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EDGE-CONNECTIVITY AND EDGES OF EVEN FACTORS OF GRAPHS

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Abstract

An even factor of a graph is a spanning subgraph in which each vertex has a positive even degree. Jackson and Yoshimoto showed that if G is a 3-edge-connected graph with $|G| \geq 5$ and v is a vertex with degree 3, then G has an even factor F containing two given edges incident with v in which each component has order at least 5. We prove that this theorem is satisfied for each pair of adjacent edges. Also, we show that each 3-edge-connected graph has an even factor F containing two given edges e and f such that every component containing neither e nor f has order at least 5. But we construct infinitely many 3-edge-connected graphs that do not have an even factor F containing two arbitrary prescribed edges in which each component has order at least 5.

Keywords: 3-edge-connected graph, 2-edge-connected graph, even factor, component.

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1. Introduction

In this paper, a graph means a multi-graph, which may have multiple edges but has no loops. A graph having neither multiple edges nor loops is called a *simple graph*. An *even factor* of a graph $G = (V(G), E(G))$ is a spanning subgraph in which each vertex has a positive even degree. The minimum order of components of G is denoted by $\sigma(G)$.

It is known that every 2-edge connected graph (i.e., a multi-graph) with minimum degree at least 3 has an even factor L. Lovász, Problem 42, Section 7 of Combinatorial Problems and Exercises, North-Holland, Amsterdam, 1979). This result was strengthened by Jackson and Yoshimoto [3]. They showed that every

2-edge-connected simple graph with n vertices and minimum degree at least 3 has an even factor F with $\sigma(F) \geq \min\{n, 4\}$. They proved better results for 3-edge-connected graphs.

Theorem 1 (Jackson and Yoshimoto, [4]). *Let G be a 3-edge-connected graph with n vertices, v be a vertex of G with $d_G(v) = 3$, and $e = vx, f = vy$ be edges of G. (We allow the posibility that $x = y$.) Then G has an even factor F containing e and f and satisfying $\sigma(F) \geq \min\{n, 5\}$.*

Theorem 2 (Jackson and Yoshimoto, [4]). *Let G be a 3-edge-connected graph with n vertices. Then G has an even factor F with $\sigma(F) \geq \min\{n, 5\}$.*

In [2] we prove the following theorem.

Theorem 3 [2]. *Let G be a 2-edge-connected simple graph with $\delta(G) \geq 3$. Then for each pair of edges e and f of G, G has an even factor F that contains e and f and satisfies $\sigma(F) \geq 4$.*

We show that Theorem 1 is satisfied for each pair of adjacent edges. Moreover, we prove that every 3-edge-connected graph has an even factor F containing two given edges e and f such that every component containing neither e nor f has order at least 5. But we construct infinitely many 3-edge-connected graphs having no even factor F containing two arbitrary prescribed edges in which $\sigma(F) \geq 5$.

Every 4-edge-connected graph has a connected even factor [5]. Also, it has a connected even factor F containing two arbitrary prescribed edges [7].

Kano *et al.* [6] proved that every cubic bipartite graph has a $\{C_n | n \geq 6\}$-factor. We extend this result to every r-regular bipartite graph. But we show that there are infinitely many 2-edge-connected simple bipartite graphs with minimum degree at least 3 having no even factor F in which $\sigma(F) \geq 6$.

All concepts not defined in this paper can be found in [1]. We denote the set of edges incident to a vertex v by $E_G(v)$. If $v \in V(G)$ and $e \in E(G)$, then the graphs $(V(G) - v, E(G) - E_G(v))$ and $(V(G), E(G) - e)$ are denoted by $G - v$ and $G - e$, respectively. Similarly, $G + e$ is defined. For a subset $X \subseteq V(G)$, the subgraph of G *induced* by X is denoted by $\langle X \rangle_G$. Also, for a connected subgraph H of G, we denote by G/H the graph obtained from G by contracting every edge in H. The vertex of G/H corresponding to H is denote by H^*. An *edge cut* of a connected graph G is a set $S \subseteq E(G)$ such that $G - S$ is disconnected. The minimum size of edge cuts of G is denoted by $\kappa'(G)$.

