figure 1(a)).

**Operation**  $\mathcal{O}_2$ . If v is a support vertex of  $T_i$ , then the tree  $T_{i+1}$  is obtained from  $T_i$  by adding a pendant  $P_2 = xy$  and adding the edge vx (see Figure 1(b)).

**Operation**  $\mathcal{O}_3$ . If  $v \in T_i$ , then the tree  $T_{i+1}$  is obtained from  $T_i$  by adding a healthy spider with at least two feet headed at x and adding the edge vx (see Figure 1(c)).



Figure 1. (a) Operation  $\mathcal{O}_1$ . (b) Operation  $\mathcal{O}_2$ . (c) Operation  $\mathcal{O}_3$ .

**Lemma 12.** If  $T_i$  is a tree with  $\gamma(T_i) = \gamma_{cr}(T_i)$  and  $T_{i+1}$  is a tree obtained from  $T_i$  by Operation  $\mathcal{O}_1$ , then  $\gamma(T_{i+1}) = \gamma_{cr}(T_{i+1})$ .

**Proof.** Let f be a  $\gamma_{cr}(T_i)$ -function and v a vertex of  $T_i$  with f(v) = 1. Then the function  $f' : V(T_{i+1}) \to \{0, 1, 2\}$  by f'(y) = 1, f'(x) = f'(z) = 0 and f'(u) = f(u) for  $u \in V(T_i)$ , is a co-Roman dominating function on  $T_{i+1}$  and so  $\gamma_{cr}(T_{i+1}) \leq \gamma_{cr}(T_i) + 1$ .

It is easy to see that  $\gamma(T_{i+1}) = \gamma(T_i) + 1$ . Now we have

$$\gamma(T_i) + 1 = \gamma(T_{i+1}) \le \gamma_{cr}(T_{i+1}) \le \gamma_{cr}(T_i) + 1 = \gamma(T_i) + 1$$

yielding  $\gamma(T_{i+1}) = \gamma_{cr}(T_{i+1})$ .

**Lemma 13.** If  $T_i$  is a tree with  $\gamma(T_i) = \gamma_{cr}(T_i)$  and  $T_{i+1}$  is a tree obtained from  $T_i$  by Operation  $\mathcal{O}_2$ , then  $\gamma(T_{i+1}) = \gamma_{cr}(T_{i+1})$ .

**Proof.** Clearly, any  $\gamma_{cr}(T_i)$ -function can be extended to a co-Roman dominating function by assigning 1 to x and 0 to y implying that  $\gamma_{cr}(T_{i+1}) \leq \gamma_{cr}(T_i) + 1$ .

Since v is a support vertex, one can easily check that  $\gamma(T_{i+1}) = \gamma(T_i) + 1$ . Now the result follows as in the proof of Lemma 12.

**Lemma 14.** If  $T_i$  is a tree with  $\gamma(T_i) = \gamma_{cr}(T_i)$  and  $T_{i+1}$  is a tree obtained from  $T_i$  by Operation  $\mathcal{O}_3$ , then  $\gamma(T_{i+1}) = \gamma_{cr}(T_{i+1})$ .

**Proof.** Let the added spider has exactly k feet. Obviously, any  $\gamma_{cr}(T_i)$ -function can be extended to a co-Roman dominating function by assigning 1 to the support vertices of spider and 0 to the remaining vertices of spider and this implies that  $\gamma_{cr}(T_{i+1}) \leq \gamma_{cr}(T_i) + k$ . Moreover, it is easy to verify that  $\gamma(T_{i+1}) = \gamma(T_i) + k$ and the result follows as in the proof of Lemma 12.

**Lemma 15.** If  $T \in \mathcal{T}$ , then  $\gamma(T) = \gamma_{cr}(T)$ .

**Proof.** Let  $T \in \mathcal{T}$ . By definition, there exists a sequence of trees  $T_1, T_2, \ldots, T_k$  $(k \geq 1)$  such that  $T_1 = K_2$ , and if  $k \geq 2$ ,  $T_{i+1}$  can be obtained recursively from  $T_i$ by Operation  $\mathcal{O}_1, \mathcal{O}_2$  or  $\mathcal{O}_3$  for  $i = 1, 2, \ldots, k-1$ . We proceed by induction on k. If  $T = K_2$ , then clearly  $\gamma(T) = \gamma_{cr}(T) = 1$ . Suppose  $k \geq 2$  and the result holds for each tree  $T \in \mathcal{T}$  which can be obtained from a sequence of operations of length k-1 and let  $T' = T_{k-1}$ . By the induction hypothesis, we have  $\gamma(T') = \gamma_{cr}(T')$ . Since  $T = T_k$  is obtained from T' by one of the Operations  $\mathcal{O}_1, \mathcal{O}_2$  or  $\mathcal{O}_3$  from T', we have  $\gamma(T) = \gamma_{cr}(T)$  by Lemmas 12, 13 and 14.

**Theorem 16.** Let T be a tree of order  $n \ge 2$ . Then  $\gamma(T) = \gamma_{cr}(T)$  if and only if  $T \in \mathcal{T}$ .

**Proof.** The sufficiency follows from Lemma 15. We use induction on n to prove the necessity. If n = 2, then  $T = P_2$  that belongs to  $\mathcal{T}$ . Assume  $n \geq 3$  and that the result holds for any tree of order less than n. Let T be a tree of order n with  $\gamma(T) = \gamma_{cr}(T)$ . Let  $P = v_1 v_2 \cdots v_\ell$  be a diametral path in T and root T at  $v_\ell$ . By Corollary 11, we have  $d(v_2) = 2$ . Consider the following cases.

Case 1.  $v_3$  is a support vertex. Let w be a leaf adjacent to  $v_3$  and let  $T' = T - \{v_1, v_2\}$ . If f is a  $\gamma_{cr}(T)$ -function, then clearly  $f(v_1) + f(v_2) \ge 1$  and  $f(v_3) + f(w) \ge 2$ . It is easy to verify that the function f, restricted to T' is a co-Roman dominating function implying that  $\gamma_{cr}(T) \ge \gamma_{cr}(T') + 1$ . Clearly  $\gamma(T) = \gamma(T') + 1$ , and we deduce from

$$\gamma(T) = \gamma_{cr}(T) \ge \gamma_{cr}(T') + 1 = \gamma(T') + 1 = \gamma(T)$$

that  $\gamma_{cr}(T') = \gamma(T')$ . By the induction hypothesis, we have  $T' \in \mathcal{T}$ . Now T can be obtained from T' by Operation  $\mathcal{O}_2$ .

