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GLOBAL DOMINATOR COLORING OF GRAPHS

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Abstract

Let $S \subseteq V$. A vertex $v \in V$ is a *dominator* of S if v dominates every vertex in S and v is said to be an *anti-dominator* of S if v dominates none of the vertices of S. Let $\mathcal{C} = (V_1, V_2, \ldots, V_k)$ be a coloring of G and let $v \in V(G)$. A color class V_i is called a *dom-color class* or an *anti dom-color class* of the vertex v according as v is a dominator of V_i or an anti-dominator of V_i. The coloring $\mathcal{C}$ is called a *global dominator coloring* of G if every vertex of G has a dom-color class and an anti dom-color class in $\mathcal{C}$. The minimum number of colors required for a global dominator coloring of G is called the *global dominator chromatic number* and is denoted by $\chi_{gd}(G)$. This paper initiates a study on this notion of global dominator coloring.

Keywords: global domination, coloring, global dominator coloring, dominator coloring.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite, non-trivial, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n

and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [3].

A subset S of V is called a *dominating set* of G if every vertex in $V \setminus S$ is adjacent to a vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G. A set D of vertices is said to be a *global dominating set* of G if D is a dominating set of both G and $\overline{G}$. The *global domination number* $\gamma_g(G)$ is the minimum cardinality of a global dominating set of G. For more details on domination related parameters, see [6]. A *coloring* of a graph G is an assignment of colors to the vertices of G in such a way that no two adjacent vertices receive the same color. In other words, a coloring of G is a partition $(V_1, V_2, \ldots, V_k)$ of $V(G)$ into independent sets; here $V_i^{'}$s are called the *color classes*. The *chromatic number* $\chi(G)$ is the minimum number of colors required for a coloring of G and such a coloring is called χ-*coloring* of G. Let $(V_1, V_2, \ldots, V_k)$ be a coloring of G. A vertex $v \in V_i$ is called *solitary* if $|V_i| = 1$.

Several concepts relating domination and coloring have been introduced and well-studied. For example, Fall coloring [7], dominating-χ-coloring [2], dominator coloring [5] and chromatic transversal domination [9] are some such concepts. A *dominator coloring* of a graph G is a coloring of G in which every vertex dominates every vertex of at least one color class. The minimum number of colors required for a dominator coloring of G is called the *dominator chromatic number* of G and is denoted by $\chi_d(G)$ and a dominator coloring that uses χ_d colors is called a χ_d-*coloring* of G.

This paper introduces such a variation connecting coloring and global domination namely global dominator coloring. If $S \subseteq V$, we say that a vertex $v \in V$ is a *dominator* of S if v dominates every vertex in S and v is said to be an *anti-dominator* of S if v dominates none of the vertices of S. Let $\mathcal{C} = (V_1, V_2, \ldots, V_k)$ be a coloring of G and let $v \in V(G)$. A color class V_i is called a *dom-color class* or an *anti dom-color class* of the vertex v according as v is a dominator of V_i or an anti-dominator of V_i. With these terminologies, a dominator coloring is a coloring with the property that every vertex has a dom-color class. We define a coloring $\mathcal{C}$ of G to be a *global dominator coloring* of G if every vertex of G has both a dom-color class and an anti dom-color class in $\mathcal{C}$. The minimum number of colors required for a global dominator coloring of G is called the *global dominator chromatic number* of G and is denoted by $\chi_{gd}(G)$.

As a vertex v dominates itself, the vertex v is a dominator of $\{v\}$, whereas it is not an anti-dominator of $\{v\}$. Hence a graph G does not admit a global dominator coloring when $\Delta(G) = n-1$. When $\Delta(G) < n-1$, the trivial coloring (that assigns distinct colors to distinct vertices) would serve as a global dominator coloring. Thus, a graph G admits a global dominator coloring if and only if $\Delta(G) < n-1$, and so throughout this paper, all the graphs G for which $\chi_{gd}(G)$ is discussed are assumed to have maximum degree at most $|V(G)| - 2$.

2. Common Classes of Graphs

Here, we determine the value of χ_{gd} for some common classes of graphs such as paths, cycles, complete multipartite graphs and the Petersen graph. For this, we state the following propositions proved in [4].

Proposition 1 [4]**.** *The path P_n of order $n \geq 2$ has*

$$\chi_d(P_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil + 1 & \text{if } n = 2, 3, 4, 5, 7, \\ \left\lceil \frac{n}{3} \right\rceil + 2 & \textit{elsewhere}. \end{cases}$$

Proposition 2 [4]**.** *The cycle C_n of order $n \geq 4$ has*

$$\chi_d(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil & \text{if } n = 4, \\ \left\lceil \frac{n}{3} \right\rceil + 1 & \text{if } n = 5, \\ \left\lceil \frac{n}{3} \right\rceil + 2 & \textit{elsewhere}. \end{cases}$$

The following lemma is an useful tool in determining the value of χ_{gd} for several graphs. Note that a global dominator coloring of a graph G is obviously a dominator coloring of G and so one has $\chi_d(G) \leq \chi_{gd}(G)$. Further, $\chi_{gd}(G) = \chi_d(G)$ if and only if there is a χ_d-coloring of G that is also a global dominator coloring for G.

