# ON SOME CHARACTERIZATIONS OF ANTIPODAL PARTIAL CUBES 

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#### Abstract

We prove that any harmonic partial cube is antipodal, which was conjectured by Fukuda and K. Handa, Antipodal graphs and oriented matroids, Discrete Math. 111 (1993) 245-256. Then we prove that a partial cube $G$ is antipodal if and only if the subgraphs induced by $W_{a b}$ and $W_{b a}$ are isomorphic for every edge $a b$ of $G$. This gives a positive answer to a question of Klavžar and Kovše, On even and harmonic-even partial cubes, Ars Combin. 93 (2009) 77-86. Finally we prove that the distance-balanced partial cube that are antipodal are those whose pre-hull number is at most 1.


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## 1. Introduction

If $x, y$ are two vertices of a connected graph $G$, then $y$ is said to be a relative antipode of $x$ if $d_{G}(x, y) \geq d_{G}(x, z)$ for every neighbor $z$ of $x$, where $d_{G}$ denotes the usual distance in $G$; and it is said to be an absolute antipode of $x$ if $d_{G}(x, y)=$ $\operatorname{diam}(G)$ (the diameter of $G$ ). The graph $G$ is said to be antipodal if every vertex $x$ of $G$ has exactly one relative antipode; it is diametrical if every vertex $x$ of $G$ has exactly one absolute antipode $\bar{x}$; and it is harmonic (or automorphically diametrical [27]) if it is diametrical and the antipodal map $x \mapsto \bar{x}, x \in V(G)$, is an automorphism of $G$, i.e., $\overline{x y} \in E(G)$ whenever $x y \in E(G)$. Note that, if $G$ is antipodal, then the unique relative antipode of a vertex $x$ is an absolute antipode of $x$, and thus is denoted by $\bar{x}$.

Bipartite antipodal graphs were introduced by Kotzig [18] under the name of $S$-graphs. Later Glivjak, Kotzig and Plesník [10] proved in particular that $a$ graph $G$ is antipodal if and only if for any $x \in V(G)$ there is an $\bar{x} \in V(G)$ such that

$$
\begin{equation*}
d_{G}(x, y)+d_{G}(y, \bar{x})=d_{G}(x, \bar{x}) \quad \text { for all } \quad y \in V(G) \tag{1}
\end{equation*}
$$

where $d_{G}$ denotes the usual distance in $G$. The definition was extended to the non-bipartite case by Kotzig and Laufer [19]. Several papers followed.

On the other hand diametrical graphs were introduced by Mulder [22] in the case of median graphs. They were later studied by Parthasarathy and Nandakumar [24] under the name of self-centered unique eccentric point graphs, then by Göbel and Veldman [11] under the name of even graphs, by Fukuda and Handa [9] who proved that the tope graphs of oriented matroids are harmonic partial cubes (i.e., isometric subgraphs of hypercubes), and more recently by Klavžar and Kovše [16] who gave a partial solution to a problem set in [9].

Any antipodal graph is clearly harmonic, and thus diametrical. Two results [9, Proposition 4.1 and Theorem 4.2] of Fukuda and Handa implicitely imply that any harmonic partial cube is antipodal. Partial cubes, i.e., isometric subgraphs of hypercubes, which were introduced by Firson [8] and characterized by Djoković [5] and Winkler [28], have been extensively studied, see [20, 3] for recent papers. Actually the aim of Fukuda and Handa in [9] was the characterization of the tope graph of an acycloid, and the fact that any harmonic partial cube is antipodal, which is clearly a consequence of their results, is not plainly expressed in their paper. This is why some had thought that this property was not proved in [9]. In Section 3, we give a direct proof of this property by using the fact that a diametrical partial cube is antipodal if and only if its diameter is equal to its isometric dimension, i.e., the least non-negative integer $n$ such that this graph is an isometric subgraph of an $n$-cube (Lemma 3.2).

A graph $G$ is said to be distance-balanced if $\left|W_{a b}\right|=\left|W_{b a}\right|$ for every edge $a b$ of $G$, where $W_{a b}$ denotes the set of vertices that are closer to $a$ than to $b$. Since their introduction by Handa [13], distance-balanced graphs have played an important role, and given rise to several papers, see for example some recent ones $[15,14,7]$. Handa [13] observed that any harmonic graph is distance-balanced, but that there exist distance-balanced partial cubes that are not diametrical. In Section 6 we show that the distance-balanced partial cubes that are antipodal are those whose pre-hull number is at most 1 (see [26]).

Harmonic partial cubes have a property that is stronger than the one of being distance-balanced. Actually if a partial cube $G$ is harmonic, and thus antipodal, then its antipodal map induces an isomorphism between the subgraphs induced by $W_{a b}$ and $W_{b a}$ for every edge $a b$ of $G$. In Section 5 we prove that the converse is also true, i.e., that the above property characterizes antipodal partial cubes, which answers a question of Klavžar and Kovše [16, Section 5]. More generally,
they asked [16, Problem 5.3] whether a partial cube $G$ is harmonic if and only if the subgraphs induced by $W_{a b}$ and $W_{b a}$ are isomorphic for every edge $a b$ of $G$. The above two results give several ways of tackling this problem.

## 2. Preliminaries

The graphs we consider are undirected, without loops or multiple edges, and are finite and connected. For a set $S$ of vertices of a graph $G$ we denote by $G[S]$ the subgraph of $G$ induced by $S$, and $G-S:=G[V(G)-S]$. A path $P$ with $V(P)=$ $\left\{x_{0}, \ldots, x_{n}\right\}, x_{i} \neq x_{j}$ if $i \neq j$, and $E(P)=\left\{x_{i} x_{i+1}: 0 \leq i<n\right\}$ is denoted by $\left\langle x_{0}, \ldots, x_{n}\right\rangle$ and is called an $\left(x_{0}, x_{n}\right)$-path. A cycle $C$ with $V(C)=\left\{x_{1}, \ldots, x_{n}\right\}$, $x_{i} \neq x_{j}$ if $i \neq j$, and $E(C)=\left\{x_{i} x_{i+1}: 1 \leq i<n\right\} \cup\left\{x_{n} x_{1}\right\}$, is denoted by $\left\langle x_{1}, \ldots, x_{n}, x_{1}\right\rangle$.