Case 2.  $d(v_3) = 2$ . Let  $T' = T - \{v_1, v_2, v_3\}$ . By Proposition 3,  $n \ge 4$ . Clearly  $\gamma(T) = \gamma(T') + 1$ . Assume  $f = (V_0, V_1, V_2)$  is a  $\gamma_{cr}(T)$ -function. By Corollary 10,  $V_2 = \emptyset$ . Clearly  $f(v_1) + f(v_2) = 1$  and  $f(v_3) + f(v_4) \ge 1$ . If  $f(v_3) = f(v_4) = 1$ , then the function  $g : V(T) \to \{0, 1, 2\}$  defined by  $g(v_4) = g(v_2) = 1$ ,  $g(v_1) = g(v_3) = 0$  and g(x) = f(x) otherwise, is a co-Roman dominating function of T of weight less than  $\omega(f)$  which is a contradiction. Hence  $f(v_3) = 0$  or  $f(v_4) = 0$  and so  $f(v_3) + f(v_4) = 1$ . Consider the following.

•  $f(v_3) = 1$  and  $f(v_4) = 0$ . If f(x) = 1 for some  $x \in N_{T'}(v_4)$ , then the function  $g: V(G) \to \{0,1\}$  defined by  $g(v_2) = 1, g(v_1) = g(v_3) = 0$  and g(x) = f(x) otherwise, is a dominating function of T of weight less than  $\omega(f)$  which contradicts  $\gamma(T) = \gamma_{cr}(T)$ . Thus f(x) = 0 for each  $x \in N_{T'}(v_4)$ . Now the function  $h: V(T') \to \{0,1\}$  defined by  $h(v_4) = 1$  and h(x) = f(x) otherwise, is a co-Roman dominating function of T of weight  $\omega(f) - 1$ . It follows from

$$\gamma(T) = \gamma_{cr}(T) \ge \gamma_{cr}(T') + 1 = \gamma(T') + 1 = \gamma(T)$$

that  $\gamma_{cr}(T') = \gamma(T')$  and that h is a  $\gamma_{cr}(T')$ -function with  $h(v_4) = 1$ . By the induction hypothesis, we have  $T' \in \mathcal{T}$  and so T can be obtained from T' by Operation  $\mathcal{O}_1$ . Thus  $T \in \mathcal{T}$ .

•  $f(v_3) = 0$  and  $f(v_4) = 1$ . As above we have f(x) = 0 for some  $x \in N_{T'}(v_4)$ . Then the function f restricted to T' is a co-Roman dominating function of T'and so  $\gamma_{cr}(T) \ge \gamma_{cr}(T') + 1$ . Using above argument, we obtain  $T \in \mathcal{T}$ .

Case 3.  $v_3$  is not a support vertex and  $d(v_3) \geq 3$ . Let T' be the component of  $T - v_3v_4$  containing  $v_3$ . Then T' is a spider with at least k feet where  $k = \deg(v_3)-1$ . Clearly  $\gamma(T) = \gamma(T')+k$ . Now we show that  $\gamma_{cr}(T) \geq \gamma_{cr}(T')+k$ . Let  $u_1, \ldots, u_k$  be the children of  $v_3$  and  $w_i$  be the leaf adjacent to  $u_i$  for  $i = 1, \ldots, k$ . Let  $f = (V_0, V_1, V_2)$  be a  $\gamma_{cr}(T)$ -function. By Corollary 10,  $V_2 = \emptyset$ . Obviously  $f(u_i) + f(w_i) = 1$  for each i. As Case 2, we can see that  $f(v_4) = 0$  or  $f(v_3) = 0$ . If  $f(v_4) = f(v_3) = 0$ , then the function f restricted to T' is a co-Roman dominating function of weight  $\gamma_{cr}(T) - k$  and so  $\gamma_{cr}(T) \geq \gamma_{cr}(T') + k$ . Consider the following subcases.

Subcase 3.1.  $f(v_3) = 1$  and  $f(v_4) = 0$ . If f(x) = 1 for some  $x \in N_{T'}(v_4)$ , then the function  $g: V(G) \to \{0, 1\}$  defined by  $g(v_3) = g(w_i) = 0$ ,  $g(u_i) = 1$  for  $1 \le i \le k$  and g(x) = f(x) otherwise, is a dominating function of T of weight less than  $\omega(f)$  contradicting  $\gamma(T) = \gamma_{cr}(T)$ . Thus f(x) = 0 for each  $x \in N_{T'}(v_4)$ . Now the function  $h: V(T') \to \{0, 1\}$  defined by  $h(v_4) = 1$  and h(x) = f(x)otherwise, is a co-Roman dominating function of T of weight  $\omega(f) - k$  and hence  $\gamma_{cr}(T) \ge \gamma_{cr}(T') + k$ .

Subcase 3.2.  $f(v_3) = 0$  and  $f(v_4) = 1$ . As above we have f(x) = 0 for some  $x \in N_{T'}(v_4)$ . Then the function f, restricted to T' is a co-Roman dominating function of T' and so  $\gamma_{cr}(T) \ge \gamma_{cr}(T') + k$ .

Thus in all cases  $\gamma_{cr}(T) \geq \gamma_{cr}(T') + k$ . As Case 2, we deduce that  $\gamma_{cr}(T') = \gamma(T')$  and so by the induction hypothesis we have  $T' \in \mathcal{T}$ . Now T can be obtained from T' by Operation  $\mathcal{O}_3$  and hence  $T \in \mathcal{T}$ . This completes the proof.

### 2. Bounds on Co-Roman Domination

In this section, we present some sharp bounds on the co-Roman domination number. First we prove two upper bounds on the co-Roman domination number in terms of matching number.