Lemma 3. *If G is a connected graph with $\chi_d(G) \geq \Delta(G) + 2$, then $\chi_{gd}(G) = \chi_d(G)$.*

Proof. Consider a χ_d-coloring $(V_1, V_2, \ldots, V_{\chi_d})$ of G. If u is an arbitrary vertex of G, then it has a dom-color class in this coloring. Further, for instance if $u \in V_1$, then at most $\Delta(G)$ color classes other than V_1 can have a neighbour of u. But $\chi_d(G) \geq \Delta(G) + 2$. Therefore, there is a color class containing no neighbour of the vertex u; this class would serve as an anti dom-color class of u. Hence this χ_d-coloring is also a global dominator coloring of G so that $\chi_{gd}(G) = \chi_d(G)$. ■

Corollary 4. (i) *For the path P_n on $n \geq 4$ vertices, we have*

$$\chi_{gd}(P_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil + 1 & \text{if } n = 7, \\ \left\lceil \frac{n}{3} \right\rceil + 2 & \textit{elsewhere}. \end{cases}$$

(ii) *For the cycle C_n on $n \geq 4$ vertices, $\chi_{gd}(C_n) = \lceil n/3 \rceil + 2$.*

Proof. For $n = 4, 5$, the proof is a simple verification. Further, if G is either a path or a cycle on $n \geq 6$ vertices, it follows from the Propositions 1 and 2 that $\chi_d(G) \geq \Delta(G) + 2$ and so the result follows from Lemma 3. ■

Corollary 5. *The global dominator chromatic number of the Petersen graph PG is* 5.

Proof. Since $\Delta(PG) = 3$ and $\chi_d(PG) = 5$, by Lemma 3 $\chi_{gd}(PG) = 5$. ■

Theorem 6. *The global dominator chromatic number of a complete m-partite graph is $2m$.*

Proof. Let G be a complete m-partite graph with partition $(X_1, X_2, \ldots, X_m)$. By our convention, $\Delta(G) < |V(G) - 1|$; that is, $|X_i| \geq 2$, for each $i = 1, 2, \ldots, m$. Now for each i with $1 \leq i \leq m$, choose a vertex in X_i, say x_i. Then $(\{x_1\}, \{x_2\}, \ldots, \{x_m\}, V_1 \backslash \{x_1\}, V_2 \backslash \{x_2\}, \ldots, V_m \backslash \{x_m\})$ is a global dominator coloring of G so that $\chi_{gd}(G) \leq 2m$. Since a part cannot be an anti dom-color class of a vertex lying in a different part it follows that $\chi_{gd}(G) \geq 2m$. ■

For disconnected graphs G, $\chi_{gd}(G)$ coincides with $\chi_d(G)$ as shown below.

Proposition 7. *For a disconnected graph G, we have $\chi_{gd}(G) = \chi_d(G)$.*

Proof. Under any χ_d-coloring of a disconnected graph G, for each component of G, there exists a color class that intersects only the vertex set of this component. In other words, for a vertex v of G belonging to a component G_i, there will be a color class V_i which does not intersect $V(G_i)$ so that V_i is an anti dom-color class of v. As a result, every χ_d-coloring of G is a global dominator coloring as well. Hence $\chi_{gd}(G) = \chi_d(G)$. ■

3. Bounds

In this section, we characterize the graphs for which $\chi_{gd} = 2$ and $\chi_{gd} = 3$. We also establish some bounds for the global dominator chromatic number.

Lemma 8. *Let G be a graph. Then $\chi_{gd}(G) = 2$ if and only if $G = \overline{K_2}$.*

Proof. Suppose $\chi_{gd}(G) = 2$. Let (V_1, V_2) be a global dominator coloring of G. If $|V_1| \geq 2$, then V_1 cannot be a dom-color class of a vertex $v \in V_1$ so that V_2 is the only dom-color class of the vertex v. This implies that the vertex v has no anti dom-color class, which is a contradiction. Hence V_1 and V_2 are singleton sets, say $V_1 = \{x\}$ and $V_2 = \{y\}$. Certainly, x and y are non-adjacent and therefore $G = \overline{K_2}$. Converse is obvious. ■

Lemma 9. *Let G be a graph. Then $\chi_{gd}(G) = 3$ if and only if $G \in \{K_{a,b} \cup K_1, \overline{K_3}, K_{1,r} \cup K_{1,s}\}$, where $a, b, r, s \in Z^+$.*

Proof. Suppose (V_1, V_2, V_3) be a χ_{gd}-coloring of G. We first prove that at least one color class must be singleton. If not, then $|V_i| \geq 2$, for each $i = 1, 2, 3$. Consider a vertex u in V_1. Then V_1 cannot be a dom-color class of u. Therefore, one of the remaining color classes must be a dom-color class of u and the other one would be an anti dom-color class of u. Let us assume without loss of generality that V_2 is the dom-color class of u and V_3 is the anti dom-color class of u. Then V_2 is the only dom-color class of a vertex $w \in V_3$. Thus every vertex of V_2 has a neighbour in V_1 and a neighbour in V_3 which implies that no vertex of V_2 has an anti dom-color class, which is a contradiction.