The usual distance between two vertices $x$ and $y$ of a graph $G$, that is, the length of any $(x, y)$-geodesic ( $=$ shortest $(x, y)$-path) in $G$, is denoted by $d_{G}(x, y)$. A connected subgraph $H$ of $G$ is isometric in $G$ if $d_{H}(x, y)=d_{G}(x, y)$ for all vertices $x$ and $y$ of $H$. The (geodesic) interval $I_{G}(x, y)$ between two vertices $x$ and $y$ of $G$ consists of the vertices of all $(x, y)$-geodesics in $G$.

In the geodesic convexity, that is, the convexity on the vertex set of a graph $G$ which is induced by the geodesic interval operator $I_{G}$, a subset $C$ of $V(G)$ is convex provided it contains the geodesic interval $I_{G}(x, y)$ for all $x, y \in C$. The convex hull $\operatorname{co}_{G}(A)$ of a subset $A$ of $V(G)$ is the smallest convex set which contains $A$. A subset $H$ of $V(G)$ is a half-space if $H$ and $V(G)-H$ are convex. We denote by $\mathcal{I}_{G}$ the pre-hull operator of the geodesic convex structure of $G$, i.e., the self-map of $\mathcal{P}(V(G))$ such that $\mathcal{I}_{G}(A):=\bigcup_{x, y \in A} I_{G}(x, y)$ for each $A \subseteq V(G)$. The convex hull of a set $A \subseteq V(G)$ is then $\sigma_{G}(A)=\bigcup_{n \in \mathbb{N}} \mathcal{I}_{G}^{n}(A)$.

For an edge $a b$ of a graph $G$, let

$$
W_{a b}:=\left\{x \in V(G): d_{G}(a, x)<d_{G}(b, x)\right\} .
$$

Note that the sets $W_{a b}$ and $W_{b a}$ are disjoint and that $V(G)=W_{a b} \cup W_{a b}$ if $G$ is bipartite.

Two edges $x y$ and $u v$ are in the Djoković-Winkler relation $\Theta$ if

$$
d_{G}(x, u)+d_{G}(y, v) \neq d_{G}(x, v)+d_{G}(y, u) .
$$

The relation $\Theta$ is clearly reflexive and symmetric.
Remark 2.1. If $G$ is bipartite, then, by [12, Lemma 11.2], the notation can be chosen so that the edges $x y$ and $u v$ are in relation $\Theta$ if and only if

$$
d_{G}(x, u)=d_{G}(y, v)=d_{G}(x, v)-1=d_{G}(y, u)-1,
$$

or equivalently if and only if

$$
y \in I_{G}(x, v) \text { and } x \in I_{G}(y, u)
$$

From now on, we will always use this way of defining the relation $\Theta$. Note that, in this way, the edges $x y$ and $y x$ are not in relation $\Theta$ because $y \notin I_{G}(x, x)$ and $x \notin G(y, y)$. In other word, each time the relation $\Theta$ is used, the notation of an edge induces an orientation of this edge.

We recall that, by Djoković [5, Theorem 1] and Winkler [28], a connected bipartite graph $G$ is a partial cube, that is, an isometric subgraph of some hypercube, if it has the following equivalent properties:
(Conv.) For every edge $a b$ of $G$, the sets $W_{a b}$ and $W_{b a}$ are convex.
(Trans.) The relation $\Theta$ is transitive, and thus is an equivalence relation.
It follows in particular that the non-trivial (i.e., distinct from $\emptyset$ and $V(G)$ ) half-spaces of a partial cube $G$ are the sets $W_{a b}, a b \in E(G)$. In the following lemma we recall two well-known properties of partial cubes that we will need later.

Lemma 2.2. Let $G$ be a partial cube. We have the following properties.
(i) Let $x, y$ be two vertices of $G, P$ an $(x, y)$-geodesic and $W$ an $(x, y)$-path of $G$. Then each edge of $P$ is $\Theta$-equivalent to some edge of $W$.
(ii) A path $P$ in $G$ is a geodesic if and only if no two distinct edges of $P$ are $\Theta$-equivalent.

## 3. Harmonicity Versus Antipodality

In this section we give an alternative proof of the following property.
Theorem 3.1. Any harmonic partial cube is antipodal.
Recall that the isometric dimension of a finite partial cube $G$, i.e., the least non-negative integer $n$ such that $G$ is an isometric subgraph of an $n$-cube, coincides with the number of $\Theta$-classes of $E(G)$. We denote it by $\operatorname{idim}(G)$. By Lemma 2.2(ii) we clearly have $\operatorname{diam}(G) \leq \operatorname{idim}(G)$.

We need the following lemma which is an immediate consequence of Desharnais [4, Lemme 1.6.9]. However we give a short proof of it. Note that, by (1), a graph $G$ is antipodal if and only if

$$
\begin{equation*}
I_{G}(x, \bar{x})=V(G) \quad \text { for all } \quad x \in V(G) \tag{2}
\end{equation*}
$$

Lemma 3.2. Let $G$ be a diametrical partial cube. Then $G$ is antipodal if and only if $\operatorname{diam}(G)=\operatorname{idim}(G)$.

