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ON THE CO-ROMAN DOMINATION IN GRAPHS

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Abstract

Let $G = (V, E)$ be a graph and let $f : V(G) \to \{0, 1, 2\}$ be a function. A vertex v is said to be protected with respect to f, if $f(v) > 0$ or $f(v) = 0$ and v is adjacent to a vertex of positive weight. The function f is a *co-Roman dominating function* if (i) every vertex in V is protected, and (ii) each $v \in V$ with positive weight has a neighbor $u \in V$ with $f(u) = 0$ such that the function $f_{uv} : V \to \{0, 1, 2\}$, defined by $f_{uv}(u) = 1$, $f_{uv}(v) = f(v) - 1$ and $f_{uv}(x) = f(x)$ for $x \in V \setminus \{v, u\}$, has no unprotected vertex. The *weight* of f is $\omega(f) = \sum_{v\in V} f(v)$. The *co-Roman domination number* of a graph G, denoted by $\gamma_{cr}(G)$, is the minimum weight of a co-Roman dominating function on G. In this paper, we give a characterization of graphs of order n for which co-Roman domination number is $\frac{2n}{3}$ or $n - 2$, which settles two open problem in [S. Arumugam, K. Ebadi and M. Manrique, *Co-Roman domination in graphs*, Proc. Indian Acad. Sci. Math. Sci. 125 (2015) 1–10]. Furthermore, we present some sharp bounds on the co-Roman domination number.

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For terminology and notation on graph theory not given here, the reader is referred to [9, 10]. In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. A *universal vertex* is a vertex that is adjacent to all other vertices of G. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. For a set $S \subseteq V(G)$ and a vertex $v \in S$, the *private neighborhood* of v with respect to S is the set $pn(v; S) = \{u|u \in N(v), N(u) \cap S = \{v\}$. A *leaf* is a vertex of degree one, and a *support vertex* is a vertex adjacent to a leaf. We also denote by L_v the set of all leaves adjacent to a support vertex v. For a vertex v in a rooted tree T, let $D(v)$ denote the set of *descendants* of v and $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. A *subdivision* of an edge uv is obtained by replacing the edge uv with a path uwv, where w is a new vertex. The *subdivision graph* $S(G)$ is the graph obtained from G by subdividing each edge of G. The subdivision star $S(K_{1,t})$ for $t \geq 2$, is called a *healthy spider*. We write P_n for a *path* of length $n-1$ and $K_{1,n}$ for a *star*. For integers $r \geq s \geq 1$, the *double star* $DS(r, s)$ is the tree obtained by connecting the centers of two stars $K_{1,r}$ and $K_{1,s}$ with an edge. The *diameter* of G, denoted by $\mathrm{diam}(G)$, is the maximum value among minimum distances between all pairs of vertices of G. For a subset S of vertices of G, we denote by $G[S]$ the subgraph induced by S. For a subset $S \subseteq V(G)$ of vertices of a graph G and a function $f : V(G) \to R$, we define $f(S) = \sum_{x \in S} f(x)$. For a function $f : V(G) \to \{0, 1, 2\}$, let $V_i = \{v \in V \mid f(v) = i\}$ for $i = 0, 1, 2$. Since these three sets determine f, we can equivalently write $f = (V_0, V_1, V_2)$ (or $f = (V_0^f, V_1^f, V_2^f)$ to refer f). We note that $\omega(f) = |V_1| + 2|V_2|$.

A *Roman dominating function* on a graph G, abbreviated RD-function, is a function $f : V(G) \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight, $\omega(f)$, of f is defined as $\omega(f) = \sum_{v \in V} f(v)$. The *Roman domination number*, denoted by $\gamma_R(G)$. An RD-function with minimum weight $\gamma_R(G)$ in G is called a $\gamma_R(G)$-function. The definition of the Roman dominating function was given multiplicity by Steward [14] and ReVelle and Rosing [13]. Roman domination is now well studied in graph theory [1, 3–7, 15].

Let $f : V(G) \to \{0, 1, 2\}$ be a function. A vertex v is said to be *protected* with respect to f, if $f(v) > 0$ or $f(v) = 0$ and v is adjacent to a vertex of positive weight. The function f is a *weak Roman dominating function* if for every vertex u with $f(u) = 0$ there exists a vertex v adjacent to u such that $f(v) \in \{1, 2\}$ and the function $f_{uv} : V \to \{0, 1, 2\}$, defined by $f_{uv}(u) = 1$, $f_{uv}(v) = f(v) - 1$ and $f_{uv}(x) = f(x)$ for $x \in V \setminus \{v, u\}$, has no unprotected vertex. The *weak Roman domination number* of a graph G, denoted by $\gamma_r(G)$, is the minimum weight among all weak Roman dominating functions on G. The weak Roman domination number was introduced by Henning and Hedetniemi in [11].

The function $f : V(G) \to \{0, 1, 2\}$ is a *co-Roman dominating function*, abbreviated CRDF if (i) every vertex in V is protected, and (ii) each $v \in V$ with positive weight has a neighbor $u \in V$ with $f(u) = 0$ such that the function $f_{uv} : V \to \{0, 1, 2\}$, defined by $f_{uv}(u) = 1$, $f_{uv}(v) = f(v)-1$ and $f_{uv}(x) = f(x)$ for $x \in V \setminus \{v, u\}$, has no unprotected vertex. The weight of f is $\omega(f) = \sum_{v\in V} f(v)$. The *co-Roman domination number* of a graph G, denoted by $\gamma_{cr}(G)$, is the minimum weight of a co-Roman dominating function on G. It follows from the definitions that for any connected graph G of order $n \geq 2$,

$$\gamma_{cr}(G) \leq n - 1. \tag{1}$$

The co-Roman domination in graphs was investigated by Arumugam *et al.* in [2]. The proof of the next results can be found in [2].

Proposition 1. *If H is a spanning subgraph of a graph H, then $\gamma_{cr}(G) \leq \gamma_{cr}(H)$.*

Proposition 2. *For $n \geq 2$, $\gamma_{cr}(K_{1,n}) = 2$.*

Proposition 3. *For $n \geq 4$, $\gamma_{cr}(P_n) = \gamma_{cr}(C_n) = \lceil \frac{2n}{5} \rceil$.*

Proposition 4. *For every tree T of order $n \geq 2$, $\gamma_{cr}(T) \leq \frac{2n}{3}$.*

The next result is an immediate consequence of Propositions 1 and 4.

Corollary 5. *For every connected graph G of order $n \geq 2$, $\gamma_{cr}(G) \leq \frac{2n}{3}$.*

Observation 6. *Let G be a graph of order $n \geq 2$. Then $\gamma_{cr}(G) = 1$ if and only if G has two vertices of degree $n - 1$.*

Theorem 7. *For every graph G, $\gamma_{cr}(G) \leq \gamma_r(G)$.*

In [2], the authors posed the following open problems.

Problem 1. Characterize graphs G of order n such that $\gamma_{cr}(G) = n - 2$.

Problem 2. Characterize trees T of order n such that $\gamma_{cr}(T) = \frac{2n(T)}{3}$.

Problem 3. Characterize graphs G such that $\gamma_{cr}(T) = \gamma(G)$.

In this paper, we settle the above open problems. Furthermore, we establish some sharp bounds on the co-Roman domination number.