If each V_i, where $1 \leq i \leq 3$, is singleton, then G is either $\overline{K_3}$ or $K_1 \cup K_{1,1}$ as $\Delta(G) < |V(G)| - 1$. Suppose exactly two color classes are singleton, say V_1 and V_2. Let $V_1 = \{u\}$ and $V_2 = \{v\}$. Now, if u is adjacent to all the vertices of V_3, then V_2 is the only anti dom-color class of u and also for each vertex of V_3. Therefore G is isomorphic to $K_1 \cup K_{1,b}$, where $b \geq 2$. If u is adjacent to none of the vertices of V_3, then V_2 is the dom-color class of each vertex in V_3 and also V_1 is the anti dom-color class of v. Therefore G is isomorphic to $K_1 \cup K_{1,b}$, where $b \geq 2$. If u has some neighbours and non-neighbours in V_3, then V_2 is the dom-color class of non-neighbours of u in V_3 and V_1 is the anti dom-color class of v. Therefore G is isomorphic to $K_{1,s} \cup K_{1,t}$, where $s, t \geq 1$. Suppose exactly one color class is singleton, say $V_1 = \{u\}$. As u has an anti dom-color class, $N(u)$ can intersect at most one of the color classes V_2 and V_3. Further, if $N(u)$ intersects exactly one of V_2 and V_3, say V_2; let $v \in N(u) \cap V_2$. Then V_3 is the anti dom-color class of v. This means that no vertex of V_3 has a dom-color class, a contradiction. Hence u is an isolate vertex of G which in turn implies that $\langle V_2 \cup V_3 \rangle$ is a complete bipartite graph and so G is isomorphic to $K_1 \cup K_{r,s}$, where $r, s > 1$. Converse is a simple verification. ∎

Theorem 10. *For any connected graph G of order $n \geq 4$, we have $4 \leq \chi_{gd}(G) \leq n$. Further, given integers k and n with $4 \leq k \leq n$, there exists a connected graph G of order n with $\chi_{gd}(G) = k$.*

Proof. The inequalities follow from Lemma 8 and Lemma 9. Now, suppose n and k are the integers with $4 \leq k \leq n$. We construct a required graph G as follows. Consider the complete graph K_{k-2} with the vertex set $\{v_1, v_2, \ldots, v_{k-2}\}$. Now, attach a pendant edge at exactly one of the vertices of K_{k-2}, say v_1; let x_1 be the corresponding pendant vertex. Now, attach $n - k + 1$ pendant edges at one of the vertices of K_{k-2} other than v_1, say v_2; let $x_2, x_3, \ldots, x_{n-k+2}$ be the corresponding pendant vertices. Let G be the resultant graph. For $n = 12$ and $k = 8$, the graph G is given in Figure 1. Now, $(\{v_1\}, \{v_2\}, \ldots, \{v_{k-2}\}, \{x_1\}, \{x_2, x_3, \ldots, x_{n-k+2}\})$ is a global dominator coloring of G so that $\chi_{gd}(G) \leq k$. Since K_{k-2} is a subgraph of G, we need at least $k - 2$ colors to color the vertices of G. Also, to get an anti dom-color class for v_1, we should give an unique color to at least one of the pendant vertices of $x_2, x_3, \ldots, x_{n-k+2}$. Further, x_1 needs a unique color as it is the only non-neighbour of v_2. This gives the inequality $\chi_{gd}(G) \geq k$. ■

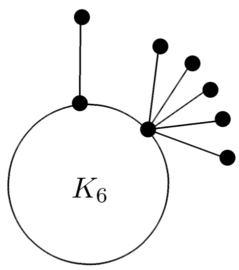

Figure 1. A graph G of order 12 with $\chi_{gd}(G) = 8$.

Even if the value of χ_{gd} for a graph G of order n is ranging from 4 to n, the upper bound of $\chi_{gd}(G)$ is substantially reduced in the case when G belongs to the class of all bipartite graphs as shown in the following theorem.

Theorem 11. *Let G be a connected bipartite graph on $n \geq 4$ vertices. Then $4 \leq \chi_{gd}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor + 2$ and these bounds are sharp.*

Proof. Let G be a connected bipartite graph with partition (V_1, V_2) with $|V_1| \geq |V_2|$. Certainly, $|V_2| \geq 2$ as $\Delta(G) < n - 1$. Now, assign an unique color to each vertex of V_2 and also to exactly one vertex of V_1. Also, assign a new color to all the remaining vertices of V_1. Then this assignment of colors is a global dominator coloring of G with $|V_2| + 2$ colors so that $\chi_{gd}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor + 2$. For complete bipartite graphs, the value of χ_{gd} is 4.

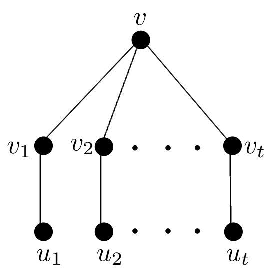

Figure 2. A graph G of order n with $\chi_{gd}(G) = \left\lfloor \frac{n}{2} \right\rfloor + 2$.