Proof. Necessity. By Lemma 2.2(ii), $\operatorname{diam}(G) \leq \operatorname{idim}(G)$. Conversely, let $u v$ be an edge of $G$, and $P$ a $(u, \bar{u})$-geodesic. Then $u v$ is $\Theta$-equivalent to an edge of $P$, since otherwise $\langle v, u\rangle \cup P$ is a geodesic, contrary to the fact that $P$ is a geodesic of maximal length. Therefore $\operatorname{idim}(G) \leq \operatorname{diam}(G)$. Whence the equality.

Sufficiency. Assume that $\operatorname{diam}(G)=\operatorname{idim}(G)=: d$. Then $G$ is an isometric subgraph of some $d$-cube $H$. Let $u$ be some vertex of $G$, and $\bar{u}$ its antipode in $G$. Note that $H$ is antipodal, and that $\bar{u}$ is the antipode of $u$ in $H$, since $G$ is an isometric subgraph of $H$ and $\operatorname{diam}(H)=d=\operatorname{diam}(G)$. It follows that $I_{H}(u, \bar{u})=$ $V(H)$ by (2). Therefore $I_{G}(u, \bar{u})=I_{H}(u, \bar{u}) \cap V(G)=V(H) \cap V(G)=V(G)$, which proves that $G$ is antipodal.

Proof of Theorem 3.1. Let $G$ be a harmonic partial cube. Denote by $d$ its diameter, and by $\bar{u}$ the unique absolute antipode of any vertex $u$ of $G$. Then the antipodal map $\alpha: u \rightarrow \bar{u}$ is an automorphism of $G$. Moreover denote by $P_{u}$ some $(u, \bar{u})$-geodesic for every $u \in V(G)$. Finally, for an edge $e$ of $G$ and a subgraph $F$ of $G$, let $\Theta[e]$ be the $\Theta$-class of $e$, and

$$
\Theta[F]:=\{\Theta[e]: e \in E(F)\} .
$$

Let $u \in V(G)$.
Claim 1. Any neighbor $v$ of $u$ belongs to $I_{G}(u, \bar{u})$, and thus $\Theta[u v] \in \Theta\left[P_{u}\right]$.
This is obvious if $v \in V\left(P_{u}\right)$. Suppose that $v \notin V\left(P_{u}\right)$. Then $d_{G}(v, \bar{u})<d$ since $\bar{v} \neq \bar{u}$ because $\alpha$ is an automorphism. It follows that $v \in I_{G}(u, \bar{u})$. Hence the edge $u v$ is $\Theta$-equivalent to some edge of $P_{u}$ by Lemma 2.2(i).
Claim 2. $\Theta\left[P_{u}\right]=\Theta\left[P_{v}\right]$ for any neighbor $v$ of $u$ in $G$.
Note that $\bar{v}$ is a neighbor of $\bar{u}$ since $\alpha$ is an automorphism. Let $Q$ be some $(v, \bar{u})$-geodesic. Then $R:=\langle u, v\rangle \cup Q$ and $R^{\prime}:=Q \cup\langle\bar{u}, \bar{v}\rangle$ are a $(u, \bar{u})$-geodesic and a $(v, \bar{v})$-geodesic, respectively. On the other hand, the edges $u v$ and $\overline{v u}$ are $\Theta$-equivalent.

It follows that $\Theta\left[P_{u}\right]=\Theta[R]=\Theta\left[R^{\prime}\right]=\Theta\left[P_{v}\right]$ by Lemma 2.2(i).
Claim 3. $\Theta\left[P_{u}\right]=\Theta\left[P_{v}\right]$ for any $v \in V(G)$.
This is Claim 2 if $v$ is a neighbor of $u$. Suppose that $d_{G}(u, v)=n>1$, and let $\left\langle x_{0}, \ldots, x_{n}\right\rangle$ be a $(u, v)$-geodesic with $x_{0}=u$ and $x_{n}=v$. Then $\left\langle\overline{x_{0}}, \ldots, \overline{x_{n}}\right\rangle$ is a $(\bar{u}, \bar{v})$-geodesic since $\alpha$ is an automorphism.

By a successive application of Claim 2, we obtain

$$
\Theta\left[P_{u}\right]=\Theta\left[P_{x_{0}}\right]=\Theta\left[P_{x_{1}}\right]=\cdots=\Theta\left[P_{x_{n}}\right]=\Theta\left[P_{v}\right] .
$$

Now, each edge $x y$ of $G$ is such that $\Theta[x y] \in \Theta\left[P_{x}\right]=\Theta\left[P_{u}\right]$ by Claims 1 and 3 , that is, each edge of $G$ is $\Theta$-equivalent to some edge of $P_{u}$. Therefore $\operatorname{idim}(G)$ is equal to the number of edges of $P_{u}$, because any two distinct edges of a geodesic are non- $\Theta$-equivalent by Lemma 2.2 (ii). It follows that $\operatorname{idim}(G)=d=\operatorname{diam}(G)$. Hence $G$ is antipodal by Lemma 3.2.

## 4. Expansion

In this section we recall some properties of expansions of a graph, a concept that we will need in the next section and which was introduced by Mulder [21] to characterize median graphs and which was later generalized by Chepoi [2].

Definition 4.1. A pair $\left(V_{0}, V_{1}\right)$ of sets of vertices of a graph $G$ is called a proper cover of $G$ if it satisfies the following conditions

- $V_{0} \cap V_{1} \neq \emptyset$ and $V_{0} \cup V_{1}=V(G)$;
- there is no edge between a vertex in $V_{0}-V_{1}$ and a vertex in $V_{1}-V_{0}$;
- $G\left[V_{0}\right]$ and $G\left[V_{1}\right]$ are isometric subgraphs of $G$.