1. Graphs G with $\gamma_{cr}(G) = \gamma_r(G)$ or $\gamma_{cr}(G) = \gamma(G)$

In this section, we study the properties of graphs G for which $\gamma_{cr}(G) = \gamma_r(G)$ or $\gamma_{cr}(G) = \gamma(G)$.

Proposition 8. *Let G be a connected graph of order at least two. Then $\gamma_{cr}(G) = \gamma_r(G)$ if and only if there exists a $\gamma_{cr}(G)$-function $f = (V_0, V_1, V_2)$ such that each vertex $x \in V_0$, either has a neighbor x' in V_2 or has a neighbor x' in V_1 for which $pn(x', V_1 \cup V_2) \subseteq N[x]$.*

Proof. If there exists a $\gamma_{cr}(G)$-function $f = (V_0, V_1, V_2)$ such that each vertex $x \in V_0$, has a neighbor $x' \in V_1 \cup V_2$ with $pn(x', V_1 \cup V_2) \subseteq N[x]$, then clearly f is a weak Roman dominating function of G and so $\gamma_r(G) \leq \gamma_{cr}(G)$. It follows from Theorem 7 that $\gamma_r(G) = \gamma_{cr}(G)$.

Conversely, let $\gamma_r(G) = \gamma_{cr}(G)$. There exists a $\gamma_r(G)$-function $f = (V_0, V_1, V_2)$ such that f is a co-Roman dominating function of G (see Theorem 3.3 of [2]). By assumption, f is a $\gamma_{cr}(G)$-function. Assume $x \in V_0$ is an arbitrary vertex. Since f is a weak Roman dominating function, x has a neighbor x' in $V_1 \cup V_2$ such that the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(x) = 1, g(x') = f(x') - 1$ and $g(u) = f(u)$ otherwise, is safe. If x has a neighbor in V_2, then we are done. Assume x has no neighbor in V_2. It follows that $x' \in V_1$. Since f is safe, we must have $pn(x', V_1 \cup V_2) \subseteq N[x]$ and the proof is complete. ∎

Proposition 9. *Let G be a connected graph of order at least two. Then $\gamma(G) = \gamma_{cr}(G)$ if and only if there exists a $\gamma(G)$-set S such that each vertex $x \in S$ has a neighbor $x' \in V \setminus S$ with $pn(x, S) \subseteq N[x']$.*

Proof. Let $\gamma(G) = \gamma_{cr}(G)$. Assume $f = (V_0, V_1, V_2)$ is a $\gamma_{cr}(G)$-function. Since $V_1 \cup V_2$ is a dominating set, we deduce from $\gamma(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_{cr}(G)$ that $V_2 = \emptyset$ and V_1 is a $\gamma(G)$-set. Let $x \in V_1$ be an arbitrary vertex. Since f is a co-Roman dominating function, there is a vertex $x' \in V_0 \cap N(x)$ such that $((V_0 \setminus \{x'\}) \cup \{x\}, (V_1 \setminus \{x\}) \cup \{x'\}, \emptyset)$ is a $\gamma_{cr}(G)$-function. It follows that $(V_1 \setminus \{x\}) \cup \{x'\}$ is a $\gamma(G)$-set and this implies that $pn(x, V_1) \subseteq N[x']$.

Conversely, let S be a $\gamma(G)$-set such that each vertex $x \in S$ has a neighbor $x' \in V \setminus S$ with $pn(x, S) \subseteq N[x']$. Then the function $f = (V(G) \setminus S, S, \emptyset)$ is clearly a co-Roman dominating function of weight $\gamma(G)$ and so $\gamma_{cr}(G) \leq \gamma(G)$. It follows that $\gamma_{cr}(G) = \gamma(G)$. ∎

Corollary 10. *Let G be a connected graph of order at least two with $\gamma(G) = \gamma_{cr}(G)$. Then for any $\gamma_{cr}(G)$-function $f = (V_0, V_1, V_2)$, $V_2 = \emptyset$.*

Corollary 11. *Let G be a connected graph of order at least two. If $\gamma(G) = \gamma_{cr}(G)$, then G has no strong support vertex.*

For a tree T, let $M(T) = \{v \mid$ there exists a $\gamma_{cr}(T)$-function f such that $f(v) = 1\}$. In what follows, we present a constructive characterization of trees T with $\gamma(T) = \gamma_{cr}(T)$. In order to do this, we define a family of trees as follows. Let $\mathcal{T}$ be the collection of trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ of trees for some $k \geq 1$, where T_1 is a P_2 and $T = T_k$. If $k \geq 2$, T_{i+1} can be obtained from T_i by one of the following three operations. Let one vertex of P_2 be considered as a support vertex.

Operation $\mathcal{O}_1$. If $v \in M(T_i)$, then the tree T_{i+1} is obtained from T_i by adding a pendant $P_3 = xyz$ and adding the edge vx (see Figure 1(a)).

Operation $\mathcal{O}_2$. If v is a support vertex of T_i, then the tree T_{i+1} is obtained from T_i by adding a pendant $P_2 = xy$ and adding the edge vx (see Figure 1(b)).

Operation $\mathcal{O}_3$. If $v \in T_i$, then the tree T_{i+1} is obtained from T_i by adding a healthy spider with at least two feet headed at x and adding the edge vx (see Figure 1(c)).

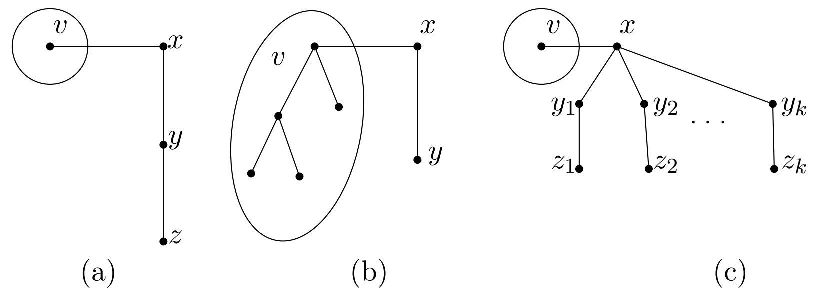

Figure 1. (a) Operation $\mathcal{O}_1$. (b) Operation $\mathcal{O}_2$. (c) Operation $\mathcal{O}_3$.

Lemma 12. *If T_i is a tree with $\gamma(T_i) = \gamma_{cr}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_1$, then $\gamma(T_{i+1}) = \gamma_{cr}(T_{i+1})$.*

Proof. Let f be a $\gamma_{cr}(T_i)$-function and v a vertex of T_i with $f(v) = 1$. Then the function $f' : V(T_{i+1}) \to \{0, 1, 2\}$ by $f'(y) = 1$, $f'(x) = f'(z) = 0$ and $f'(u) = f(u)$ for $u \in V(T_i)$, is a co-Roman dominating function on T_{i+1} and so $\gamma_{cr}(T_{i+1}) \leq \gamma_{cr}(T_i) + 1$.
It is easy to see that $\gamma(T_{i+1}) = \gamma(T_i) + 1$. Now we have

$$\gamma(T_i) + 1 = \gamma(T_{i+1}) \leq \gamma_{cr}(T_{i+1}) \leq \gamma_{cr}(T_i) + 1 = \gamma(T_i) + 1$$

yielding $\gamma(T_{i+1}) = \gamma_{cr}(T_{i+1})$. ■

Lemma 13. *If T_i is a tree with $\gamma(T_i) = \gamma_{cr}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_2$, then $\gamma(T_{i+1}) = \gamma_{cr}(T_{i+1})$.*

Proof. Clearly, any $\gamma_{cr}(T_i)$-function can be extended to a co-Roman dominating function by assigning 1 to x and 0 to y implying that $\gamma_{cr}(T_{i+1}) \leq \gamma_{cr}(T_i) + 1$.