For the sharpness of the upper bound, consider the graph G obtained from a star $K_{1,t}$, where $t \geq 2$, by attaching exactly one edge at each pendant vertex of the star. Let the vertices of G be labeled as given in Figure 2. It is clear that $(\{v\}, \{v_1\}, \{v_2\}, \ldots, \{v_t\}, \{u_1, u_2, \ldots, u_t\})$ is a global dominator coloring of

G so that $\chi_{gd}(G) \le t+2 = \lfloor \frac{n}{2} \rfloor + 2$. For the other inequality, consider a global dominator coloring $\mathcal{C}$ of G. Then, for all i, $1 \le i \le t$, either v_i or u_i is solitary. Suppose each v_i is solitary with respect to $\mathcal{C}$. Then the vertex v must receive a new color $t+1$ in $\mathcal{C}$. Also, in order to get an anti dom-color class for v, at least one of the pendant vertices must receive a new color $t+2$ in $\mathcal{C}$ and so $\chi_{gd}(G) \ge t+2 = \lfloor \frac{n}{2} \rfloor + 2$. On the other hand, if at least one v_i is not solitary, say v_1 is not solitary, then u_1 must be solitary. Also, for each i with $2 \le i \le t$, one of u_i and v_i is solitary. Hence, we need at least t colors to color the vertices of $G - \{v, v_1\}$. Thus G needs at least $t+2$ colors as $vv_1 \in E(G)$. ■

4. Relationship

Here, we discuss some relationships of the parameter χ_{gd} with the parameters χ, χ_d and γ_g. We establish the following two such relations.

Theorem A. *For any graph G, we have* $\max\{\gamma_g(G), \chi(G)+1\} \le \chi_{gd}(G) \le \chi(G) + \gamma_g(G)$.

Theorem B. *For any graph G, we have $\chi_d(G) \le \chi_{gd}(G) \le 2\chi_d(G)$.*

We prove these theorems with the aid of the following lemmas. We say that a vertex v is a *colorful vertex* with respect to a coloring $\mathcal{C}$ of a graph G if v has a neighbour in every color class of $\mathcal{C}$ other than the color class where v lies. Here, we say that the coloring $\mathcal{C}$ admits such a vertex.

Lemma 12. *Every χ-coloring of G admits a colorful vertex.*

Proof. Suppose $\mathcal{C} = (V_1, V_2, \ldots, V_\chi)$ is a χ-coloring of G not admitting a colorful vertex. In particular, no vertex of V_1 is colorful with respect to $\mathcal{C}$. That is, every vertex belonging to V_1 has an anti dom-color class and consequently we can have a coloring of G with $\chi - 1$ colors by putting each vertex of V_1 into one of its anti dom-color classes. ■

Corollary 13. *For any graph G, we have $\chi(G) + 1 \le \chi_{gd}(G)$.*

Proof. Certainly, no global dominator coloring of G admits a colorful vertex. So, by Lemma 12, there is no global dominator coloring using χ colors and so the inequality follows. ■

Lemma 14. *Let G be any graph. Then $\gamma_g(G) \le \chi_{gd}(G)$.*

Proof. Consider a χ_{gd}-coloring of G. Choose exactly one vertex from each color class. Let D be the set of those vertices. Now, it is enough to show that the set D is a global dominating set of G. This is certainly true because each vertex

$v \in V(G) \setminus D$ has both a dom-color class and an anti dom-color class and so v has a neighbour as well as a non-neighbour in D. ■

Lemma 15. *For any graph G, we have $\chi_{gd}(G) \leq \gamma_g(G) + \chi(G)$.*

Proof. Let $(V_1, V_2, \ldots, V_\chi)$ be a χ-coloring of G. Consider a minimum global dominating set D of G. Let $D_i = D \cap V_i$ for each i with $1 \leq i \leq \chi$. Consider the coloring $\mathcal{C} = \{\{x\} : x \in D\} \cup \{V_1 \setminus D_1, \ldots, V_\chi \setminus D_\chi\}$ of G. Let v be an arbitrary vertex of G. If v is not in D, then it has a neighbour as well as non-neighbour in D, say x and $x^{'}$, respectively, and so $\{x\}$ is a dom-color class of v and $\left\{x^{'}\right\}$ is an anti dom-color class of v. On the other hand, if v lies in D, then $\{v\}$ is a dom-color class of itself. Certainly, $v \in V_i$ for some $i = 1, 2, \ldots, \chi$. Now, if $|D_i| > 1$, then the singleton color class (under the coloring $\mathcal{C}$) consisting of a vertex of D_i other than v, is an anti dom-color class of v. Suppose $|D_i| = 1$. Now, if $|V_i| \geq 2$, then $V_i \setminus D_i$ is an anti dom-color class of v. Therefore $\mathcal{C}$ serves a global dominator coloring of G if for no i with $1 \leq i \leq \chi$, it is not true that $|V_i| = |D_i| = 1$. Further, when $|V_i| = |D_i| = 1$ for at least one i, the coloring $\mathcal{C}$ need not be a global dominator coloring of G. In this case, we look for a new coloring as follows. If, for instance, $V_i = D_i = \{u_i\}$ for $i = 1, 2, \ldots, k$, let $u_i^{'}$ be a non-neighbour of u_i as $\Delta(G) < n - 1$. If $u_i^{'} \in D$ for all i, then $\mathcal{C}$ serves a global dominator coloring of G. In case, at least one such $u_i^{'}$ is not in D, say, for instance $u_i^{'} \notin D$ for all i with $1 \leq i \leq l \leq k$ and $u_j^{'} \in D$ for all $j > l$. In this case, one can verify that the coloring $\mathcal{C}^{'} = \big\{\{x\} : x \in D\big\} \cup \big\{\{u_i^{'}\} : 1 \leq i \leq l\big\} \cup \big\{V_{k+1} \setminus D_{k+1} \setminus U^{'}, \ldots, V_\chi \setminus D_\chi \setminus U^{'}\big\}$, where $U^{'} = \{u_i^{'} : 1 \leq i \leq l\}$ is a global dominator coloring of G (of course, some of the sets $V_{k+1} \setminus D_{k+1} \setminus U^{'}, \ldots, V_\chi \setminus D_\chi \setminus U^{'}$ may be empty; in that case we can manage with less number of colors). Hence the result. ■