**Theorem 17.** For any connected graph G of order  $n \ge 2$ ,

$$\gamma_{cr}(G) \le n - \alpha'(G).$$

**Proof.** Let  $M = \{u_1v_1, \ldots, u_{\alpha'}v_{\alpha'}\}$  be a maximum matching of G and let X be the independent set of M-unsaturated vertices. If y and z are vertices of X and  $yu_i \in E(G)$ , then since the matching M is maximum,  $zv_i \notin E(G)$ . Therefore, for all  $i \in \{1, 2, \ldots, \alpha'\}$  there are at most two edges between the sets  $\{u_i, v_i\}$  and  $\{y, z\}$ . Assume S is the set of all vertices in X which belongs to a triangle with an edge in M. Let  $S = \{x_1, \ldots, x_s\}$  if  $S \neq \emptyset$  and  $X \setminus S = \{y_1, \ldots, y_k\}$  if  $X \setminus S \neq \emptyset$ .

First let  $S = \emptyset$ . Then  $vu_i \notin E(G)$  or  $vv_i \notin E(G)$  for each  $v \in X$  and each  $i \in \{1, \ldots, \alpha'\}$ . We may assume  $N(x) \subseteq \{u_1, \ldots, u_{\alpha'}\}$  for each  $x \in X$ . Define  $f: V(G) \to \{0, 1, 2\}$  by  $f(u_i) = 0$  for  $1 \leq i \leq \alpha'$  and f(x) = 1 otherwise. Clearly, f is a co-Roman dominating function of G of weight  $\alpha' + |X|$  and hence

$$\gamma_{cr}(G) \le \alpha'(G) + |X| = \alpha'(G) + (n - 2\alpha'(G)) = n - \alpha'(G).$$

Now let  $S \neq \emptyset$ . We may assume, without loss of generality, that  $x_i u_i, x_i v_i \in E(G)$  for  $i = 1, \ldots, s$ . As above, we can assume that  $N(x) \subseteq \{u_1, \ldots, u_{\alpha'}\}$  for each  $x \in X \setminus S$ . Define  $f: V(G) \to \{0, 1, 2\}$  by f(x) = 0 for  $x \in S \cup \{u_1, \ldots, u_{\alpha'}\}$  and f(x) = 1 otherwise. Obviously, f is a co-Roman dominating function of G of weight  $\alpha'(G) + |X| - |S|$  and hence

(2) 
$$\gamma_{cr}(G) \le \alpha'(G) + |X| - |S| = \alpha'(G) + (n - 2\alpha') - |S| \le n - \alpha'(G) - |S|$$

This completes the proof.

**Theorem 18.** For any connected graph G of order  $n \ge 2$  with  $\delta(G) \ge 2$ ,

$$\gamma_{cr}(G) \le \alpha'(G).$$

**Proof.** Let M, X and S be the sets defined in the proof of Theorem 17. Assume first that  $S = \emptyset$ . Then as above we may assume  $N(x) \subseteq \{u_1, \ldots, u_{\alpha'}\}$  for each  $x \in X$ . Define  $f: V(G) \to \{0, 1, 2\}$  by  $f(u_i) = 1$  for  $1 \leq i \leq \alpha'$  and f(x) = 0otherwise. Since  $\delta(G) \geq 2$ , the function  $f_i: V(G) \to \{0, 1, 2\}$  defined by  $f(u_i) =$  $0, f(v_i) = 1$  and  $f_i(x) = f(x)$  otherwise, is safe for each *i*. Thus *f* is a co-Roman dominating function of *G* of weight  $\alpha'(G)$  and so  $\gamma_{cr}(G) \leq \alpha'(G)$ .

Now let  $S = \{x_1, \ldots, x_s\}$ . We may assume, without loss of generality, that  $x_i u_i, x_i v_i \in E(G)$  for  $i = 1, \ldots, s$ . As above, we can assume that  $N(x) \subseteq \{u_1, \ldots, u_{\alpha'}\}$  for each  $x \in X \setminus S$ . It is easy to see that the function f defined above is a co-Roman dominating function of G. Thus  $\gamma_{cr}(G) \leq \alpha'(G)$  and the proof is complete.

**Theorem 19.** For any connected graph G of order  $n \ge 2$ ,

$$\gamma_{cr}(G) \le 2\alpha'(G).$$

**Proof.** Let M, X and S be the sets defined in the proof of Theorem 17. As Theorem 17, we may assume that  $x_i u_i, x_i v_i \in E(G)$  for  $i = 1, \ldots, s$  if  $S \neq \emptyset$  and  $N(x) \subseteq \{u_1, \ldots, u_{\alpha'}\}$  for each  $x \in X \setminus S$ . Then the function  $f: V(G) \to \{0, 1, 2\}$ defined by  $f(u_i) = 1$  if  $u_i$  is adjacent to a vertex in S,  $f(u_i) = 2$  if  $u_i$  is adjacent to a vertex in  $X \setminus S$  and f(x) = 0 otherwise, is a co-Roman dominating function of G and so  $\gamma_{cr}(G) \leq |S| + 2|X - S| = 2\alpha'(G) - |S| \leq 2\alpha'(G)$ .

A set  $X \subseteq V(G)$  is called a 2-packing if d(u, v) > 2 for any different vertices u and v of X. The 2-packing number  $\rho(G)$  is the maximum cardinality of a 2-packing of G.

**Theorem 20.** For any connected graph G of order  $n \ge 2$  with  $\delta(G) \ge 2$ ,

$$\gamma_{cr}(G) \le n - \rho(G)(\delta(G) - 1).$$

**Proof.** Let S be a 2-packing of G of size  $\rho(G)$ . Define  $f: V(G) \to \{0, 1, 2\}$  by f(x) = 2 for  $x \in S$ , f(x) = 0 for  $x \in \bigcup_{v \in S} N(v)$  and f(x) = 1 otherwise. Clearly, f is a co-Roman dominating function of G and hence

$$\gamma_{cr}(G) \leq (n - |\bigcup_{v \in S} N[v]|) + 2|S| = n - \sum_{v \in S} |N[v]| + 2\rho(G)$$
  
$$\leq n - \rho(G)(\delta(G) + 1) - 2\rho(G) = n - \rho(G)(\delta(G) - 1),$$

as desired.