Since v is a support vertex, one can easily check that $\gamma(T_{i+1}) = \gamma(T_i) + 1$. Now the result follows as in the proof of Lemma 12. ∎

Lemma 14. *If T_i is a tree with $\gamma(T_i) = \gamma_{cr}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_3$, then $\gamma(T_{i+1}) = \gamma_{cr}(T_{i+1})$.*

Proof. Let the added spider has exactly k feet. Obviously, any $\gamma_{cr}(T_i)$-function can be extended to a co-Roman dominating function by assigning 1 to the support vertices of spider and 0 to the remaining vertices of spider and this implies that $\gamma_{cr}(T_{i+1}) \leq \gamma_{cr}(T_i) + k$. Moreover, it is easy to verify that $\gamma(T_{i+1}) = \gamma(T_i) + k$ and the result follows as in the proof of Lemma 12. ∎

Lemma 15. *If $T \in \mathcal{T}$, then $\gamma(T) = \gamma_{cr}(T)$.*

Proof. Let $T \in \mathcal{T}$. By definition, there exists a sequence of trees $T_1, T_2, \ldots, T_k$ ($k \geq 1$) such that $T_1 = K_2$, and if $k \geq 2$, T_{i+1} can be obtained recursively from T_i by Operation $\mathcal{O}_1$, $\mathcal{O}_2$ or $\mathcal{O}_3$ for $i = 1, 2, \ldots, k-1$. We proceed by induction on k. If $T = K_2$, then clearly $\gamma(T) = \gamma_{cr}(T) = 1$. Suppose $k \geq 2$ and the result holds for each tree $T \in \mathcal{T}$ which can be obtained from a sequence of operations of length $k-1$ and let $T' = T_{k-1}$. By the induction hypothesis, we have $\gamma(T') = \gamma_{cr}(T')$. Since $T = T_k$ is obtained from T' by one of the Operations $\mathcal{O}_1$, $\mathcal{O}_2$ or $\mathcal{O}_3$ from T', we have $\gamma(T) = \gamma_{cr}(T)$ by Lemmas 12, 13 and 14. ∎

Theorem 16. *Let T be a tree of order $n \geq 2$. Then $\gamma(T) = \gamma_{cr}(T)$ if and only if $T \in \mathcal{T}$.*

Proof. The sufficiency follows from Lemma 15. We use induction on n to prove the necessity. If $n = 2$, then $T = P_2$ that belongs to $\mathcal{T}$. Assume $n \geq 3$ and that the result holds for any tree of order less than n. Let T be a tree of order n with $\gamma(T) = \gamma_{cr}(T)$. Let $P = v_1 v_2 \cdots v_\ell$ be a diametral path in T and root T at v_ℓ. By Corollary 11, we have $d(v_2) = 2$. Consider the following cases.

Case 1. v_3 is a support vertex. Let w be a leaf adjacent to v_3 and let $T' = T - \{v_1, v_2\}$. If f is a $\gamma_{cr}(T)$-function, then clearly $f(v_1) + f(v_2) \geq 1$ and $f(v_3) + f(w) \geq 2$. It is easy to verify that the function f, restricted to T' is a co-Roman dominating function implying that $\gamma_{cr}(T) \geq \gamma_{cr}(T') + 1$. Clearly $\gamma(T) = \gamma(T') + 1$, and we deduce from

$$\gamma(T) = \gamma_{cr}(T) \geq \gamma_{cr}(T') + 1 = \gamma(T') + 1 = \gamma(T)$$

that $\gamma_{cr}(T') = \gamma(T')$. By the induction hypothesis, we have $T' \in \mathcal{T}$. Now T can be obtained from T' by Operation $\mathcal{O}_2$.

Case 2. $d(v_3) = 2$. Let $T' = T - \{v_1, v_2, v_3\}$. By Proposition 3, $n \geq 4$. Clearly $\gamma(T) = \gamma(T') + 1$. Assume $f = (V_0, V_1, V_2)$ is a $\gamma_{cr}(T)$-function. By Corollary 10, $V_2 = \emptyset$. Clearly $f(v_1) + f(v_2) = 1$ and $f(v_3) + f(v_4) \geq 1$. If $f(v_3) = f(v_4) = 1$, then the function $g : V(T) \to \{0, 1, 2\}$ defined by $g(v_4) = g(v_2) = 1$, $g(v_1) = g(v_3) = 0$ and $g(x) = f(x)$ otherwise, is a co-Roman dominating function of T of weight less than $\omega(f)$ which is a contradiction. Hence $f(v_3) = 0$ or $f(v_4) = 0$ and so $f(v_3) + f(v_4) = 1$. Consider the following.

• $f(v_3) = 1$ and $f(v_4) = 0$. If $f(x) = 1$ for some $x \in N_{T'}(v_4)$, then the function $g : V(G) \to \{0, 1\}$ defined by $g(v_2) = 1, g(v_1) = g(v_3) = 0$ and $g(x) = f(x)$ otherwise, is a dominating function of T of weight less than $\omega(f)$ which contradicts $\gamma(T) = \gamma_{cr}(T)$. Thus $f(x) = 0$ for each $x \in N_{T'}(v_4)$. Now the function $h : V(T') \to \{0, 1\}$ defined by $h(v_4) = 1$ and $h(x) = f(x)$ otherwise, is a co-Roman dominating function of T of weight $\omega(f) - 1$. It follows from

$$\gamma(T) = \gamma_{cr}(T) \geq \gamma_{cr}(T') + 1 = \gamma(T') + 1 = \gamma(T)$$

that $\gamma_{cr}(T') = \gamma(T')$ and that h is a $\gamma_{cr}(T')$-function with $h(v_4) = 1$. By the induction hypothesis, we have $T' \in \mathcal{T}$ and so T can be obtained from T' by Operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$.

• $f(v_3) = 0$ and $f(v_4) = 1$. As above we have $f(x) = 0$ for some $x \in N_{T'}(v_4)$. Then the function f restricted to T' is a co-Roman dominating function of T' and so $\gamma_{cr}(T) \geq \gamma_{cr}(T') + 1$. Using above argument, we obtain $T \in \mathcal{T}$.

Case 3. v_3 is not a support vertex and $d(v_3) \geq 3$. Let T' be the component of $T - v_3v_4$ containing v_3. Then T' is a spider with at least k feet where $k = \deg(v_3) - 1$. Clearly $\gamma(T) = \gamma(T') + k$. Now we show that $\gamma_{cr}(T) \geq \gamma_{cr}(T') + k$. Let $u_1, \ldots, u_k$ be the children of v_3 and w_i be the leaf adjacent to u_i for $i = 1, \ldots, k$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{cr}(T)$-function. By Corollary 10, $V_2 = \emptyset$. Obviously $f(u_i) + f(w_i) = 1$ for each i. As Case 2, we can see that $f(v_4) = 0$ or $f(v_3) = 0$. If $f(v_4) = f(v_3) = 0$, then the function f restricted to T' is a co-Roman dominating function of weight $\gamma_{cr}(T) - k$ and so $\gamma_{cr}(T) \geq \gamma_{cr}(T') + k$. Consider the following subcases.