Now, Theorem A is an immediate consequence of Corollary 13, Lemma 14 and Lemma 15.

***Proof of Theorem* B.** The first inequality is already seen. We proceed to prove the rest. For this, consider a χ_d-coloring $(V_1, V_2, \ldots, V_{\chi_d})$ of G. Now, for each i with $1 \leq i \leq \chi_d$, choose a vertex in V_i, say v_i. Suppose $|V_i| \geq 2$, for all i with $1 \leq i \leq \chi_d$. Then $\{V_i \setminus \{v_i\} : 1 \leq i \leq \chi_d\} \cup \{\{v_i\} : 1 \leq i \leq \chi_d\}$ is a global dominator coloring of G. Suppose $|V_i| = 1$, for all i with $1 \leq i \leq k < \chi_d$. Consider a non-neighbour $v_i^{'}$ of v_i for all i with $1 \leq i \leq k$. Then $\{V_i : 1 \leq i \leq k\} \cup \big\{V_i \setminus \{v_i\} \setminus U^{'} : k+1 \leq i \leq \chi_d\big\} \cup \big\{\{v_i\} : k+1 \leq i \leq \chi_d\big\} \cup \big\{\{v_i^{'}\} : 1 \leq i \leq k\big\}$, where $U^{'} = \{v_i^{'} : 1 \leq i \leq k\}$ is a global dominator coloring of G (of course some of the sets $V_{k+1} \setminus \{v_{k+1}\} \setminus U^{'}, \ldots, V_{\chi_d} \setminus \{v_{\chi_d}\} \setminus U^{'}$ may be empty; in that case we can manage with less number of colors). Hence $\chi_{gd}(G) \leq 2\chi_d(G)$. ■

The bounds established in Theorem A and Theorem B are sharp. That is, there exist graphs G_1, G_2, G_3, G_4 and G_5 such that

(a) $\chi(G_1)+1 > \gamma_g(G_1)$ and $\chi(G_1)+1 = \chi_{gd}(G_1)$.
(b) $\gamma_g(G_2) > \chi(G_2)+1$ and $\gamma_g(G_2) = \chi_{gd}(G_2)$.
(c) $\chi_{gd}(G_3) = \chi(G_3)+\gamma_g(G_3)$.
(d) $\chi_{gd}(G_4) = \chi_d(G_4)$.
(e) $\chi_{gd}(G_5) = 2\chi_d(G_5)$.

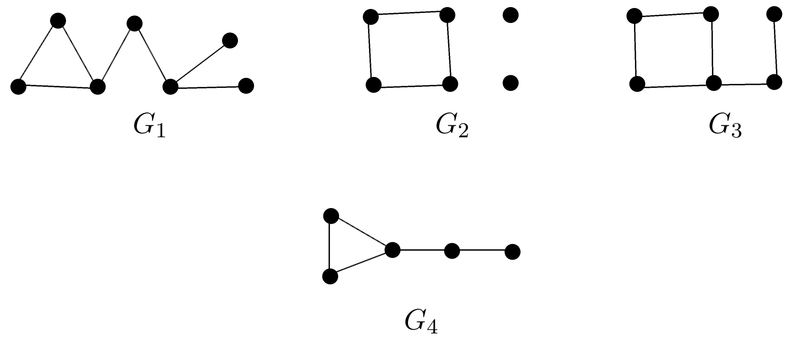

Figure 3. Examples of graphs achieving the bounds of Theorem A and Theorem B.

The graphs G_1 to G_4 are given in Figure 3. Let the graph G_5 be a complete multipartite graph. Also, it seems that complete multipartite graphs are the only graphs in which the global dominator chromatic number equals twice the dominator chromatic number. So we pose the following Conjecture.

Conjecture 16. *Let G be a graph with $\Delta(G) < n-1$. Then $\chi_{gd}(G) = 2\chi_d(G)$ if and only if G is a complete multipartite graph.*

5. Trees

Here, we determine the value of χ_{gd} for trees in terms of γ_g and χ_d independently as shown in Theorem C and Theorem D. In this connection, we define the following classes of trees.

I. Let $\Im_1$ be the collection of all trees T of diameter 4 which are constructed from two or more stars with at least two vertices by joining the centers of these stars to a common vertex.