**Proposition 21.** Let G be a simple connected graph of order n with  $\delta(G) \ge 2$ and  $g(G) \ge 5$ . Then  $\gamma_{cr}(G) \le \frac{2(n-g(G))}{3} + \left\lceil \frac{2g(G)}{5} \right\rceil$ .

**Proof.** If G is an n-cycle, then the result follows by Proposition 3. Assume G is not a cycle and C is a cycle of length g(G) in G. Let G' be the graph obtained from G by removing the vertices of V(C). Since  $g(G) \ge 5$ , each vertex of G' can be adjacent to at most one vertex of C which implies  $\delta(G') \ge 1$ . By Corollary 5, we have  $\gamma_{cr}(G') \le \frac{2(n-g(G))}{3}$ . Let g be a  $\gamma_{cr}(G')$ -function and h be a  $\gamma_{cr}(C)$ -function. Define  $f: V(G) \to \{0, 1, 2\}$  by f(v) = g(v) for  $v \in V(G')$  and f(v) = h(v) for  $v \in V(C)$ . Obviously, f is a co-Roman dominating function and so

$$\gamma_{cr}(G) \le \frac{2(n-g(G))}{3} + \left\lceil \frac{2g(G)}{5} \right\rceil.$$

3. CHARACTERIZATION OF GRAPHS G OF ORDER n WITH  $\gamma_{cr}(G) = \frac{2n}{3}$ 

In this section, we characterize the graphs attaining the upper bound in Corollary 5. For any arbitrary tree T, let  $T_{cr}$  be the tree obtained from T by adding exactly two pendant edges at each vertex of T. Note that  $n(T_{cr}) = 3n(T)$ . Let  $\mathcal{F}$  be the family of all trees  $T_{cr}$ . In fact,  $\mathcal{F}$  is the family of trees T such that V(T) can be partitioned into sets inducing  $P_3$  such that the subgraph induced by the central vertices of these paths is connected.

**Lemma 22.** If  $T \in \mathcal{F}$ , then  $\gamma_{cr}(T) = \frac{2n(T)}{3}$ .

**Proof.** Let  $T \in \mathcal{F}$  and let f be a  $\gamma_{cr}$ -function on T. Then T is obtained from a tree T' by adding exactly two pendant edges at each vertex of T'. For each non-leaf vertex  $v \in V(T)$ , let  $L_v = \{v_1, v_2\}$ . It is easy to see that for any non-leaf vertex  $v \in V(T)$ ,  $f(v) + f(v_1) + f(v_2) \geq 2$ , otherwise we have an unprotected vertex in either f or  $f_{vv_i}$  for some i = 1, 2. Hence,  $\gamma_{cr}(T) = \omega(f) = \sum_{v \in V(T')} (f(v) + f(v_1) + f(v_2)) \geq 2n(T') = \frac{2n(T)}{3}$ . Now the result follows from Proposition 4.

**Lemma 23.** Let  $q \ge p \ge 1$  and let T = DS(p,q). Then  $\gamma_{cr}(T) = \frac{2n(T)}{3}$  if and only if q = p = 2.

**Proof.** If q = p = 2, then Lemma 22 implies  $\gamma_{cr}(T) = \frac{2n(T)}{3}$ . Conversely, let  $\gamma_{cr}(T) = \frac{2n(T)}{3}$ . It follows from Proposition 3 that  $q \ge 2$ . If p = 1, then clearly  $\gamma_{cr}(T) = 3 < \frac{2n(T)}{3}$ , a contradiction. Suppose that  $p \ge 2$ . If q > 2, then we have  $\gamma_{cr}(T) \le 4 < \frac{2n(T)}{3}$ , a contradiction again. Thus q = p = 2 and the proof is complete.

**Theorem 24.** Let T be a tree of order  $n \ge 3$ . Then  $\gamma_{cr}(T) = \frac{2n}{3}$  if and only if  $T \in \mathcal{F}$ .

**Proof.** According to Lemma 22, we only need to prove the necessity. Let T be a tree of order  $n \geq 3$  with  $\gamma_{cr}(T) = \frac{2n}{3}$ . Note that n is a multiple of 3. The proof is by induction on n. If n = 3, then the only tree T of order 3 and  $\gamma_{cr}(T) = 2$  is  $P_3 \in \mathcal{F}$ . Let  $n \geq 4$  and let the statement hold for all trees of order less than n. Suppose that T is a tree of order n with  $\gamma_{cr}(T) = \frac{2n}{3}$ . If diam(T) = 2, then  $T = K_{1,s}$  and we deduce from Proposition 2 that  $T = P_3$  and so  $T \in \mathcal{F}$ . If diam(T) = 3, then we deduce from Lemma 23 that T = DS(2, 2) and so  $T \in \mathcal{F}$ . Henceforth we assume that diam $(T) \geq 4$ . Let  $v_1v_2\cdots v_k$   $(k \geq 5)$  be a diametral path in T and root T at  $v_k$ . We show that  $\deg_T(v_2) = 3$ . Let  $T' = T - T_{v_2}$  and f be a  $\gamma_{cr}(T')$ -function. If  $\deg_T(v_2) \geq 4$ , then the function  $g: V(T) \to \{0, 1, 2\}$  defined by  $g(v_2) = 2$ , g(x) = 0 if  $x \in L_{v_2}$  and g(x) = f(x) for  $x \in T'$ , is a CRDF on T of weight  $\omega(f) + 2$ . By Proposition 4, we have  $\gamma_{cr}(T) \leq \omega(g) \leq \gamma_{cr}(T') + 2 \leq \frac{2n(T')}{3} + 2 \leq \frac{2(n-4)}{3} + 2 < \frac{2n}{3}$ , which is a contradiction. If  $\deg_T(v_2) = 2$ , then the function  $g: V(T) \to \{0, 1, 2\}$  defined by  $g(v_2) = 3$ . Assume that  $T = T - T_{v_2}$ . As above, we have  $\gamma_{cr}(T) \leq \omega(g) \leq \gamma_{cr}(T') + 1 \leq \frac{2(n-2)}{3} + 1 < \frac{2n}{3}$ , a contradiction again. Thus  $\deg(v_2) = 3$ . Assume that  $T' = T - T_{v_2}$ .