Subcase 3.1. $f(v_3) = 1$ and $f(v_4) = 0$. If $f(x) = 1$ for some $x \in N_{T'}(v_4)$, then the function $g : V(G) \to \{0, 1\}$ defined by $g(v_3) = g(w_i) = 0$, $g(u_i) = 1$ for $1 \leq i \leq k$ and $g(x) = f(x)$ otherwise, is a dominating function of T of weight less than $\omega(f)$ contradicting $\gamma(T) = \gamma_{cr}(T)$. Thus $f(x) = 0$ for each $x \in N_{T'}(v_4)$. Now the function $h : V(T') \to \{0, 1\}$ defined by $h(v_4) = 1$ and $h(x) = f(x)$ otherwise, is a co-Roman dominating function of T of weight $\omega(f) - k$ and hence $\gamma_{cr}(T) \geq \gamma_{cr}(T') + k$.

Subcase 3.2. $f(v_3) = 0$ and $f(v_4) = 1$. As above we have $f(x) = 0$ for some $x \in N_{T'}(v_4)$. Then the function f, restricted to T' is a co-Roman dominating function of T' and so $\gamma_{cr}(T) \geq \gamma_{cr}(T') + k$.

Thus in all cases $\gamma_{cr}(T) \geq \gamma_{cr}(T') + k$. As Case 2, we deduce that $\gamma_{cr}(T') = \gamma(T')$ and so by the induction hypothesis we have $T' \in \mathcal{T}$. Now T can be obtained from T' by Operation $\mathcal{O}_3$ and hence $T \in \mathcal{T}$. This completes the proof. ■

2. Bounds on Co-Roman Domination

In this section, we present some sharp bounds on the co-Roman domination number. First we prove two upper bounds on the co-Roman domination number in terms of matching number.

Theorem 17. *For any connected graph G of order $n \geq 2$,*

$$\gamma_{cr}(G) \leq n - \alpha'(G).$$

Proof. Let $M = \{u_1v_1, \ldots, u_{\alpha'}v_{\alpha'}\}$ be a maximum matching of G and let X be the independent set of M-unsaturated vertices. If y and z are vertices of X and $yu_i \in E(G)$, then since the matching M is maximum, $zv_i \notin E(G)$. Therefore, for all $i \in \{1, 2, \ldots, \alpha'\}$ there are at most two edges between the sets $\{u_i, v_i\}$ and $\{y, z\}$. Assume S is the set of all vertices in X which belongs to a triangle with an edge in M. Let $S = \{x_1, \ldots, x_s\}$ if $S \neq \emptyset$ and $X \setminus S = \{y_1, \ldots, y_k\}$ if $X \setminus S \neq \emptyset$.

First let $S = \emptyset$. Then $vu_i \notin E(G)$ or $vv_i \notin E(G)$ for each $v \in X$ and each $i \in \{1, \ldots, \alpha'\}$. We may assume $N(x) \subseteq \{u_1, \ldots, u_{\alpha'}\}$ for each $x \in X$. Define $f : V(G) \to \{0, 1, 2\}$ by $f(u_i) = 0$ for $1 \leq i \leq \alpha'$ and $f(x) = 1$ otherwise. Clearly, f is a co-Roman dominating function of G of weight $\alpha' + |X|$ and hence

$$\gamma_{cr}(G) \leq \alpha'(G) + |X| = \alpha'(G) + \big(n - 2\alpha'(G)\big) = n - \alpha'(G).$$

Now let $S \neq \emptyset$. We may assume, without loss of generality, that $x_iu_i, x_iv_i \in E(G)$ for $i = 1, \ldots, s$. As above, we can assume that $N(x) \subseteq \{u_1, \ldots, u_{\alpha'}\}$ for each $x \in X \setminus S$. Define $f : V(G) \to \{0, 1, 2\}$ by $f(x) = 0$ for $x \in S \cup \{u_1, \ldots, u_{\alpha'}\}$ and $f(x) = 1$ otherwise. Obviously, f is a co-Roman dominating function of G of weight $\alpha'(G) + |X| - |S|$ and hence

$$\gamma_{cr}(G) \leq \alpha'(G) + |X| - |S| = \alpha'(G) + (n - 2\alpha') - |S| \leq n - \alpha'(G) - |S|. \quad (2)$$

This completes the proof. ■

Theorem 18. *For any connected graph G of order $n \geq 2$ with $\delta(G) \geq 2$,*

$$\gamma_{cr}(G) \leq \alpha'(G).$$

Proof. Let M, X and S be the sets defined in the proof of Theorem 17. Assume first that $S = \emptyset$. Then as above we may assume $N(x) \subseteq \{u_1, \ldots, u_{\alpha'}\}$ for each $x \in X$. Define $f : V(G) \to \{0,1,2\}$ by $f(u_i) = 1$ for $1 \leq i \leq \alpha'$ and $f(x) = 0$ otherwise. Since $\delta(G) \geq 2$, the function $f_i : V(G) \to \{0,1,2\}$ defined by $f(u_i) = 0, f(v_i) = 1$ and $f_i(x) = f(x)$ otherwise, is safe for each i. Thus f is a co-Roman dominating function of G of weight $\alpha'(G)$ and so $\gamma_{cr}(G) \leq \alpha'(G)$.

Now let $S = \{x_1, \ldots, x_s\}$. We may assume, without loss of generality, that $x_iu_i, x_iv_i \in E(G)$ for $i = 1, \ldots, s$. As above, we can assume that $N(x) \subseteq \{u_1, \ldots, u_{\alpha'}\}$ for each $x \in X \setminus S$. It is easy to see that the function f defined above is a co-Roman dominating function of G. Thus $\gamma_{cr}(G) \leq \alpha'(G)$ and the proof is complete. ■

Theorem 19. *For any connected graph G of order $n \geq 2$,*

$$\gamma_{cr}(G) \leq 2\alpha'(G).$$

Proof. Let M, X and S be the sets defined in the proof of Theorem 17. As Theorem 17, we may assume that $x_iu_i, x_iv_i \in E(G)$ for $i = 1, \ldots, s$ if $S \neq \emptyset$ and $N(x) \subseteq \{u_1, \ldots, u_{\alpha'}\}$ for each $x \in X \setminus S$. Then the function $f : V(G) \to \{0,1,2\}$ defined by $f(u_i) = 1$ if u_i is adjacent to a vertex in S, $f(u_i) = 2$ if u_i is adjacent to a vertex in $X \setminus S$ and $f(x) = 0$ otherwise, is a co-Roman dominating function of G and so $\gamma_{cr}(G) \leq |S| + 2|X - S| = 2\alpha'(G) - |S| \leq 2\alpha'(G)$. ■

A set $X \subseteq V(G)$ is called a 2-*packing* if $d(u,v) > 2$ for any different vertices u and v of X. The 2-*packing number* $\rho(G)$ is the maximum cardinality of a 2-packing of G.