II. Let $\Im_2$ be the collection of all trees T of diameter 4 that are constructed as follows. Consider $r \geq 2$ stars $K_{1,t_1}, K_{1,t_2}, \ldots, K_{1,t_r}$, where $t_i \geq 2$ and let $v_1, v_2, \ldots, v_r$ be the respective centers. Now, consider another star $K_{1,t}$, where $t \geq 1$ and let v be its center. Join the vertex v with all the centers $v_1, v_2, \ldots, v_r$. Let T be resultant tree.

Some trees belonging to the classes $\Im_1$ and $\Im_2$ are given in Figure 4.

Theorem C. *For any tree T, the value of χ_{gd} is either $\gamma_g(T)+1$ or $\gamma_g(T)+2$.*

Theorem D. *For a tree T, $\chi_{gd}(T)$ is either $\chi_d(T)$ or $\chi_d(T)+1$. Further, $\chi_{gd}(T)=\chi_d(T)+1$ if and only if T is either a double star or $T\in\Im_1\cup\Im_2$.*

Theorem C is an immediate consequence of Lemma 15 and the following lemma.

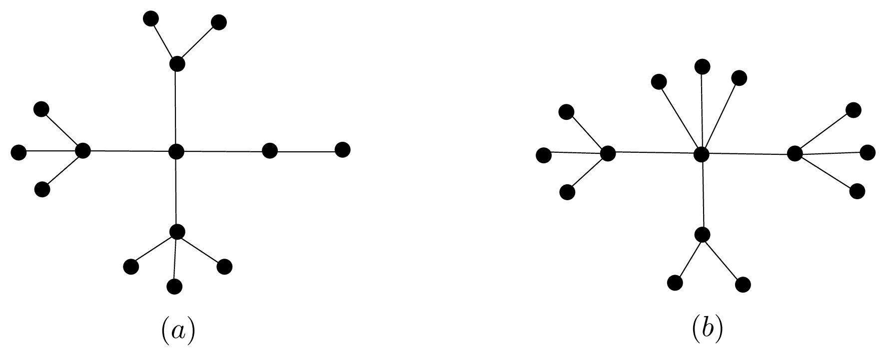

Figure 4. (a) A tree in $\Im_1$, (b) a tree in $\Im_2$.

Lemma 17. *Let G be a graph with $\delta(G)=1$. Then $\chi_{gd}(G)\ge\gamma_g(G)+1$ and the bound is sharp.*

Proof. Consider a χ_{gd}-coloring $\big(V_1,V_2,\ldots,V_{\chi_{gd}}\big)$ of G. Let v be a support vertex of G and let u be one of its pendant neighbour. Assume without loss of generality that $u\in V_1$ and $v\in V_2$. Now for each $i\ge 2$, choose a vertex v_i from V_i. Let $S=\big\{v=v_2,v_3,\ldots,v_{\chi_{gd}}\big\}$. It is enough to prove that S is a global dominating set of G.

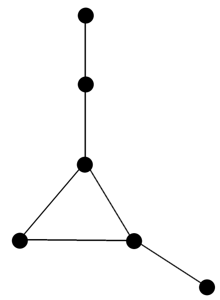

Figure 5. A graph G with $\delta(G)=1$ and $\chi_{gd}(G)=\gamma_g(G)+1$.

Let $x\in V\setminus S$. Suppose $x\in V_1$. Now, if $x=u$, then v is the neighbour of u in S and every vertex of S other than v is a non-neighbour of u. If $x\neq u$, then

$|V_1| \geq 2$ and so V_1 cannot be a dom-color class of x. Further, V_1 is not an anti-dom-color class of x. This means that x has a neighbour and a non-neighbour in S.

Now, suppose $x \notin V_1$, say $x \in V_k$, where $k > 1$. Then $|V_k| \geq 2$ as $x \in V \setminus S$ and so the vertex v_k is a non-neighbour of x in S. Also if V_l is a dom-color class of x, then $l \notin \{1, k\}$ and so v_l is a neighbour of x in S. Thus every vertex outside S has both a neighbour and a non-neighbour in S. That is, S is a global dominating set of G. Sharpness of the bound follows from Figure 5. ■

Now, Theorem D is proved with the aid of the following lemmas.

Lemma 18. *If G is a graph with $\delta(G) = 1$, then $\chi_{gd}(G) \leq \chi_d(G) + 2$.*

Proof. Consider a χ_d-coloring of G. Let v be a support vertex of G and let u be a non-neighbour of v in G. Consider a pendant neighbour w of v. Now recolor w with the color χ_d+1 and u with the color χ_d+2 and keep the colors of the remaining vertices unchanged. Let $\mathcal{C}$ be the resultant coloring. We now claim that $\mathcal{C}$ is a global dominator coloring of G. Clearly, every vertex of G has a dom-color class in $\mathcal{C}$. Now, $\{w\}$ is an anti dom-color class for every vertex of G other than v and w. Also, for v and w, $\{u\}$ is an anti dom-color class. This implies that $\mathcal{C}$ is a global dominator coloring and so $\chi_{gd}(G) \leq \chi_d(G) + 2$. ■