2. Even Factors of 3-Edge-Connected Graphs

We state some results about even factors of 3-edge-connected graphs that contain or do not contain some given edges.

Theorem 4. *Let G be a 3-edge-connected graph. Then for each pair of edges e and f of G, there is an even factor containing e and f in which every component containing neither e nor f has order at least* $\min\{|G|, 5\}$.

Proof. Let $e = xx'$ and $f = yy'$. We construct the graph G' by subdividing two edges $e = xx'$ and $f = yy'$ and put new vertices x'' and y'' on e and f, respectively, then connect x'' to y'' with the new edge h.

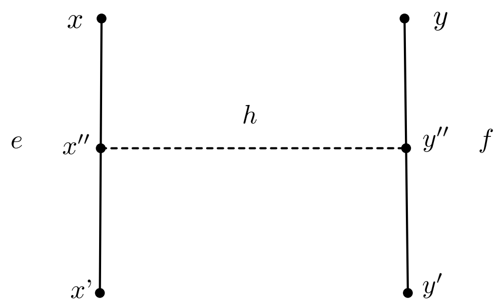

Figure 1. G'.

Now, we have $d_{G'}(x'') = d_{G'}(y'') = 3$. It is easy to see that the graph obtained from a 3-edge-connected by dividing one edge is still 2-edge-connected. Then G' is 2-edge-connected. Let $W = \{xx'', x'x'', x''y'', yy'', y'y''\}$. If there is a minimum edge cut S with $|S| = 2$, then by considering three states for $S' = W \cap S$, we can easily find the edge cut S' such that $|S'| \leq 2$ and $G - S'$ is disconnected. It is a contradiction. Hence, G' is 3-edge-connected. Now, by Theorem 1, there is an even factor F' of G' avoiding h in which $\sigma(F') \geq \min\{|G'|, 5\}$. It is clear that $F = F' - \{x'', y''\} \cup \{e, f\}$ is a desired even factor of G. ■

Theorem 5. *Let G be a 3-edge-connected graph. Then for every given edge e of G, G has an even factor F that does not contain e and satisfies $\sigma(F) \geq$* $\min\{|G|, 5\}$.

Proof. Let $e = xy$ be an edge. If $d_G(x) = 3$ or $d_G(y) = 3$, then the assertion is clear by Theorem 1. Therefore, $d_G(x) \geq 4$ and $d_G(y) \geq 4$. If $G - e$ is 3-edge-connected, then by Theorem 2, $G - e$ has an even factor F in which $\sigma(F) \geq \min\{|G - e|, 5\}$. Hence, F is a desired even factor of G. Then we may assume that $S' = \{e_1, e_2\}$ is a minimum edge cut of $G - e$. Also, $S = \{e_1, e_2, e\}$ is a minimum edge cut in G. Let G_1 and G_2 be two components of $G - S$ and $G_1' = G/G_1$ and $G_2' = G/G_2$. We can assume that $S \subseteq E(G_1')$ and $S \subseteq E(G_2')$. We have $d_{G_1'}(G_1^*) = d_{G_2'}(G_2^*) = 3$. By Theorem 1, there are even factors F_1 of G_1' and F_2 of G_2' that contain e_1 and e_2, respectively, but do not contain e and satisfy $\sigma(F_1) \geq \min\{|G_1'|, 5\}$ and $\sigma(F_2) \geq \min\{|G_2'|, 5\}$. It is easy to see that $F = ((F_1 - G_1^*) \cup (F_2 - G_2^*)) \cup \{e_1, e_2\}$ is a desired even factor of G. ■

Now, we have the following theorem.