$$\frac{2n(T)}{3} = \gamma_{cr}(T) \le \gamma_{cr}(T') + 2 \le \frac{2n(T')}{3} + 2 = \frac{2(n-3)}{3} + 2 = \frac{2n}{3}.$$

Thus all inequalities in the above inequality chain must be equalities and so  $\gamma_{cr}(T') = \frac{2n(T')}{3}$ . By the induction hypothesis we have  $T' \in \mathcal{F}$ . Now we show that  $v_3$  is not a leaf of T'. If  $v_3$  is a leaf in T', then let  $T'' = T - T_{v_3}$  and let h be a  $\gamma_{cr}(T'')$ -function. Define the function  $g: V(T) \to \{0, 1, 2\}$  by  $g(v_2) = 2$ , g(v) = 0 if  $v \in N_T(v_2)$  and g(x) = h(x) for  $x \in T''$ . Clearly, g is a CRDF on T of weight  $\omega(f)+2$ . By Proposition 4, we have  $\gamma_{cr}(T) \leq \omega(g) = \gamma_{cr}(T')+2 \leq \frac{2(n-4)}{3}+2 < \frac{2n}{3}$ , a contradiction. Thus  $v_3$  is a non-leaf vertex of T' and so  $T \in \mathcal{F}$ . This completes the proof.

**Theorem 25.** Let G be a connected n-vertex graph with  $n \ge 3$ . Then  $\gamma_{cr}(G) = \frac{2n}{3}$  if and only if G is obtained from  $\frac{n}{3}P_3$  by adding edges between the centers of the paths  $P_3$  such that the resulting graph is connected.

**Proof.** If G has the specified form, then clearly every CRDF puts weight at least 2 on the vertex set of each copy of  $P_3$ .

Now suppose that  $\gamma_{cr}(G) = \frac{2n}{3}$ . Since adding edges cannot increase  $\gamma_{cr}(G)$ , every spanning tree of G belongs to  $\mathcal{F}$ . Given a spanning tree T, let  $S_1, S_2, \ldots, S_{\frac{n}{3}}$  be the 3-sets in the special partition of V(T). The assignment of weight 2 that guards  $S_i$  can be chosen independently of any other  $S_j$ . If any edge of G joins vertices of  $S_i$  and  $S_j$  that are not the centers of the paths they induce, then a CRDF with weight less than  $\frac{2n}{3}$  can be built as in the proof of Theorem 24. This completes the proof.

## 4. GRAPHS WITH LARGE CO-ROMAN DOMINATION NUMBER

In this section, we characterize all graphs of order n with co-Roman domination number n-2 and n-3. The first result is an immediate consequence of Theorem 17.

**Corollary 26** (Theorem 4.2 in [2]). Let G be a connected graph on  $n \ge 2$  vertices. Then  $\gamma_{cr}(G) = n - 1$  if and only if  $G = K_2$  or  $K_{1,2}$ .

Arumugam et al. [2] posed the following problem.

**Problem.** Characterize graphs G such that  $\gamma_{cr}(G) = n - 2$ . Next we solve this problem.

**Theorem 27.** Let G be a connected graph on  $n \ge 2$  vertices. Then  $\gamma_{cr}(G) = n-2$  if and only if G is a graph on four vertices different from  $K_4$  and  $K_4 - e$ , or  $G \cong DS(2,1)$ , or  $G \cong DS(2,2)$ .

**Proof.** By Theorem 17, we have  $\alpha'(G) \leq 2$ . If  $\alpha'(G) = 1$ , then G is the star  $K_{1,n-1}$  and we conclude from Proposition 2 that  $G = K_{1,3}$ . Assume that  $\alpha'(G) = 2$ . Let M, X and S be the sets defined in the proof of Theorem 17. By (2), we have  $S = \emptyset$ . As above, we may assume  $N(x) \subseteq \{u_1, \ldots, u_{\alpha'}\}$  for each  $x \in X$ . If  $u_i$  has at least two neighbors in X for some i, say i = 1, then the function  $f: V(G) \to \{0, 1, 2\}$  defined by  $f(u_1) = 2, f(u_i) = 0$  for  $2 \leq i \leq \alpha', f(x) = 0$  if  $x = v_1$  or  $x \in N(u_1) \cap X$  and f(x) = 1 otherwise, is clearly a co-Roman dominating function of G of weight  $n - \alpha'(G) - 1$  which leads to a contradiction. Hence each  $u_i$  has at most one neighbor in X and this implies that  $|X| \leq 2$ . If |X| = 0, then n = 4 and obviously G is a connected graph on four vertices different from  $K_4$  and  $K_4 - e$ . Hence  $|X| \geq 1$ .

First let |X| = 2. Since X is independent and G is connected, we may assume that  $u_i y_i \in E(G)$  for i = 1, 2. Since each  $u_i$  has at most one neighbor in X, we deduce that  $\deg(y_i) = 1$  for i = 1, 2. Considering the matching  $M' = \{u_1 y_1, u_2 y_2\}$ instead of M, we have  $\deg(v_1) = \deg(v_2) = 1$ . Since G is connected, we have  $u_1 u_2 \in E(G)$  and hence G = DS(2, 2).

Now let |X| = 1. Since G is connected, we suppose that  $u_1y_1 \in E(G)$ . If  $u_2y_1 \in E(G)$ , then the function  $f_1 : V(G) \to \{0, 1, 2\}$  defined by  $f_1(u_1) = f_1(u_2) = 1$  and  $f_1(x) = 0$  otherwise, is clearly a co-Roman dominating function of G of weight 2, a contradiction. Thus  $\deg(y_1) = 1$ . Considering the matching  $M' = \{u_1y_1, u_2v_2\}$  instead of M, we obtain  $\deg(v_1) = 1$ . Since G is connected, we may assume that  $u_1u_2 \in E(G)$ . If  $u_1v_2 \in E(G)$ , then clearly  $\gamma_{cr}(G) \leq 2$  which is a contradiction. Thus G = DS(1, 2) and the proof is complete.