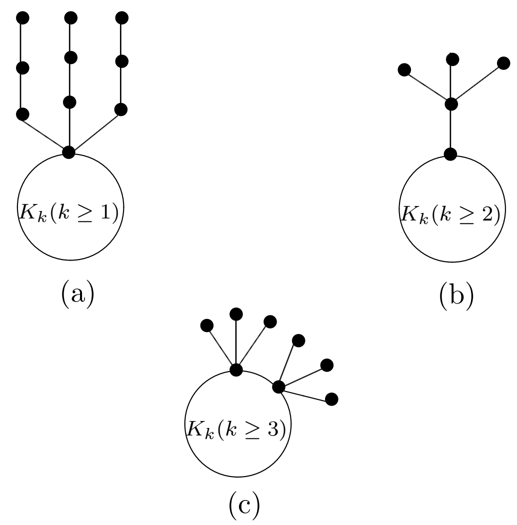

Figure 6. (a) A graph G with $\chi_{gd}(G) = \chi_d(G)$.
(b) A graph G with $\chi_{gd}(G) = \chi_d(G) + 1$.
(c) A graph G with $\chi_{gd}(G) = \chi_d(G) + 2$.

As for any graph G, $\chi_{gd}(G) \geq \chi_d(G)$, by Lemma 18, the value of $\chi_{gd}(G)$ for a graph G with $\delta(G) = 1$ is either $\chi_d(G)$ or $\chi_d(G) + 1$ or $\chi_d(G) + 2$. Further, graphs for each of these possibilities are shown in Figure 6. However, there is no tree for which the bound is attained as shown in the following lemma.

It is shown in [8] that every tree T admits a χ_d-coloring in which every support vertex is solitary and all the pendant vertices of T have the same color.

Lemma 19. *If T is a tree, then $\chi_{gd}(T)$ is either $\chi_d(T)$ or $\chi_d(T)+1$.*

Proof. Consider a χ_d-coloring $\mathcal{C}$ of T in which every support vertex is solitary and all the pendant vertices of T have the same color. Consider a support vertex v of T. Recolor all the pendant neighbours of v with the new color χ_d+1 and keep the colors of remaining vertices unchanged. Let $\mathcal{C}'$ be the resultant coloring. We claim that $\mathcal{C}'$ is a global dominator coloring of T. Clearly, every vertex of T has a dom-color class in $\mathcal{C}'$. Also, for every vertex of T other than v has the χ_d+1-color class as an anti dom-color class. For the vertex v, if there exists a support vertex u that is not adjacent to v, then $\{u\}$ is an anti dom-color class of v in $\mathcal{C}'$. On the other hand, if every support vertex of T is adjacent to v, then every non-neighbour of v is a pendant vertex in T. This shows the color class of $\mathcal{C}'$ which contains the pendant vertices of T other than the pendant neighbours of v becomes an anti dom-color class for v. Thus $\mathcal{C}'$is a global dominator coloring of G and so $\chi_{gd}(T)\leq\chi_d(T)+1$. ■

The following two theorems concerning the value of χ_d for trees are proved in [1] and [8], respectively.

Theorem 20 [1]**.** *If G is a graph with $\delta(G)=1$ and k support vertices, then $\chi_d(G)\geq k+1$, and $\chi_d(G)=k+1$ if and only if the set of non-support vertices is an independent dominating set of G.*

Theorem 21 [8]**.** *Let T be a nontrivial tree. Then for every χ_d-coloring of T, either each support is solitary or it is adjacent to exactly one pendant and that pendant is solitary.*

Lemma 22. *If $T\in\Im_1\cup\Im_2$, then $\chi_{gd}(T)=\chi_d(T)+1$.*

Proof. Suppose $T\in\Im_1$. Let $v_1,v_2,\ldots,v_r$, where $r\geq 2$, be the support vertices of T and let v be the vertex of T that is adjacent to all the supports. We claim that every support vertex is solitary in any χ_d-coloring of T. If not, consider a χ_d-coloring $\mathcal{C}$ of T in which not all support vertices are solitary. If $v_1,v_2,\ldots,v_l$ ($l\leq r$) are the support vertices of T which are not solitary in $\mathcal{C}$, then by Theorem 21, each of these support vertices is adjacent to exactly one pendant vertex and that pendant is solitary in $\mathcal{C}$; let them be $u_1,u_2,\ldots,u_l$, respectively. That is, $u_1,u_2,\ldots,u_l,v_{l+1},v_{l+2},\ldots,v_r$ are solitary in $\mathcal{C}$. Since v and v_1 are adjacent in T, they must receive two different colors other than the colors of the vertices $u_1,u_2,\ldots,u_l,v_{l+1},v_{l+2},\ldots,v_r$. Therefore $\chi_d(T)\geq r+2$, which is a contradiction to Theorem 20. So, what we have proved is that in every χ_d-coloring of T, all the support vertices are solitary. Therefore, under any χ_d-coloring of T, the vertex v does not have an anti dom-color class and consequently $\chi_{gd}(T) > \chi_d(T)$. This implies from Lemma 19 that $\chi_{gd}(T) = \chi_d(T) + 1$.