Theorem 20. *For any connected graph G of order $n \geq 2$ with $\delta(G) \geq 2$,*

$$\gamma_{cr}(G) \leq n - \rho(G)(\delta(G) - 1).$$

Proof. Let S be a 2-packing of G of size $\rho(G)$. Define $f : V(G) \to \{0,1,2\}$ by $f(x) = 2$ for $x \in S$, $f(x) = 0$ for $x \in \bigcup_{v \in S} N(v)$ and $f(x) = 1$ otherwise. Clearly, f is a co-Roman dominating function of G and hence

$$\begin{aligned}\gamma_{cr}(G) &\leq (n - |\textstyle\bigcup_{v \in S} N[v]|) + 2|S| = n - \sum_{v \in S} |N[v]| + 2\rho(G) \\ &\leq n - \rho(G)(\delta(G) + 1) - 2\rho(G) = n - \rho(G)(\delta(G) - 1),\end{aligned}$$

as desired. ■

Proposition 21. *Let G be a simple connected graph of order n with $\delta(G) \geq 2$ and $g(G) \geq 5$. Then $\gamma_{cr}(G) \leq \frac{2(n - g(G))}{3} + \left\lceil \frac{2g(G)}{5} \right\rceil$.*

Proof. If G is an n-cycle, then the result follows by Proposition 3. Assume G is not a cycle and C is a cycle of length $g(G)$ in G. Let G' be the graph obtained from G by removing the vertices of $V(C)$. Since $g(G) \geq 5$, each vertex of G' can be adjacent to at most one vertex of C which implies $\delta(G') \geq 1$. By Corollary 5, we have $\gamma_{cr}(G') \leq \frac{2(n-g(G))}{3}$. Let g be a $\gamma_{cr}(G')$-function and h be a $\gamma_{cr}(C)$-function. Define $f : V(G) \to \{0, 1, 2\}$ by $f(v) = g(v)$ for $v \in V(G')$ and $f(v) = h(v)$ for $v \in V(C)$. Obviously, f is a co-Roman dominating function and so

$$\gamma_{cr}(G) \leq \frac{2(n-g(G))}{3} + \left\lceil \frac{2g(G)}{5} \right\rceil.$$

■

3. Characterization of Graphs G of Order n with $\gamma_{cr}(G) = \frac{2n}{3}$

In this section, we characterize the graphs attaining the upper bound in Corollary 5. For any arbitrary tree T, let T_{cr} be the tree obtained from T by adding exactly two pendant edges at each vertex of T. Note that $n(T_{cr}) = 3n(T)$. Let $\mathcal{F}$ be the family of all trees T_{cr}. In fact, $\mathcal{F}$ is the family of trees T such that $V(T)$ can be partitioned into sets inducing P_3 such that the subgraph induced by the central vertices of these paths is connected.

Lemma 22. *If $T \in \mathcal{F}$, then $\gamma_{cr}(T) = \frac{2n(T)}{3}$.*

Proof. Let $T \in \mathcal{F}$ and let f be a γ_{cr}-function on T. Then T is obtained from a tree T' by adding exactly two pendant edges at each vertex of T'. For each non-leaf vertex $v \in V(T)$, let $L_v = \{v_1, v_2\}$. It is easy to see that for any non-leaf vertex $v \in V(T)$, $f(v) + f(v_1) + f(v_2) \geq 2$, otherwise we have an unprotected vertex in either f or f_{vv_i} for some $i = 1, 2$. Hence, $\gamma_{cr}(T) = \omega(f) = \sum_{v \in V(T')} (f(v) + f(v_1) + f(v_2)) \geq 2n(T') = \frac{2n(T)}{3}$. Now the result follows from Proposition 4. ■

Lemma 23. *Let $q \geq p \geq 1$ and let $T = DS(p, q)$. Then $\gamma_{cr}(T) = \frac{2n(T)}{3}$ if and only if $q = p = 2$.*

Proof. If $q = p = 2$, then Lemma 22 implies $\gamma_{cr}(T) = \frac{2n(T)}{3}$. Conversely, let $\gamma_{cr}(T) = \frac{2n(T)}{3}$. It follows from Proposition 3 that $q \geq 2$. If $p = 1$, then clearly $\gamma_{cr}(T) = 3 < \frac{2n(T)}{3}$, a contradiction. Suppose that $p \geq 2$. If $q > 2$, then we have $\gamma_{cr}(T) \leq 4 < \frac{2n(T)}{3}$, a contradiction again. Thus $q = p = 2$ and the proof is complete. ■

Theorem 24. *Let T be a tree of order $n \geq 3$. Then $\gamma_{cr}(T) = \frac{2n}{3}$ if and only if $T \in \mathcal{F}$.*

Proof. According to Lemma 22, we only need to prove the necessity. Let T be a tree of order $n \geq 3$ with $\gamma_{cr}(T) = \frac{2n}{3}$. Note that n is a multiple of 3. The proof is by induction on n. If $n = 3$, then the only tree T of order 3 and $\gamma_{cr}(T) = 2$ is $P_3 \in \mathcal{F}$. Let $n \geq 4$ and let the statement hold for all trees of order less than n. Suppose that T is a tree of order n with $\gamma_{cr}(T) = \frac{2n}{3}$. If $\mathrm{diam}(T) = 2$, then $T = K_{1,s}$ and we deduce from Proposition 2 that $T = P_3$ and so $T \in \mathcal{F}$. If $\mathrm{diam}(T) = 3$, then we deduce from Lemma 23 that $T = DS(2,2)$ and so $T \in \mathcal{F}$. Henceforth we assume that $\mathrm{diam}(T) \geq 4$. Let $v_1 v_2 \cdots v_k$ $(k \geq 5)$ be a diametral path in T and root T at v_k. We show that $\deg_T(v_2) = 3$. Let $T' = T - T_{v_2}$ and f be a $\gamma_{cr}(T')$-function. If $\deg_T(v_2) \geq 4$, then the function $g : V(T) \to \{0,1,2\}$ defined by $g(v_2) = 2$, $g(x) = 0$ if $x \in L_{v_2}$ and $g(x) = f(x)$ for $x \in T'$, is a CRDF on T of weight $\omega(f)+2$. By Proposition 4, we have $\gamma_{cr}(T) \leq \omega(g) \leq \gamma_{cr}(T')+2 \leq \frac{2n(T')}{3} + 2 \leq \frac{2(n-4)}{3} + 2 < \frac{2n}{3}$, which is a contradiction. If $\deg_T(v_2) = 2$, then the function $g : V(T) \to \{0,1,2\}$ defined by $g(v_2) = 1$, $g(v_1) = 0$ and $g(x) = f(x)$ for $x \in T'$, is a CRDF on T of weight $\omega(f) + 1$. By Proposition 4, we have $\gamma_{cr}(T) \leq \omega(g) \leq \gamma_{cr}(T') + 1 \leq \frac{2(n-2)}{3} + 1 < \frac{2n}{3}$, a contradiction again. Thus $\deg(v_2) = 3$. Assume that $T' = T - T_{v_2}$. As above, we have

$$\frac{2n(T)}{3} = \gamma_{cr}(T) \leq \gamma_{cr}(T') + 2 \leq \frac{2n(T')}{3} + 2 = \frac{2(n-3)}{3} + 2 = \frac{2n}{3}.$$

Thus all inequalities in the above inequality chain must be equalities and so $\gamma_{cr}(T') = \frac{2n(T')}{3}$. By the induction hypothesis we have $T' \in \mathcal{F}$. Now we show that v_3 is not a leaf of T'. If v_3 is a leaf in T', then let $T'' = T - T_{v_3}$ and let h be a $\gamma_{cr}(T'')$-function. Define the function $g : V(T) \to \{0,1,2\}$ by $g(v_2) = 2$, $g(v) = 0$ if $v \in N_T(v_2)$ and $g(x) = h(x)$ for $x \in T''$. Clearly, g is a CRDF on T of weight $\omega(f)+2$. By Proposition 4, we have $\gamma_{cr}(T) \leq \omega(g) = \gamma_{cr}(T')+2 \leq \frac{2(n-4)}{3}+2 < \frac{2n}{3}$, a contradiction. Thus v_3 is a non-leaf vertex of T' and so $T \in \mathcal{F}$. This completes the proof. ∎

Theorem 25. *Let G be a connected n-vertex graph with $n \geq 3$. Then $\gamma_{cr}(G) = \frac{2n}{3}$ if and only if G is obtained from $\frac{n}{3}P_3$ by adding edges between the centers of the paths P_3 such that the resulting graph is connected.*

Proof. If G has the specified form, then clearly every CRDF puts weight at least 2 on the vertex set of each copy of P_3.