Recall that the prism over a graph $G$ is the Cartesian product of $G$ and $K_{2}$, i.e., the graph denoted by $G \square K_{2}$ whose vertex set is $V(G) \times V\left(K_{2}\right)$, and such that, for all $x, y \in V(G)$ and $i, j \in V\left(K_{2}\right)=\{0,1\},(x, i)(y, j) \in E\left(G \square K_{2}\right)$ if $x y \in E(G)$ and $i=j$, or $x=y$ and $i \neq j$.

Definition 4.2. An expansion of a graph $G$ with respect to a proper cover $\left(V_{0}, V_{1}\right)$ of $G$ is the subgraph of the prism over $G$ induced by the set $\left(V_{0} \times\{0\}\right) \cup$ $\left(V_{1} \times\{1\}\right)$.

An expansion of a bipartite graph (respectively, a partial cube) is a bipartite graph (respectively, a partial cube (see [2])). If $G^{\prime}$ is an expansion of a partial cube $G$, then we say that $G$ is a $\Theta$-contraction of $G^{\prime}$, because, as we can easily see, $G$ is obtained from $G^{\prime}$ by contracting each element of some $\Theta$-class of edges of $G^{\prime}$. More precisely, let $G$ be a partial cube different from $K_{1}$ and let $u v$ be an edge of $G$. Let $G / u v$ be the quotient graph of $G$ whose vertex set $V(G / u v)$ is the partition of $V(G)$ such that $x$ and $y$ belong to the same block of this partition if and only if $x=y$ or $x y$ is an edge which is $\Theta$-equivalent to $u v$. The natural surjection $\gamma_{u v}$ of $V(G)$ onto $V(G / u v)$ is a contraction (weak homomorphism in [12]) of $G$ onto $G / u v$, that is, an application which maps any two adjacent vertices to adjacent vertices or to a single vertex. Then clearly the graph $G / u v$ is a partial cube and $\left(\gamma_{u v}\left(W_{u v}^{G}\right), \gamma_{u v}\left(W_{v u}^{G}\right)\right)$ is a proper cover of $G / u v$ with respect to which $G$ is an expansion of $G / u v$. We will say that $G / u v$ is the $\Theta$-contraction of $G$ with respect to the $\Theta$-class of $u v$.

Let $G^{\prime}$ be an expansion of a graph $G$ with respect to a proper cover $\left(V_{0}, V_{1}\right)$ of $G$. We will use the following notation.

- For $i=0,1$ denote by $\psi_{i}: V_{i} \rightarrow V\left(G^{\prime}\right)$ the natural injection $\psi_{i}: x \mapsto(x, i)$, $x \in V_{i}$, and let $V_{i}^{\prime}:=\psi_{i}\left(V_{i}\right)$. Note that $V_{0}^{\prime}$ and $V_{1}^{\prime}$ are complementary half-spaces of $G^{\prime}$.
- For $A \subseteq V(G)$ put $\psi(A):=\psi_{0}\left(A \cap V_{0}\right) \cup \psi_{1}\left(A \cap V_{1}\right)$.

The following lemma is a restatement with more precisions of $[25$, Lemma 4.5] (also see [23, Lemma 8.1]).

Lemma 4.3. Let $G$ be a connected bipartite graph and $G^{\prime}$ an expansion of $G$ with respect to a proper cover $\left(V_{0}, V_{1}\right)$ of $G$, and let $P=\left\langle x_{0}, \ldots, x_{n}\right\rangle$ be a path in $G$. We have the following properties
(i) If $x_{0}, x_{n} \in V_{i}$ for some $i=0$ or 1 , then

- if $P$ is a geodesic in $G$, then there exists an $\left(x_{0}, x_{n}\right)$-geodesic $R$ in $G\left[V_{i}\right]$ such that $V(P) \cap V_{i} \subseteq V(R)$;
- $P$ is a geodesic in $G\left[V_{i}\right]$ if and only if $P^{\prime}=\left\langle\psi_{i}\left(x_{0}\right), \ldots, \psi_{i}\left(x_{n}\right)\right\rangle$ is a geodesic in $G^{\prime}$;

$$
\begin{aligned}
& -d_{G^{\prime}}\left(\psi_{i}\left(x_{0}\right), \psi_{i}\left(x_{n}\right)\right)=d_{G}\left(x_{0}, x_{n}\right) ; \\
& -d_{G^{\prime}}\left(\psi_{i}\left(x_{0}\right), \psi_{i}\left(x_{n}\right)\right)=d_{G}\left(x_{0}, x_{n}\right) ; \\
& -I_{G^{\prime}}\left(\psi_{i}\left(x_{0}\right), \psi_{i}\left(x_{n}\right)\right)=\psi_{i}\left(I_{G\left[V_{i}\right]}\left(x_{0}, x_{n}\right)\right) \subseteq \psi\left(I_{G}\left(x_{0}, x_{n}\right)\right) .
\end{aligned}
$$

(ii) If $x_{0} \in V_{i}$ and $x_{n} \in V_{1-i}$ for some $i=0$ or 1 , then

- if there exists $p$ such that $x_{0}, \ldots, x_{p} \in V_{i}$ and $x_{p}, \ldots, x_{n} \in V_{1-i}$, then $P$ is a geodesic in $G$ if and only if the path $P^{\prime}=\left\langle\psi_{i}\left(x_{0}\right), \ldots, \psi_{i}\left(x_{p}\right), \psi_{1-i}\left(x_{p}\right), \ldots\right.$, $\left.\psi_{1-i}\left(x_{n}\right)\right\rangle$ is a geodesic in $G^{\prime}$;
$-d_{G^{\prime}}\left(\psi_{i}\left(x_{0}\right), \psi_{1-i}\left(x_{n}\right)\right)=d_{G}\left(x_{0}, x_{n}\right)+1 ;$
$-I_{G^{\prime}}\left(\psi_{i}\left(x_{0}\right), \psi_{1-i}\left(x_{n}\right)\right)=\psi\left(I_{G}\left(x_{0}, x_{n}\right)\right)$.
Now we introduce a variety of expansions that are related to antipodal partial cubes. We need the following notation. If $A$ is a set of vertices of an antipodal graph $G$, we write

$$
\bar{A}:=\{\bar{x}: x \in A\} .
$$

Lemma 4.4. If $G$ is an antipodal partial cube, then $\overline{W_{a b}}=W_{b a}$ for every edge $a b$ of $G$.