The corona graph cor(H) of a graph H is the graph obtained from H by attaching a leaf to every vertex of H. We recall the following result established by Payan and Xuong [12] (see also Fink *et al.* [8]).

**Theorem 28.** For a graph G with even order n and with no isolated vertices,  $\gamma(G) = \frac{n}{2}$  if and only if the components of G are the cycle  $C_4$  or the corona cor(H) for any connected graph H.

Now we characterize all connected graphs G of order  $n \ge 4$  with  $\gamma_{cr}(G) = n-3$ . To do this, we introduced some families of graphs.



Figure 2. The graphs G of order 7 with  $\gamma_{cr}(G) = 4$ .

Let

- $\mathcal{G}_1 = \{K_4, K_4 e, K_{1,4}\},\$
- $\mathcal{G}_2$  be the family of connected graphs G obtained from a triangle and a path  $P_2$  by adding some edges between them so that the resulting graph has at most one universal vertex,

- $\mathcal{G}_3$  be the family of connected graphs G obtained from a path  $P_3$  and a path  $P_2$  by adding some edges between them such that the resulting graph is different from DS(1,2) and has at most one universal vertex,
- $\mathcal{G}_4$  be the family of connected graphs  $G \not\cong DS(2,2)$  of order 6 consisting of  $cor(P_3), cor(C_3)$  and all graphs G with  $\Delta(G) \leq 4$ , for which every  $\gamma(G)$ -set S has a vertex x such that x has no neighbor  $x' \in V \setminus S$  with  $pn(x,S) \subseteq N[x']$ .
- $\mathcal{G}_5 = \{G_1, G_2, \dots, G_{13}\},\$
- $\mathcal{G}_6$  be the family of connected graphs G obtained from three paths  $v_1u_1y_1$ ,  $v_2u_2y_2$  and  $v_3u_3$  by adding edges between  $u_1, u_2, u_3$  such that the resulting graph is connected,
- $\mathcal{G}_7$  be the family of connected graphs G obtained from  $3P_3$  by adding edges between the centers of the paths  $P_3$  such that the resulting graph is connected.



Figure 3. Two graphs G of order 6 with  $\gamma_{cr}(G) = 3$ .

**Theorem 29.** Let G be a connected graph on  $n \ge 4$  vertices, then  $\gamma_{cr}(G) = n-3$  if and only if  $G \in \bigcup_{i=1}^{7} \mathcal{G}_i$ .

**Proof.** Let  $G \in \bigcup_{i=1}^{7} \mathcal{G}_i$ . We deduce from (1), Corollary 26 and Theorem 27 that  $\gamma_{cr}(G) \leq n-3$ . If  $G = K_{1,4}$ , then  $\gamma_{cr}(G) = 2 = n-3$  by Proposition 2, and if  $G \in \mathcal{G}_1 \setminus \{K_{1,4}\}$  then  $\gamma_{cr}(G) = 1 = n-3$  by Observation 6. If  $G \in \mathcal{G}_2 \cup \mathcal{G}_3$ , then we conclude from Observation 6 that  $\gamma_{cr}(G) \geq 2 = n-3$  and so  $\gamma_{cr}(G) = n-3$ . If  $G \in \{cor(P_3), cor(C_3)\}$ , then by Proposition 9 and Theorem 28 we have  $\gamma_{cr}(G) = \gamma(G) = 3$ , and if  $G \in \mathcal{G}_4 - \{cor(P_3), cor(C_3)\}$ , then clearly  $\gamma(G) = 2$  and Proposition 9 implies that  $\gamma_{cr}(G) \geq \gamma(G) + 1 = 3 = n-3$  and so  $\gamma_{cr}(G) = n-3$ . If  $G \in \mathcal{G}_5 \cup \mathcal{G}_6$ , then it is easy to see that  $\gamma_{cr}(G) = n-3$ . Finally, if  $G \in \mathcal{G}_7$ , then by Theorem 25, we have  $\gamma_{cr}(G) = 6 = n-3$ .

Conversely, let  $\gamma_{cr}(G) = n - 3$ . By Corollary 5 and Theorem 17, we obtain  $n \leq 9$  and  $\alpha'(G) \leq 3$ . If  $\alpha'(G) = 1$ , then G is the star  $K_{1,n-1}$  and we conclude from Proposition 2 that  $G = K_{1,4} \in \mathcal{G}_1$ . Assume that  $\alpha'(G) \geq 2$ . Suppose M, X and S are the sets defined in the proof of Theorem 17. We consider the following cases.

Case 1.  $\alpha'(G) = 3$ . Since  $n \leq 9$ , we must have  $|X| \leq 3$ . If |X| = 3, then n = 9 and we conclude from Theorem 25 that  $G \in \mathcal{G}_7$ . Let  $|X| \leq 2$ . By (2), we have  $S = \emptyset$ . As above, we may assume  $N(x) \subseteq \{u_1, u_2, u_3\}$  for each  $x \in X$ . Consider the following subcases.