Now suppose that $\gamma_{cr}(G) = \frac{2n}{3}$. Since adding edges cannot increase $\gamma_{cr}(G)$, every spanning tree of G belongs to $\mathcal{F}$. Given a spanning tree T, let $S_1, S_2, \ldots, S_{\frac{n}{3}}$ be the 3-sets in the special partition of $V(T)$. The assignment of weight 2 that guards S_i can be chosen independently of any other S_j . If any edge of G joins vertices of S_i and S_j that are not the centers of the paths they induce, then a CRDF with weight less than $\frac{2n}{3}$ can be built as in the proof of Theorem 24. This completes the proof. ∎

4. Graphs with Large Co-Roman Domination Number

In this section, we characterize all graphs of order n with co-Roman domination number $n-2$ and $n-3$. The first result is an immediate consequence of Theorem 17.

Corollary 26 (Theorem 4.2 in [2]). *Let G be a connected graph on $n \geq 2$ vertices. Then $\gamma_{cr}(G) = n-1$ if and only if $G = K_2$ or $K_{1,2}$.*

Arumugam *et al.* [2] posed the following problem.

Problem. Characterize graphs G such that $\gamma_{cr}(G) = n-2$.

Next we solve this problem.

Theorem 27. *Let G be a connected graph on $n \geq 2$ vertices. Then $\gamma_{cr}(G) = n-2$ if and only if G is a graph on four vertices different from K_4 and $K_4 - e$, or $G \cong DS(2,1)$, or $G \cong DS(2,2)$.*

Proof. By Theorem 17, we have $\alpha'(G) \leq 2$. If $\alpha'(G) = 1$, then G is the star $K_{1,n-1}$ and we conclude from Proposition 2 that $G = K_{1,3}$. Assume that $\alpha'(G) = 2$. Let M, X and S be the sets defined in the proof of Theorem 17. By (2), we have $S = \emptyset$. As above, we may assume $N(x) \subseteq \{u_1, \ldots, u_{\alpha'}\}$ for each $x \in X$. If u_i has at least two neighbors in X for some i, say $i = 1$, then the function $f : V(G) \to \{0,1,2\}$ defined by $f(u_1) = 2, f(u_i) = 0$ for $2 \leq i \leq \alpha'$, $f(x) = 0$ if $x = v_1$ or $x \in N(u_1) \cap X$ and $f(x) = 1$ otherwise, is clearly a co-Roman dominating function of G of weight $n - \alpha'(G) - 1$ which leads to a contradiction. Hence each u_i has at most one neighbor in X and this implies that $|X| \leq 2$. If $|X| = 0$, then $n = 4$ and obviously G is a connected graph on four vertices different from K_4 and $K_4 - e$. Hence $|X| \geq 1$.

First let $|X| = 2$. Since X is independent and G is connected, we may assume that $u_i y_i \in E(G)$ for $i = 1, 2$. Since each u_i has at most one neighbor in X, we deduce that $\deg(y_i) = 1$ for $i = 1, 2$. Considering the matching $M' = \{u_1y_1, u_2y_2\}$ instead of M, we have $\deg(v_1) = \deg(v_2) = 1$. Since G is connected, we have $u_1u_2 \in E(G)$ and hence $G = DS(2,2)$.

Now let $|X| = 1$. Since G is connected, we suppose that $u_1y_1 \in E(G)$. If $u_2y_1 \in E(G)$, then the function $f_1 : V(G) \to \{0,1,2\}$ defined by $f_1(u_1) = f_1(u_2) = 1$ and $f_1(x) = 0$ otherwise, is clearly a co-Roman dominating function of G of weight 2, a contradiction. Thus $\deg(y_1) = 1$. Considering the matching $M' = \{u_1y_1, u_2v_2\}$ instead of M, we obtain $\deg(v_1) = 1$. Since G is connected, we may assume that $u_1u_2 \in E(G)$. If $u_1v_2 \in E(G)$, then clearly $\gamma_{cr}(G) \leq 2$ which is a contradiction. Thus $G = DS(1,2)$ and the proof is complete. ■

The *corona graph* $cor(H)$ of a graph H is the graph obtained from H by attaching a leaf to every vertex of H. We recall the following result established by Payan and Xuong [12] (see also Fink *et al.* [8]).

Theorem 28. *For a graph G with even order n and with no isolated vertices, $\gamma(G) = \frac{n}{2}$ if and only if the components of G are the cycle C_4 or the corona $cor(H)$ for any connected graph H.*

Now we characterize all connected graphs G of order $n \geq 4$ with $\gamma_{cr}(G) = n - 3$. To do this, we introduced some families of graphs.

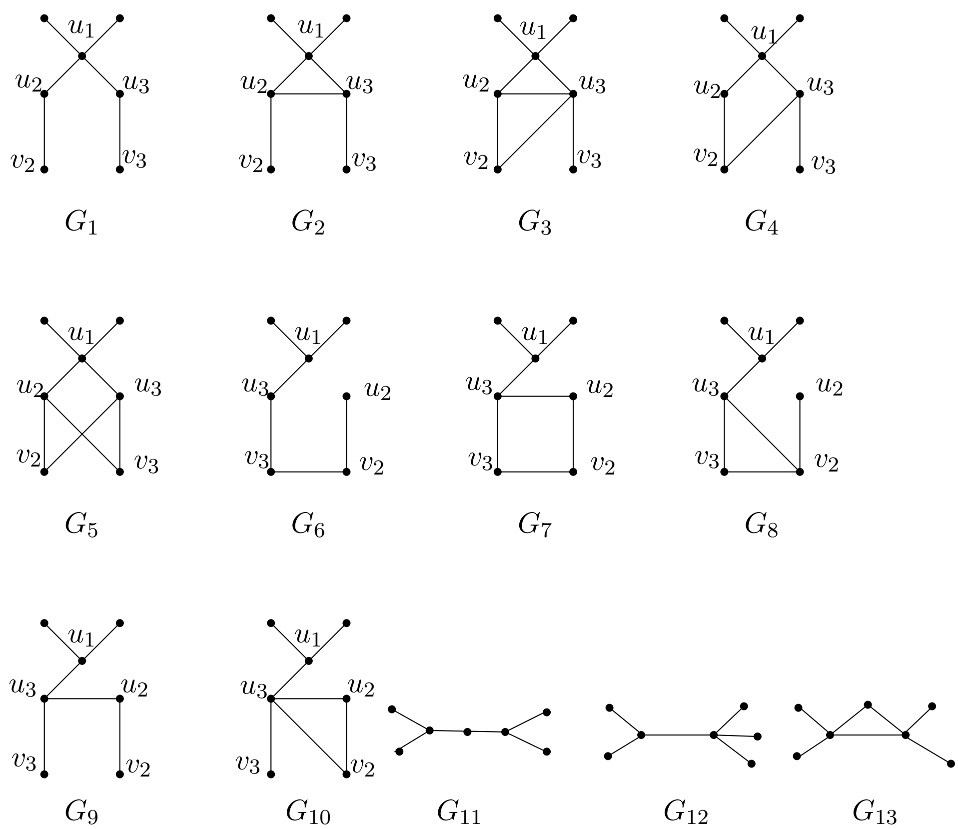

Figure 2. The graphs G of order 7 with $\gamma_{cr}(G) = 4$.