Suppose $T \in \Im_2$. Let $v_1, v_2, \ldots, v_r$ and v be the support vertices of T as described in the construction of $\Im_2$. By Theorem 20, $\chi_d(T) = r + 2$. Now, since each v_i has at least two pendant neighbours, by Theorem 21, they are solitary in any χ_d-coloring of T. Therefore the vertex v must receive a new color. Certainly, one of the pendant neighbours of v also gets a new color. So, the vertex v is adjacent to a vertex in every color class under any χ_d-coloring of T. That is, there is no χ_d - coloring for T that is also a global dominator coloring so that by Lemma 19, $\chi_{gd}(T) = \chi_d(T) + 1$. ∎

Lemma 23. *If a tree T has two support vertices with the property that the distance between the support vertices is at least three, then $\chi_d(T) = \chi_{gd}(T)$.*

Proof. Let u and v be two support vertices in T such that $d(u, v) \geq 3$. Then there is no vertex in T which is adjacent to both u and v. Consider a χ_d-coloring $\mathcal{C}$ of T in which u and v are solitary. Then $\mathcal{C}$ is a global dominator coloring of T. If a vertex x of T does not lie on $N[u]$, then $\{u\}$ is an anti dom-color class of x and if x does not lie on $N[v]$, then $\{v\}$ is an anti dom-color class of x. Hence the result follows. ∎

Converse of the above lemma is not true. For the tree T given in Figure 7, $\chi_d(T) = \chi_{gd}(T)$. But the distance between every pair of support vertices is at most 2 in T.

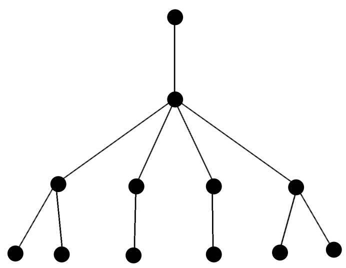

Figure 7. A counterexample to the converse of the Lemma 23.

***Proof of Theorem* D.** By Lemma 19, $\chi_{gd}(T)$ is either $\chi_d(T)$ or $\chi_d(T)+1$. Now, suppose $\chi_{gd}(T) = \chi_d(T) + 1$. Then by Lemma 23, the distance between every pair of support vertices is at most 2 in T. Therefore either there exists exactly one vertex in T which is adjacent to all the support vertices of T or there exists exactly one support vertex which is adjacent to all the remaining support vertices of T. If the earlier case happens, then T is a tree belonging to the family $\Im_1$. If the later case happens, let v be the respective support vertex (that is adjacent to all the remaining support vertices of T). Let k be the number of support vertices of T other than v. If $k = 1$, then T is a double star. Assume $k \geq 2$. In this case we prove that $T \in \Im_2$. Let $v_1, v_2, \ldots, v_k$ be the support vertices of T other than v. So, we need to prove that each v_i has at least two pendant neighbours. If not, let v_1 be a support vertex having exactly one pendant neighbour, say u_1. Now, consider a χ_d-coloring of T in which every support vertex is solitary. In particular, the vertex v_1 has a unique color. Let $\mathcal{C}$ be the coloring of T obtained from this χ_d-coloring by interchanging the colors of u_1 and v_1. Certainly, $\mathcal{C}$ is a dominator coloring of T. Also, the color class $\{u_1\}$ of $\mathcal{C}$ is an anti dom-color class of every vertex of T other than v_1. For the vertex v_1, the color class $\{v_2\}$ of $\mathcal{C}$ is an anti dom-color class (this is possible as $k \geq 2$). So, $\mathcal{C}$ is a global dominator coloring of T with χ_d colors, which is a contradiction to our assumption. Therefore $T \in \Im_2$.

Conversely, if T is a double star, then $\chi_d(T) = 3$ and $\chi_{gd}(T) = 4$. On the other hand, if $T \in \Im_1 \cup \Im_2$, then by Lemma 22, $\chi_{gd}(T) = \chi_d(T) + 1$. ∎

Open Problems

This paper introduces a new variation of coloring namely global dominator coloring connecting the concepts of domination and coloring. We have just initiated a study on this coloring parameter. However, there are abundant scope for further research on χ_{gd} and we list some of them.

1. Find a characterization of connected bipartite graphs of order $n \geq 4$, for which $\chi_{gd}(G) = \lfloor \frac{n}{2} \rfloor + 2$. One can try this problem in the case when G is a tree.

2. Characterize connected graphs G for which

   (i) $\chi_{gd}(G) = 4$.

   (ii) $\chi_{gd}(G) = \chi_d(G)$.

   (iii) $\chi_{gd}(G) = 2\chi_d(G)$.

   (iv) $\chi_{gd}(G) = \chi(G) + 1$.

   (v) $\chi_{gd}(G) = \gamma_g(G)$.

   (vi) $\chi_{gd}(G) = \chi(G) + \gamma_g(G)$.

3. By virtue of Theorem C, the family of trees can be split into two classes, namely Class 1 and Class 2. A tree T is of Class 1 or Class 2 according as $\chi_{gd}(T) = \gamma_g(T) + 1$ or $\chi_{gd}(T) = \gamma_g(T) + 2$. The Class 1 is non-empty, as for the family of subdivision of stars $K_{1,t} (t \geq 3)$, we have $\chi_{gd} = \gamma_g + 1$. Further, for the double star, $\chi_{gd} = \gamma_g + 2$ and so Class 2 is also non-empty. However, the problem of characterizing trees of Class 1 or Class 2 seems to be little challenging.

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