Subcase 1.1. |X| = 2. If  $u_i y_1, u_i y_2 \in E(G)$  for some i, say i = 1, then the function  $f_1: V(G) \to \{0, 1, 2\}$  defined by  $f_1(u_1) = 2, f_1(u_2) = f_1(u_3) = 1$ and  $f_1(x) = 0$  otherwise, is clearly a co-Roman dominating function of G of weight 4 which is a contradiction. Thus each  $u_i$  has at most one neighbor in X. Assume without loss of generality that  $u_1y_1, u_2y_2 \in E(G)$ . If  $y_1u_3 \in E(G)$  (the case  $y_2u_3 \in E(G)$  is similar), then the function  $f_2: V(G) \to \{0, 1, 2\}$  defined by  $f_2(u_1) = f_2(u_3) = 1, f_2(u_2) = 2$  and  $f_2(x) = 0$  otherwise, is clearly a co-Roman dominating function of G of weight 4 which is a contradiction again. Hence  $y_1u_3, y_2u_3 \notin E(G)$ . It follows that  $\deg(y_1) = \deg(y_2) = 1$ . Considering the matching  $M' = \{u_1y_1, u_2y_2, u_3v_3\}$  instead of M, we obtain  $\deg(v_1) = \deg(v_2) = 1$ . Since G is connected, we may assume, without loss of generality, that  $u_1u_3 \in$ E(G). If  $u_1v_3 \in E(G)$  or  $u_2v_3 \in E(G)$ , then the function  $f_3: V(G) \to \{0, 1, 2\}$ defined by  $f_3(u_1) = f_3(u_2) = 2$  and  $f_3(x) = 0$  otherwise, is clearly a co-Roman dominating function of G of weight 4, a contradiction. Therefore,  $\deg(v_3) = 1$ . Since G is connected, we conclude that G is a graph obtained from three paths  $v_1u_1y_1, v_2u_2y_2$  and  $v_3u_3$  by adding edges between  $u_1, u_2, u_3$  such that the resulting graph is connected. Hence  $G \in \mathcal{G}_6$ .

Subcase 1.2. |X| = 1. Assume that  $u_1y_1 \in E(G)$ . If  $y_1u_3 \in E(G)$  (the case  $y_1u_2 \in E(G)$  is similar), then the function  $f_4: V(G) \to \{0, 1, 2\}$  defined by  $f_4(u_1) = f_4(u_2) = f_4(u_3) = 1$  and  $f_4(x) = 0$  otherwise, is clearly a co-Roman dominating function of G of weight 3 which is a contradiction. Hence  $y_1u_3, y_1u_2 \notin E(G)$ . Hence  $\deg(y_1) = 1$ . Regarding the matching  $M' = \{u_1y_1, u_2v_2, u_3v_3\}$  instead of M, we have  $\deg(v_1) = 1$ . Since G is connected, we may assume that  $u_1u_3 \in E(G)$ . If  $u_1v_3 \in E(G)$ , then the function  $h_1: V(G) \to \{0, 1, 2\}$  defined by  $h_1(u_1) = 2, h_1(u_2) = 1$  and  $h_1(x) = 0$  otherwise, is clearly a co-Roman dominating function of G of weight 3, a contradiction. Therefore  $u_1v_3 \notin E(G)$ . Consider the following.

•  $u_1u_2 \in E(G)$  (the case  $u_1v_2 \in E(G)$  is similar). Then as above  $u_1v_2 \notin E(G)$ . If  $v_2v_3 \in E(G)$ , then the function  $h_2 : V(G) \to \{0, 1, 2\}$  defined by  $h_2(u_1) = 2, h_2(v_2) = 1$  and  $h_2(x) = 0$  otherwise, is clearly a co-Roman dominating function of G of weight 3, a contradiction. Hence  $v_2v_3 \notin E(G)$ . If

 $\{u_2u_3, u_2v_3, u_3v_2\} \subseteq E(G)$ , then the function  $h_3: V(G) \to \{0, 1, 2\}$  defined by  $h_3(u_1) = 2, h_3(u_2) = 1$  and  $h_3(x) = 0$  otherwise, is clearly a co-Roman dominating function of G of weight 3, a contradiction. Thus  $\{u_2u_3, u_2v_3, u_3v_2\} \not\subseteq E(G)$ . It follows that  $G \in \{G_1, G_2, G_3, G_4, G_5\}$  and so  $G \in \mathcal{G}_5$ .

•  $u_1u_2, u_1v_2 \notin E(G)$ . If  $\{u_2, v_2, v_3\}$  induces a triangle, then the function  $h_4: V(G) \to \{0, 1, 2\}$  defined by  $h_4(u_1) = 2, h_4(u_2) = 1$  and  $h_4(x) = 0$  otherwise, is clearly a co-Roman dominating function of G of weight 3, a contradiction. Thus  $\{u_2, v_2, v_3\}$  does not induce a triangle. As above we have  $\{u_2u_3, u_2v_3, u_3v_2\} \not\subseteq E(G)$ . Since G is connected, the graph induced by  $u_2, v_2, u_3, v_3$  is connected. This implies that  $G \in \{G_6, G_7, G_8, G_9, G_{10}\}$  and so  $G \in \mathcal{G}_5$ .

Subcase 1.3. |X| = 0. Then n = 6. Since  $\gamma_{cr}(G) = 3$ , we have  $\Delta(G) \leq 4$  by Propositions 1 and 2. Hence  $\gamma(G) \geq 2$ . If  $\gamma(G) = 3$ , then we deduce from Theorem 28 that G is the corona  $cor(P_3)$  or  $cor(C_3)$  and so  $G \in \mathcal{G}_4$ . Assume  $\gamma(G) = 2$ . Then we conclude from Proposition 9 that every  $\gamma(G)$ -set S contains a vertex x such that x has no neighbor  $x' \in V \setminus S$  with  $pn(x, S) \subseteq N[x']$ . It follows that  $G \in \mathcal{G}_4$ .

Case 2.  $\alpha'(G) = 2$ . First let  $S \neq \emptyset$ . We deduce from (2) that |S| = 1 and so  $S = \{x_1\}$ . Let  $x_1u_1, x_1v_1 \in E(G)$ . Then we assume that each other vertex of X is adjacent only to  $u_2$ . It follows that  $\deg(x) = 1$  for each  $x \in X \setminus \{x_1\}$ . Since the function  $g: V(G) \to \{0, 1, 2\}$  defined by  $g(u_1) = 1, g(u_2) = 2$  and g(x) = 0 otherwise, is an co-Roman dominating function of G, we deduce that  $n - 3 \leq 3$  and so  $n \leq 6$ . If n = 6, then clearly  $X = \{x_1, y_1\}$ . By considering the matching  $M' = \{u_1v_1, u_2y_1\}$  instead of M, we have  $\deg(v_2) = 1$ . Since G is connected and  $\gamma_{cr}(G) = 3, u_2$  must be adjacent to at least one vertex and at most two vertices in  $\{u_1, v_1, x_1\}$ . Thus G is a graph obtained from a triangle by adding a path  $P_3$  and joining its center to at least one and at most two vertices of triangle and so  $G \simeq H_1$  or  $H_2$ . Hence  $G \in \mathcal{G}_4$ . Assume that n = 5. Since G is connected, G is a graph obtained from a triangle and so the the resulting graph has at most one universal vertex. Thus  $G \in \mathcal{G}_2$ .