Let

- $\mathcal{G}_1 = \{K_4, K_4 - e, K_{1,4}\}$,
- $\mathcal{G}_2$ be the family of connected graphs G obtained from a triangle and a path P_2 by adding some edges between them so that the resulting graph has at most one universal vertex,

- $\mathcal{G}_3$ be the family of connected graphs G obtained from a path P_3 and a path P_2 by adding some edges between them such that the resulting graph is different from $DS(1,2)$ and has at most one universal vertex,
- $\mathcal{G}_4$ be the family of connected graphs $G \not\cong DS(2,2)$ of order 6 consisting of $cor(P_3), cor(C_3)$ and all graphs G with $\Delta(G) \leq 4$, for which every $\gamma(G)$-set S has a vertex x such that x has no neighbor $x' \in V \setminus S$ with $pn(x,S) \subseteq N[x']$.
- $\mathcal{G}_5 = \{G_1, G_2, \ldots, G_{13}\}$,
- $\mathcal{G}_6$ be the family of connected graphs G obtained from three paths $v_1u_1y_1$, $v_2u_2y_2$ and v_3u_3 by adding edges between u_1, u_2, u_3 such that the resulting graph is connected,
- $\mathcal{G}_7$ be the family of connected graphs G obtained from $3P_3$ by adding edges between the centers of the paths P_3 such that the resulting graph is connected.

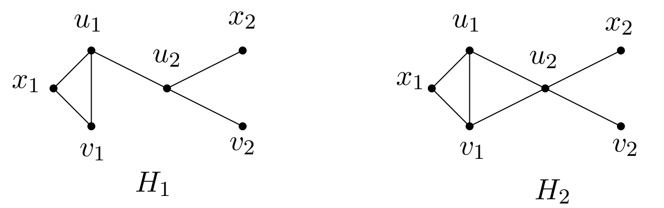

Figure 3. Two graphs G of order 6 with $\gamma_{cr}(G) = 3$.

Theorem 29. *Let G be a connected graph on $n \geq 4$ vertices, then $\gamma_{cr}(G) = n-3$ if and only if $G \in \bigcup_{i=1}^{7} \mathcal{G}_i$.*

Proof. Let $G \in \bigcup_{i=1}^{7} \mathcal{G}_i$. We deduce from (1), Corollary 26 and Theorem 27 that $\gamma_{cr}(G) \leq n-3$. If $G = K_{1,4}$, then $\gamma_{cr}(G) = 2 = n-3$ by Proposition 2, and if $G \in \mathcal{G}_1 \setminus \{K_{1,4}\}$ then $\gamma_{cr}(G) = 1 = n-3$ by Observation 6. If $G \in \mathcal{G}_2 \cup \mathcal{G}_3$, then we conclude from Observation 6 that $\gamma_{cr}(G) \geq 2 = n-3$ and so $\gamma_{cr}(G) = n-3$. If $G \in \{cor(P_3), cor(C_3)\}$, then by Proposition 9 and Theorem 28 we have $\gamma_{cr}(G) = \gamma(G) = 3$, and if $G \in \mathcal{G}_4 - \{cor(P_3), cor(C_3)\}$, then clearly $\gamma(G) = 2$ and Proposition 9 implies that $\gamma_{cr}(G) \geq \gamma(G) + 1 = 3 = n-3$ and so $\gamma_{cr}(G) = n-3$. If $G \in \mathcal{G}_5 \cup \mathcal{G}_6$, then it is easy to see that $\gamma_{cr}(G) = n-3$. Finally, if $G \in \mathcal{G}_7$, then by Theorem 25, we have $\gamma_{cr}(G) = 6 = n-3$.

Conversely, let $\gamma_{cr}(G) = n - 3$. By Corollary 5 and Theorem 17, we obtain $n \le 9$ and $\alpha'(G) \le 3$. If $\alpha'(G) = 1$, then G is the star $K_{1,n-1}$ and we conclude from Proposition 2 that $G = K_{1,4} \in \mathcal{G}_1$. Assume that $\alpha'(G) \ge 2$. Suppose M, X and S are the sets defined in the proof of Theorem 17. We consider the following cases.

Case 1. $\alpha'(G) = 3$. Since $n \le 9$, we must have $|X| \le 3$. If $|X| = 3$, then $n = 9$ and we conclude from Theorem 25 that $G \in \mathcal{G}_7$. Let $|X| \le 2$. By (2), we have $S = \emptyset$. As above, we may assume $N(x) \subseteq \{u_1, u_2, u_3\}$ for each $x \in X$. Consider the following subcases.

Subcase 1.1. $|X| = 2$. If $u_iy_1, u_iy_2 \in E(G)$ for some i, say $i = 1$, then the function $f_1 : V(G) \to \{0, 1, 2\}$ defined by $f_1(u_1) = 2, f_1(u_2) = f_1(u_3) = 1$ and $f_1(x) = 0$ otherwise, is clearly a co-Roman dominating function of G of weight 4 which is a contradiction. Thus each u_i has at most one neighbor in X. Assume without loss of generality that $u_1y_1, u_2y_2 \in E(G)$. If $y_1u_3 \in E(G)$ (the case $y_2u_3 \in E(G)$ is similar), then the function $f_2 : V(G) \to \{0, 1, 2\}$ defined by $f_2(u_1) = f_2(u_3) = 1, f_2(u_2) = 2$ and $f_2(x) = 0$ otherwise, is clearly a co-Roman dominating function of G of weight 4 which is a contradiction again. Hence $y_1u_3, y_2u_3 \notin E(G)$. It follows that $\deg(y_1) = \deg(y_2) = 1$. Considering the matching $M' = \{u_1y_1, u_2y_2, u_3v_3\}$ instead of M, we obtain $\deg(v_1) = \deg(v_2) = 1$. Since G is connected, we may assume, without loss of generality, that $u_1u_3 \in E(G)$. If $u_1v_3 \in E(G)$ or $u_2v_3 \in E(G)$, then the function $f_3 : V(G) \to \{0, 1, 2\}$ defined by $f_3(u_1) = f_3(u_2) = 2$ and $f_3(x) = 0$ otherwise, is clearly a co-Roman dominating function of G of weight 4, a contradiction. Therefore, $\deg(v_3) = 1$. Since G is connected, we conclude that G is a graph obtained from three paths $v_1u_1y_1$, $v_2u_2y_2$ and v_3u_3 by adding edges between u_1, u_2, u_3 such that the resulting graph is connected. Hence $G \in \mathcal{G}_6$.