Proof. Suppose both $x$ and $\bar{x}$ belong to $W_{a b}$ for some vertex $x$ of $G$. Then $I_{G}(x, \bar{x}) \subseteq W_{a b}$ since $W_{a b}$ is convex by (Conv.), hence $I_{G}(x, \bar{x}) \neq V(G)$, contrary to (2).

Definition 4.5. A proper cover $\left(V_{0}, V_{1}\right)$ of an antipodal partial cube $G$ is said to respect the antipodality, or to be antipodality-respectful, if $\overline{V_{0}}=V_{1}$.

Clearly, if $\overline{V_{0}}=V_{1}$, then $\overline{V_{1}}=V_{0}$ and $\overline{V_{0} \cap V_{1}}=V_{0} \cap V_{1}$. For any antipodal partial cube $G$, there always exists a proper cover that respects the antipodality. For example, the proper cover $\left(V_{0}, V_{1}\right)$ such that $V_{0}=V_{1}=V(G)$ respects the antipodality, and the expansion of $G$ with respect to this proper cover is the prism over $G$.

Definition 4.6. An expansion of an antipodal partial cube $G$ with respect to an antipodality-respectful proper cover of $G$ is called an antipodality-respectful expansion of $G$.

These antipodality-respectful expansions were already defined in [9] under the name of acycloidal expansions.

Lemma 4.7. Any antipodality-respectful expansion of an antipodal partial cube is an antipodal partial cube.

Proof. Let $G^{\prime}$ be an expansion of an antipodal partial cube $G$ with respect to an antipodality-respectful proper cover $\left(V_{0}, V_{1}\right)$ of $G$. Clearly $G^{\prime}$ is a bipartite partial cube such that $\operatorname{diam}\left(G^{\prime}\right)=\operatorname{diam}(G)+1$. Denote by pr the projection of $G^{\prime}$ onto $G$. Let $x \in V\left(G^{\prime}\right)$. Then $x \in V_{i}^{\prime}$ for some $i=0$ or 1 , and thus $\operatorname{pr}(x) \in V_{i}$ and $\overline{\operatorname{pr}(x)} \in V_{1-i}$ because $\left(V_{0}, V_{1}\right)$ respects the antipodality. Let $y \in V\left(G^{\prime}\right)$. Then $\operatorname{pr}(y) \in I_{G}(\operatorname{pr}(x), \overline{\operatorname{pr}(x)})$ since $G$ is antipodal. Hence $y \in I_{G^{\prime}}\left(x, \psi_{1-i}(\overline{\operatorname{pr}(x)})\right)$ by Lemma 4.3. It follows, by (2), that $G^{\prime}$ is antipodal with $\bar{x}=\psi_{1-i}(\overline{\operatorname{pr}(x)})$.

Lemma 4.8. Let $G^{\prime}$ be an expansion of a partial cube $G$ with respect to a proper cover $\left(V_{0}, V_{1}\right)$. If $G^{\prime}$ is antipodal, then so is $G$ and moreover $\left(V_{0}, V_{1}\right)$ is an antipodality-respectful proper cover of $G$.
Proof. Assume that $G^{\prime}$ is antipodal. We use the notations introduced above.
Claim 1. $G$ is antipodal.
Because $V_{0}^{\prime}$ and $V_{1}^{\prime}$ are complementary half-spaces of $G^{\prime}$, it follows that $\overline{V_{i}^{\prime}}=$ $V_{1-i}^{\prime}$ for $i=0,1$ by Lemma 4.4.

Let $x \in V(G)$. Without loss of generality, we can suppose that $x \in V_{0}$. Then $\psi_{0}(x) \in V_{0}^{\prime}$ and $\overline{\psi_{0}(x)} \in V_{1}^{\prime}$. Hence, by Lemma 4.3,

$$
\psi\left(I_{G}\left(x, p r\left(\overline{\psi_{0}(x)}\right)\right)=I_{G}^{\prime}\left(\psi_{0}(x), \psi_{1}\left(\overline{\psi_{0}(x)}\right)=V\left(G^{\prime}\right) .\right.\right.
$$

It follows that $I_{G}\left(x, \operatorname{pr}\left(\overline{\psi_{0}(x)}\right)=V(G)\right.$, which proves that $\operatorname{pr}\left(\overline{\psi_{0}(x)}\right)$ is the antipode of $x$ in $G$. Therefore $G$ is antipodal.
Claim 2. ( $V_{0}, V_{1}$ ) respects the antipodality.
By Lemma 4.4, $\overline{V_{i}^{\prime}}=\underline{V_{1-i}^{\prime}}$ for $i=0,1$ since $G$ is antipodal by Claim 1. Hence, by Lemma 4.3, $\overline{V_{i}}=\overline{\operatorname{pr}\left(V_{i}^{\prime}\right)}=\operatorname{pr}\left(\overline{V_{i}^{\prime}}\right)=\operatorname{pr}\left(V_{1-i}^{\prime}\right)=V_{1-i}$ for $i=0,1$, and thus ( $V_{0}, V_{1}$ ) respects the antipodality, or in other words, $G^{\prime}$ is an antipodalityrespectful expansion of $G$.