Theorem 6. *Let G be a 3-edge-connected graph and $e = vx, f = vy$ be two adjacent edges of G. Then G has an even factor F containig e and f such that $\sigma(F) \geq \min\{|G|, 5\}$.*

Proof. We can assume that $d_G(v) \geq 4$, by Theorem 1. Suppose on the contrary that G is a counterexample to the statement such that $|E(G)|$ is minimized. Consider the graph $H = G - \{f, e\} + v'x + v'y + vv'$, where v' is a new vertex. There are three cases.

Case 1. $\kappa'(H) = 3$. In this case by Theorem 1, H has an even factor F' containing $v'x$ and $v'y$ in which $\sigma(F') \geq \min\{|H|, 5\}$, since $d_H(v') = 3$. By replacing v' with v in F', we obtain an even factor F of G containing e and f. If $\sigma(F) \geq \min\{|G|, 5\}$, then we are done. Therefore, F has exactly one component D of order 4 and F' has exactly one component D' of order 5. Now, there are two subcases.

Subcase 1a. $x \neq y$. In this case there is vertex s such that $V(D) = \{x, y, v, s\}$ and there is a vertex $t \in \{x, y, s\}$ such that there is a multiple edge between t and v. Consider graph G' obtained from G by contracting this multiple edge and removing all resulted loops. Let v^* be the new vertex of G' instead of v and t. Since $x \neq y$, we can assume that $f \in E(G')$. The graph G' is 3-edge-connected and $d_{G'}(v^*) \geq 3$, since G is a 3-edge-connected graph. The graph G' has an even factor F' containing f in which $\sigma(F') \geq \min\{|G'|, 5\}$, since $|E(G')| < |E(G)|$. If F' contains even number of edges incident with v and even number of edges incident with t, then we can convert F' to a desired even factor of G by adding e and another edge of the contracted multiple edge. Otherwise, F' contains odd number of edges incident with v and odd number of edges incident with x and we can convert F' to a desired even factor of G by adding the edge e, and we are done.

Subcase 1b. $x = y$. In this case there are vertices r and t such that $E(D') = \{vr, vt, rx, tx, v'x, v'x\}$ and $E(D) = \{vr, vt, rx, tx, e, f\}$. Graph $G'' = G - e$ is 3-edge-connected, since there are three edge disjoint path between v and x in G''. By Theorem 5, G'' has an even factor F'' in which $\sigma(F'') \geq \min\{|G''|, 5\}$ and F'' does not contain f. It is obvious that $F = F'' + \{e, f\}$ is a required even factor of G.

Case 2. $\kappa'(H) = 2$. In this case assume that S is a minimum edge cut of H. It is clear that $vv' \in S$, since G is 3-edge-connected. We may suppose that $S = \{vv', zw\}$. It is possible that $\{x, y\} \cap \{z, w\} \neq \emptyset$.

Now, let G_3 and G_4 be two components of $H - S$ and we have $v, z \in V(G_3)$ and $v', w \in V(G_4)$. Assume first $v = z$. It is clear that v is a cut vertex of G. Let G_1 be a component of $G - \{e, f, zw\}$ containing v, and let $G_2 =$

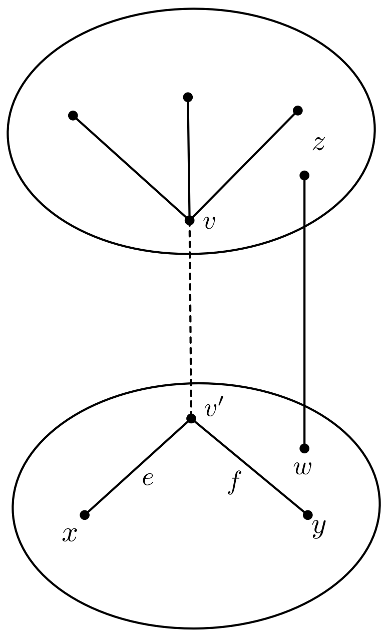

Figure 2. H.