Subcase 1.2. $|X| = 1$. Assume that $u_1y_1 \in E(G)$. If $y_1u_3 \in E(G)$ (the case $y_1u_2 \in E(G)$ is similar), then the function $f_4 : V(G) \to \{0, 1, 2\}$ defined by $f_4(u_1) = f_4(u_2) = f_4(u_3) = 1$ and $f_4(x) = 0$ otherwise, is clearly a co-Roman dominating function of G of weight 3 which is a contradiction. Hence $y_1u_3, y_1u_2 \notin E(G)$. Hence $\deg(y_1) = 1$. Regarding the matching $M' = \{u_1y_1, u_2v_2, u_3v_3\}$ instead of M, we have $\deg(v_1) = 1$. Since G is connected, we may assume that $u_1u_3 \in E(G)$. If $u_1v_3 \in E(G)$, then the function $h_1 : V(G) \to \{0, 1, 2\}$ defined by $h_1(u_1) = 2, h_1(u_2) = 1$ and $h_1(x) = 0$ otherwise, is clearly a co-Roman dominating function of G of weight 3, a contradiction. Therefore $u_1v_3 \notin E(G)$. Consider the following.

• $u_1u_2 \in E(G)$ (the case $u_1v_2 \in E(G)$ is similar). Then as above $u_1v_2 \notin E(G)$. If $v_2v_3 \in E(G)$, then the function $h_2 : V(G) \to \{0, 1, 2\}$ defined by $h_2(u_1) = 2, h_2(v_2) = 1$ and $h_2(x) = 0$ otherwise, is clearly a co-Roman dominating function of G of weight 3, a contradiction. Hence $v_2v_3 \notin E(G)$. If

$\{u_2u_3, u_2v_3, u_3v_2\} \subseteq E(G)$, then the function $h_3 : V(G) \to \{0,1,2\}$ defined by $h_3(u_1) = 2, h_3(u_2) = 1$ and $h_3(x) = 0$ otherwise, is clearly a co-Roman dominating function of G of weight 3, a contradiction. Thus $\{u_2u_3, u_2v_3, u_3v_2\} \nsubseteq E(G)$. It follows that $G \in \{G_1, G_2, G_3, G_4, G_5\}$ and so $G \in \mathcal{G}_5$.

• $u_1u_2, u_1v_2 \notin E(G)$. If $\{u_2, v_2, v_3\}$ induces a triangle, then the function $h_4 : V(G) \to \{0,1,2\}$ defined by $h_4(u_1) = 2, h_4(u_2) = 1$ and $h_4(x) = 0$ otherwise, is clearly a co-Roman dominating function of G of weight 3, a contradiction. Thus $\{u_2, v_2, v_3\}$ does not induce a triangle. As above we have $\{u_2u_3, u_2v_3, u_3v_2\} \nsubseteq E(G)$. Since G is connected, the graph induced by u_2, v_2, u_3, v_3 is connected. This implies that $G \in \{G_6, G_7, G_8, G_9, G_{10}\}$ and so $G \in \mathcal{G}_5$.

Subcase 1.3. $|X| = 0$. Then $n = 6$. Since $\gamma_{cr}(G) = 3$, we have $\Delta(G) \leq 4$ by Propositions 1 and 2. Hence $\gamma(G) \geq 2$. If $\gamma(G) = 3$, then we deduce from Theorem 28 that G is the corona $cor(P_3)$ or $cor(C_3)$ and so $G \in \mathcal{G}_4$. Assume $\gamma(G) = 2$. Then we conclude from Proposition 9 that every $\gamma(G)$-set S contains a vertex x such that x has no neighbor $x' \in V \setminus S$ with $pn(x, S) \subseteq N[x']$. It follows that $G \in \mathcal{G}_4$.

Case 2. $\alpha'(G) = 2$. First let $S \neq \emptyset$. We deduce from (2) that $|S| = 1$ and so $S = \{x_1\}$. Let $x_1u_1, x_1v_1 \in E(G)$. Then we assume that each other vertex of X is adjacent only to u_2. It follows that $\deg(x) = 1$ for each $x \in X \setminus \{x_1\}$. Since the function $g : V(G) \to \{0,1,2\}$ defined by $g(u_1) = 1, g(u_2) = 2$ and $g(x) = 0$ otherwise, is an co-Roman dominating function of G, we deduce that $n - 3 \leq 3$ and so $n \leq 6$. If $n = 6$, then clearly $X = \{x_1, y_1\}$. By considering the matching $M' = \{u_1v_1, u_2y_1\}$ instead of M, we have $\deg(v_2) = 1$. Since G is connected and $\gamma_{cr}(G) = 3$, u_2 must be adjacent to at least one vertex and at most two vertices in $\{u_1, v_1, x_1\}$. Thus G is a graph obtained from a triangle by adding a path P_3 and joining its center to at least one and at most two vertices of triangle and so $G \simeq H_1$ or H_2. Hence $G \in \mathcal{G}_4$. Assume that $n = 5$. Since G is connected, G is a graph obtained from a triangle and a path P_2 by adding some edges between them so that the resulting graph has at most one universal vertex. Thus $G \in \mathcal{G}_2$.

Now let $S = \emptyset$. As above, we may assume $N(x) \subseteq \{u_1, u_2\}$ for each $x \in X$. By Theorem 19, we have $\gamma_{cr}(G) \leq 4$ and this implies that $n \leq 7$. Thus $|X| \leq 3$. If $n = 4$, then we have $\gamma_{cr}(G) = 1$ yielding $G \in \{K_4, K_4 - e\} \subseteq \mathcal{G}_1$ by Observation 6. If $n = 5$, then G is a graph obtained from a path P_3 and a path P_2 by adding some edges between them such that the resulting graph is different from $DS(1, 2)$ and has at most one universal vertex. Thus $G \in \mathcal{G}_3$. Let $n \geq 6$. Since $\gamma_{cr}(G) \geq 3$, G has no vertex of degree $n - 1$ and so $\gamma(G) \geq 2$. Since $\{u_1, u_2\}$ is a dominating set, we have $\gamma(G) = 2$. If $n = 6$, then clearly $G \in \mathcal{G}_4$. Suppose $n = 7$. Then $X = \{y_1, y_2, y_3\}$. If u_i is adjacent to all vertices of X for some i, say $i = 1$, then the function $g : V(G) \to \{0,1,2\}$ defined by $g(u_1) = 2, g(u_2) = 1$ and $g(x) = 0$ otherwise, is an co-Roman dominating function of G of weight 3 which leads to a contradiction. Hence, each u_i is adjacent to at most two vertices in X. We may assume without loss of generality that $u_1y_1, u_1y_2, u_2y_3 \in E(G)$ and $u_1y_3 \notin E(G)$. Since $\{y_1, y_2, y_3, v_1\}$ is independent, we deduce that $\deg(y_3) = 1$. Considering the matching $M' = \{u_1v_1, u_2y_3\}$ instead of M, we obtain $\deg(v_2) = 1$. Since $\gamma_{cr}(G) = 4$, u_2 is adjacent to at most one vertex in $\{y_1, y_2, v_1\}$. Thus G is a connected graph obtained from $K_{1,3}$ and a path P_3 by joining the center of P_3 to the center or at most one leaf of $K_{1,3}$. This implies that $G \in \{G_{11}, G_{12}, G_{13}\}$ and so $G \in \mathcal{G}_5$. This completes the proof. ■

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