The following theorem, which is similar to a characterization of median graphs by Mulder [21] and of partial cubes by Chepoi [2] (also see [9, Theorem 4.6]), is easily proved by induction on the isometric dimension by using the above two lemmas.
Theorem 4.9. A finite graph is an antipodal partial cube if and only if it can be obtained from $K_{1}$ by a sequence of antipodality-respectful expansions.

The number of iterations to obtain some antipodal partial cube $G$ from $K_{1}$ is equal to the isometric dimension of $G$.

## 5. Special Automorphisms

By Lemma 4.4, if $G$ is an antipodal partial cube, then the antipodal map $x \rightarrow \bar{x}$, $x \in V(G)$, is an isomorphism of $G\left[W_{a b}\right]$ onto $G\left[W_{b a}\right]$. We will show that such a property characterizes antipodal partial cubes. Let $G$ be a partial cube. Recall that the subgraphs $G\left[W_{a b}\right], a b \in E(G)$, are called semicubes by Eppstein [6], and that the semicubes $G\left[W_{a b}\right]$ and $G\left[W_{b a}\right]$ are said to be opposite.

Definition 5.1. Let $\alpha$ be an automorphism of a partial cube $G$. We say that $\alpha$ is
(i) semicube-switching if it induces an isomorphism between the subgraphs $G\left[W_{a b}\right]$ and $G\left[W_{b a}\right]$ for each edge $a b$ of $G$.
(ii) $\Theta$-faithful if the edges $u v$ and $\alpha(v) \alpha(u)$ are $\Theta$-equivalent for each $u v \in E(G)$.

In Remark 2.1 we observed that the edges $x y$ and $y x$ are not $\Theta$-equivalent. It follows that the identity automorphism of a partial cube distinct from $K_{1}$ is not $\Theta$-faithful. Also note that, if, for example, $G$ is a 4 -cycle $\left\langle x_{1}, x_{2}, x_{3}, x_{4}, x_{1}\right\rangle$, then the only $\Theta$-faithful automorphism, and also the only semicube-switching automorphism, is the antipodal map. Indeed, the involution $\beta$ mapping $x_{1}$ to $x_{2}$, and $x_{3}$ to $x_{4}$ is not $\Theta$-faithful because the edges $x_{1} x_{4}$ and $\beta\left(x_{4}\right) \beta\left(x_{1}\right)=x_{3} x_{2}$ are not $\Theta$-equivalent since $x_{4} \notin I_{G}\left(x_{1}, x_{2}\right)$.

Theorem 5.2. Let $G$ be a partial cube. The following assertions are equivalent.
(i) $G$ is antipodal.
(ii) There exits an automorphism of $G$ that is $\Theta$-faithful.
(iii) There exits an automorphism of $G$ that is semicube-switching.

Proof. (i) $\Rightarrow$ (ii): Assume that $G$ is antipodal, and let $u v \in E(G)$. Then, by (2), $v \in I_{G}(u, \bar{u})$ and $u \in I_{G}(v, \bar{v})$. Hence the edges $u v$ and $\overline{v u}$ are $\Theta$-equivalent. Therefore the antipodal map $\alpha$ is $\Theta$-faithful.
(ii) $\Rightarrow$ (iii): Assume that there exists a $\Theta$-faithful automorphism $\alpha$ of $G$. Let $u v \in E(G)$, and let $x \in W_{u v}$. Then $u \in I_{G}(x, v)$, and thus no edge of any $(u, x)$-geodesic is $\Theta$-equivalent to $u v$. Because $\alpha$ is $\Theta$-faithful, it follows that no edge of any $(\alpha(u), \alpha(x))$-geodesic is $\Theta$-equivalent to $u v$, and thus to $\alpha(v) \alpha(u)$. Hence $\alpha(u) \in I_{G}(\alpha(x), \alpha(v))$, and thus $\alpha(x) \in W_{\alpha(u) \alpha(v)}=W_{v u}$. Therefore $\alpha$ is an isomorphism between $G\left[W_{a b}\right]$ and $G\left[W_{b a}\right]$, and thus $\alpha$ is semicube-switching.
(iii) $\Rightarrow$ (i): We prove by induction on the isometric dimension that any partial cube $G$ that has a semicube-switching automorphism $\alpha$ is antipodal and that $\alpha$ is its antipodal map. This is obvious if $\operatorname{idim}(G) \leq 2$, i.e., if $G$ is $K_{1}, K_{2}$ or a 4 -cycle. Note that a path of length 2 , which is also a partial cube of isometric dimension 2 , has no semicube-switching automorphism. Suppose that this holds for any partial cube of isometric dimension $n$ for some $n \geq 2$. Let $G$ be a partial
cube with $\operatorname{idim}(G)=n+1$ that has a semicube-switching automorphism $\alpha$. Let $u v$ be an edge of $G, F:=G / u v$ the $\Theta$-contraction of $G$ with respect to the $\Theta$-class of $u v$, and $\gamma$ the natural surjection of $V(G)$ onto $V(F)$. Then $F$ is a partial cube with $\operatorname{idim}(F)=n$.

Note that, if $x y$ is an edge of $G$ that is $\Theta$-equivalent to $u v$, then so is the edge $\alpha(v) \alpha(u)$, because $\alpha(u) \in W_{v u}, \alpha(v) \in W_{u v}$ and $\alpha(u)$ and $\alpha(v)$ are adjacent since $\alpha$ is a semicube-switching automorphism.