Now let  $S = \emptyset$ . As above, we may assume  $N(x) \subseteq \{u_1, u_2\}$  for each  $x \in X$ . By Theorem 19, we have  $\gamma_{cr}(G) \leq 4$  and this implies that  $n \leq 7$ . Thus  $|X| \leq 3$ . If n = 4, then we have  $\gamma_{cr}(G) = 1$  yielding  $G \in \{K_4, K_4 - e\} \subseteq \mathcal{G}_1$  by Observation 6. If n = 5, then G is a graph obtained from a path  $P_3$  and a path  $P_2$  by adding some edges between them such that the resulting graph is different from DS(1, 2)and has at most one universal vertex. Thus  $G \in \mathcal{G}_3$ . Let  $n \geq 6$ . Since  $\gamma_{cr}(G) \geq 3$ , G has no vertex of degree n - 1 and so  $\gamma(G) \geq 2$ . Since  $\{u_1, u_2\}$  is a dominating set, we have  $\gamma(G) = 2$ . If n = 6, then clearly  $G \in \mathcal{G}_4$ . Suppose n = 7. Then  $X = \{y_1, y_2, y_3\}$ . If  $u_i$  is adjacent to all vertices of X for some i, say i = 1, then the function  $g : V(G) \to \{0, 1, 2\}$  defined by  $g(u_1) = 2, g(u_2) = 1$  and g(x) = 0otherwise, is an co-Roman dominating function of G of weight 3 which leads to a contradiction. Hence, each  $u_i$  is adjacent to at most two vertices in X. We may assume without loss of generality that  $u_1y_1, u_1y_2, u_2y_3 \in E(G)$  and  $u_1y_3 \notin E(G)$ . Since  $\{y_1, y_2, y_3, v_1\}$  is independent, we deduce that  $\deg(y_3) = 1$ . Considering the matching  $M' = \{u_1v_1, u_2y_3\}$  instead of M, we obtain  $\deg(v_2) = 1$ . Since  $\gamma_{cr}(G) = 4, u_2$  is adjacent to at most one vertex in  $\{y_1, y_2, v_1\}$ . Thus G is a connected graph obtained from  $K_{1,3}$  and a path  $P_3$  by joining the center of  $P_3$ to the center or at most one leaf of  $K_{1,3}$ . This implies that  $G \in \{G_{11}, G_{12}, G_{13}\}$ and so  $G \in \mathcal{G}_5$ . This completes the proof.

### Acknowledgment

This work is supported by the National Key Research and Development Program under grant 2017YFB0802303, the National Natural Science Foundation of China under the grant 11361008, and Applied Basic Research (Key Project) of Sichuan Province under grant 2017JY0095.

#### References

- H. Abdollahzadeh Ahangar, M.A. Henning, V. Samodivkin and I.G. Yero, *Total Roman domination in graphs*, Appl. Anal. Discrete Math. **10** (2016) 501–517. doi:10.2298/AADM160802017A
- [2] S. Arumugam, K. Ebadi and M. Manrique, Co-Roman dominaton in graphs, Proc. Indian Acad. Sci. Math. Sci. 125 (2015) 1–10. doi:10.1007/s12044-015-0209-8
- [3] R.A. Beeler, T.W. Haynes and S.T. Hedetniemi, *Double Roman domination*, Discrete Appl. Math. **211** (2016) 23–29. doi:10.1016/j.dam.2016.03.017
- [4] E.W. Chambers, B. Kinnersley, N. Prince and D.B. West, Extremal problems for Roman domination, SIAM J. Discrete Math. 23 (2009) 1575–1586. doi:10.1137/070699688
- M. Chellali, T.W. Haynes, S.T. Hedetniemi and A. McRae, Roman {2}-domination, Discrete Appl. Math. 204 (2016) 22–28. doi:10.1016/j.dam.2015.11.013
- [6] E.J. Cockayne, P.A. Dreyer Jr., S.M. Hedetniemi and S.T. Hedetniemi, Roman domination in graphs, Discrete Math. 278 (2004) 11–22. doi:10.1016/j.disc.2003.06.004
- [7] O. Favaron, H. Karami, R. Khoeilar and S.M. Sheikholeslami, On the Roman domination number of a graph, Discrete Math. **309** (2009) 3447–3451. doi:10.1016/j.disc.2008.09.043
- [8] J. Fink, M. Jacobson, L. Kinch and J. Roberts, On graphs having domination number half their order, Period. Math. Hungar. 16 (1985) 287–293. doi:10.1007/BF01848079

- [9] T.W. Haynes and S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dakker Inc., New York, 1998).
- [10] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in Graphs: Advanced Topics (Marcel Dekker Inc., New York, 1998).
- M.A. Henning and S.T. Hedetniemi, Defending the Roman Empire—A new strategy, Discrete Math. 266 (2003) 239–251. doi:10.1016/S0012-365X(02)00811-7
- [12] C. Payan and N.H. Xuong, *Domination-balanced graphs*, J. Graph Theory 6 (1982) 23–32.
  doi:10.1002/jgt.3190060104
- [13] C.S. ReVelle and K.E. Rosing, Defendens imperium Romanum: a classical problem in military strategy, Amer. Math. Monthly 107 (2000) 585–594. doi:10.2307/2589113
- [14] I. Stewart, Defend the Roman Empire, Sci. Amer. 281 (1999) 136–138. doi:10.1038/scientificamerican1299-136
- [15] Z. Zhang, Z. Shao and X. Xu, On the Roman domination numbers of generalized Petersen graphs, J. Combin. Math. Combin. Comput. 89 (2014) 311–320.

Received 21 October 2016 Revised 13 September 2017 Accepted 13 September 2017