Let $\beta: V(F) \rightarrow V(F)$ be such that $\beta \circ \gamma=\gamma \circ \alpha$. Because $\alpha$ is an automorphism of $G$, it is sufficient to prove that $\beta$ preserves the edges to show that it is an automorphism of $F$. Let $x$ and $y$ be two adjacent vertices of $G$. If $x y$ is not $\Theta$-equivalent to $u v$, then so is $\alpha(y) \alpha(x)$, and thus $\gamma(x)$ and $\gamma(y)$ are adjacent, and so are $\gamma(\alpha(x))$ and $\gamma((\alpha(y))$, and hence so also are $\beta(\gamma(x))$ and $\beta(\gamma(y))$. If $x y$ is $\Theta$-equivalent to $u v$, then so is $\alpha(y) \alpha(x)$, and thus $\gamma(x)=\gamma(y)$ and $\gamma(\alpha(x))=\gamma((\alpha(y))$. Therefore $\beta$ is an automorphism of $F$.

We now show that $\beta$ is semicube-switching. Note that each edge of $F$ is the image by $\gamma$ of somme edge of $G$ that is not $\Theta$-equivalent to $u v$. Let $a b$ be an edge of $G$ that is not $\Theta$-equivalent to $u v$, and $x$ a vertex of $G$. Without loss of generality we can suppose that $x \in W_{a b}$. Then $a \in I_{G}(x, b)$, and thus each $(x, a)$ geodesic contains an edge that is $\Theta$-equivalent to $u v$ if and only if so does each $(x, b)$-geodesic. It follows, by Lemma 4.3, that $\gamma(a) \in I_{F}(\gamma(x), \gamma(b))$, and thus $\gamma(x) \in W_{\gamma(a) \gamma(b)}^{F}$. On the other hand, $\alpha(x) \in W_{b a}$ since $\alpha$ is semicube-switching. Hence, as above, $\gamma(\alpha(x)) \in W_{\gamma(b) \gamma(a)}^{F}$, that is, $\beta(\gamma(x)) \in W_{\gamma(b) \gamma(a)}^{F}$, which implies that $\beta$ is semicube-switching.

It follows, by the induction hypothesis and since $\operatorname{idim}(F)=n$, that the partial cube $F$ is antipodal and that $\beta$ is its antipodal map. Note that $\beta\left(\gamma\left(W_{u v}\right)\right)=$ $\gamma\left(\alpha\left(W_{u v}\right)\right)=\gamma\left(W_{v u}\right)$. Hence $\left(\gamma\left(W_{u v}\right), \gamma\left(W_{v u}\right)\right)$ is an antipodality-respectful proper cover of $F$, and $G$ is the expansion of $F$ with respect to this proper cover. Consequently $G$ is antipodal by Lemma 4.7.

## 6. Pre-Hull Number and Distance-Balanced Graphs

We now give a characterization of antipodal partial cubes that uses the concept of pre-hull number, a concept which was introduced in [26] and that we first recall.

A copoint at a vertex $x$ of a graph $G$ is a convex set $C$ which is maximal with respect to the property that $x \notin C ; x$ is an attaching point of $C$. Note that $\operatorname{co}_{G}(C \cup\{x\})=\operatorname{co}_{G}(C \cup\{y\})$ for any two attaching points $x, y$ of $C$. We denote by $\operatorname{Att}(C)$ the set of all attaching points of $C$, i.e.,

$$
\begin{equation*}
\operatorname{Att}(C):=\operatorname{co}_{G}(C \cup\{x\})-C \tag{3}
\end{equation*}
$$

By [26, Proposition 5.6], the copoints of a partial cube $G$ are precisely the
sets $W_{a b}, a b \in E(G)$, and thus are the non-trivial half-spaces of $G$.
Definition 6.1. Let $G$ be a graph. The least non-negative integer $n$ (if it exists) such that $\operatorname{co}_{G}(C \cup\{x\})=\mathcal{I}_{G}^{n}(C \cup\{x\})$ for each vertex $x$ of $G$ and each copoint $C$ at $x$, is called the pre-hull number of $G$ and is denoted by $p h(G)$. If there is no such $n$ we put $p h(G):=\infty$.

Recall that by [26, Corollary 3.8], the pre-hull number of a connected bipartite graph $G$ is zero if and only if $G$ is a tree. For a graph $G, p h(G) \leq 1$ if $c_{G}(C \cup$ $\{x\})=\mathcal{I}_{G}(C \cup\{x\})$ for each vertex $x$ of $G$ and each copoint $C$ at $x$, i.e., for all $x, y \in \operatorname{Att}(C)$ there exists some $z \in C$ such that $y \in I_{G}(x, z)$.

Lemma 6.2. If a partial cube $G$ is distance-balanced, then every non-trivial halfspace of $G$ is maximal.
Proof. Assume that $G$ is distance-balanced, and let $H$ be a non-trivial half-space of $G$. Then $H=W_{u v}$ for some $u v \in E(G)$ (see Section 2). Then $\left|W_{u v}\right|=\left|W_{v u}\right|$ since $G$ is distance-balanced. Suppose that $H$ is not maximal. Then there exists a non-trivial half-space $H^{\prime}$ that contains $H \cup\{x\}$ for some $x \in W_{v u}$. Hence $H^{\prime}=W_{a b}$ for some $a b \in E(G)$, and thus $\left|W_{b a}\right|<\left|W_{v u}\right|=\left|W_{u v}\right|<\left|W_{a b}\right|$, contrary to the assumption. Therefore $H$ is maximal.

Theorem 6.3. Let $G$ be a partial cube. The following assertions are equivalent.
(i) $G$ is antipodal.
(ii) $p h(G) \leq 1$ and $G$ is distance-balanced.
(iii) $p h(G) \leq 1$ and every non-trivial half-space of $G$ is maximal.

Proof. (i) $\Rightarrow$ (ii): Assume that $G$ is antipodal. Then $p h(G) \leq 1$ by [26, Section 8]. Because the antipodal map of $G$ is semicube-switching by Theorem 5.2, it follows that $\left|W_{u v}\right|=\left|W_{v u}\right|$ for every $u v \in E(G)$.
(ii) $\Rightarrow$ (iii) is a consequence of Lemma 6.2.
(iii) $\Rightarrow($ (i): Assume that $G$ satisfies (iii). Let $u v \in E(G)$. Because the halfspace $W_{v u}$ is maximal, it follows that $\operatorname{Att}\left(W_{v u}\right)=W_{u v}$. Indeed, if some $x \in W_{u v}$ is not an attaching point of $W_{v u}$, then there exists a copoint at $x$, and thus a half-space by what we saw above, which strictly contains $W_{v u}$, contrary to the maximality of $W_{v u}$.

Let $x, y \in W_{u v}$. Because $p h(G) \leq 1$, there is some $z \in W_{v u}$ such that $y \in I_{G}(x, z)$. This implies that no geodesic in the subgraph $G\left[W_{u v}\right]$ is maximal in $G$. Hence, for every $x \in W_{u v}$, any relative antipode of $x$ belongs to $W_{v u}$.

It follows that, if $x$ has several antipodes, then, for any edge $u v$ of $G$, if $x \in$ $W_{u v}$, then all antipodes of $x$ belong to $W_{v u}$, contrary to the fact that any partial cube has the Separation Property $S_{2}$, i.e., any two vertices can be separated by a half-space. Therefore any vertex of $G$ has exactly one relative antipode, and thus $G$ is antipodal.

In other words, the distance-balanced partial cubes that are antipodal are those whose pre-hull number is at most 1.

## 7. Crossing Graph

Let $G$ be a partial cube. We say that two $\Theta$-classes $A, B$ of edges of $G$ cross if, for $a_{0} a_{1} \in A$ and $b_{0} b_{1} \in B$,

$$
W_{a_{i} a_{1-i}} \cap W_{b_{j} b_{1-j}} \neq \emptyset \quad \text { for all } \quad i, j \in\{0,1\}
$$

Note that this definition is independent of the choice of the edges in $A$ and $B$.
The crossing graph of a partial cube $G$ is the graph $G^{\#}$ whose vertices are the $\Theta$-classes of $G$, and where two vertices are adjacent if they cross. The concept of crossing graph was introduced by Bandelt and Dress [1] under the name of incompatibility graph, and extensively studied by Klavžar and Mulder [17].

Proposition 7.1. The crossing graph $G^{\#}$ of a partial cube $G$ is a complete graph if and only if every non-trivial half-space of $G$ is maximal.

Proof. Suppose that some non-trivial half-space $H$ of $G$ is not maximal. Then there exist two edges $u v$ and $a b$ of $G$ such that $H=W_{u v}$ and $W_{u v} \subset W_{a b}$. It follows that $W_{u v} \cap W_{b a}=\emptyset$, and thus that the $\Theta$-classes of $u v$ and $a b$ do not cross. Therefore $G^{\#}$ is not complete.

Conversely suppose that two $\Theta$-classes $A$ and $B$ of $G$ do not cross. Then there exist $a_{0} a_{1} \in A$ and $b_{0} b_{1} \in B$ such that $W_{a_{0} a_{1}} \subset W_{b_{0} b_{1}}$. It follows that the half-space $W_{a_{0} a_{1}}$ is not maximal.

From Theorem 6.3 we then deduce immediately.
Theorem 7.2. A partial cube $G$ is antipodal if and only if $p h(G) \leq 1$ and $G^{\#}$ is complete.

Median graphs are particular partial cubes whose pre-hull number is at most 1 ([26, Theorem 4.4]). We recall the following result.

Proposition 7.3. (Klavžar and Mulder [17, Proposition 4.1] and Mulder [22, Corollary 5]) Let $G$ be a median graph. The following assertions are equivalent.
(i) $G$ is antipodal.
(ii) $G^{\#}$ is complete.
(iii) $G$ is a hypercube.

In [17], Klavžar and Mulder defined an all-color expansion of a partial cube $G$ as an expansion with respect to a proper cover $\left(V_{0}, V_{1}\right)$ such that each $\Theta$-classes
of $G$ has a representative occurring in $E\left(G\left[V_{i}\right]\right)$ for $i=0,1$. They prove [17, Proposition 4.4] that the crossing graph of a partial cube $G$ is complete if and only if $G$ can be obtained from $K_{1}$ by a sequence of all-color expansions. By comparing this result with Theorem 4.9, we see that any antipodality-respectful expansion is an all-color expansion, but that the converse is false.

## 8. On Two Problems of Klavžar and Kovše

At the end of their paper [16], after having noticed that, for any harmonic (and thus antipodal by Theorem 3.1) partial cube $G$, the antipodal map induces an isomorphism between each opposite semicubes, Klavžar and Kovše asked if the converse is true. Theorem 5.2 asserts that so it is. However they asked the more general question [16, Problem 5.3] whether a partial cube $G$ is harmonic if and only if $G\left[W_{a b}\right]$ is isomorphic to $G\left[W_{b a}\right]$ for every $a b \in E(G)$. We still have no definitive answer to this problem, but Theorems 5.2 and 6.3 (because a partial cube all of whose opposite semicubes are isomorphic is obviously distancebalanced) give new approaches of dealing with this question. More precisely, if a partial cube $G$ is such that the opposite semicubes $G\left[W_{a b}\right]$ and $G\left[W_{b a}\right]$ are isomorphic for every $a b \in E(G)$, then we can tackle the above problem by asking one of the following question:
(a) Does there exist an automorphism of $G$ that is $\Theta$-faithful?
(b) Does there exist an automorphism of $G$ that is semicube-switching?
(c) Do we have $p h(G) \leq 1$ ?

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