$\langle V(G) - (V(G_1) - \{v\})\rangle_G$. Since G is 3-edge-connected, G_1 and G_2 are 3-edge-connected and $\delta(G_1), \delta(G_2) \geq 3$. We can consider that $e, f \in E(G_1)$. Since $|E(G_1)| < |E(G)|$, the graph G_1 has an even factor F_1 containinig e and f such that $\sigma(F_1) \geq \min\{|G_1|, 5\}$. Also, G_2 has an even factor F_2 in which $\sigma(F_2) \geq \min\{|G_2|, 5\}$, by Theorem 2. It is clear that $F = F_1 \cup F_2$ is an even factor of G containing e and f in which $\sigma(F) \geq \min\{|G|, 5\}$. Thus we may assume $v \neq z$. It is obvious that $v' \neq w$. We show that $G_3 + vz$ and $G_4 + v'w$ are 3-edge-connected and $\delta(G_3 + vz), \delta(G_4 + v'w) \geq 3$. It is possible that we obtain multiple edges. We show that $G_3 + vz$ is 3-edge-connected and for $G_4 + v'w$ the result follows similarly. Let S' be a minimum edge cut for $G_3 + vz$. If G_3' and G_3'' are two componenets of $(G_3 + vz) - S'$, then v and z are not in the same component, since otherwise, S' is an edge cut for G and it is a contradiction. Then we may assume that $v \in V(G_3')$ and $z \in V(G_3'')$ and we have $vz \in S'$ and $|S'| = 2$, since G is 3-edge-connected. Let $S' = \{vz, e'\}$. If $vz \in E(G)$, then we have $e' = vz$. Now, according to Figure 3, it is clear that $\{e', zw\}$ is an edge cut in G and it is a contradiction. Hence, $G_3 + vz$ is 3-edge-connected.

We have $e, f \in E(G_4 + v'w)$ and $|E(G_4 + v'w)| < |E(G)|$. Then $G_4 + v'w$ has an even factor F_4 containing e and f such that $\sigma(F_4) \geq \min\{|G_4 + v'w|, 5\}$. By Theorem 5, $G_3 + vz$ has an even factor F_3 in which $\sigma(F_3) \geq \min\{|G_3 + vz|, 5\}$ and F_3 does not contain vz. Therefore, by replacing v' with v in F_4, $F = F_3 \cup F_4$

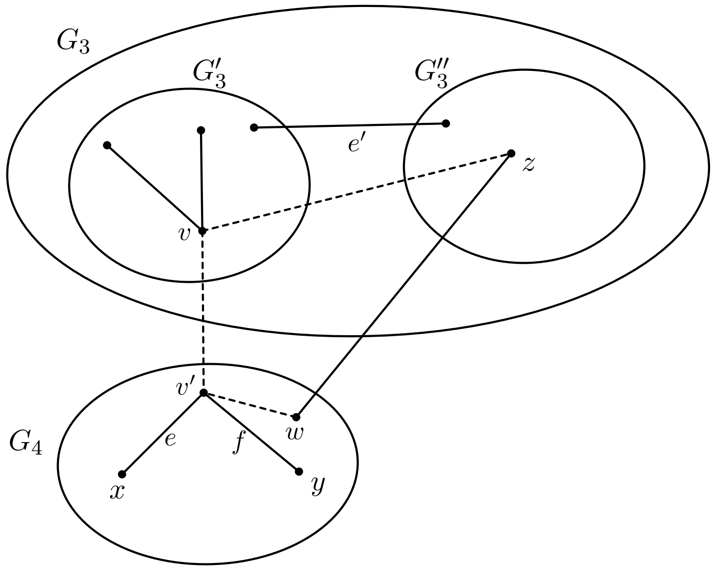

Figure 3. G_3, G_4, G_3' and G_3''.

is a desired even factor of G containing e and f such that $\sigma(F) \geq \min\{|G|, 5\}$.

Case 3. $\kappa'(H) = 1$. In this case vv' is a bridge of H. Hence, $\{e, f\}$ is an edge cut of G, a contradiction. ■

In the next theorem we show that Theorem 4 is not satisfied for each pair of edges of G.

Theorem 7. *There are infinitely many* 3*-edge-connected graphs which do not have an even factor F containing two arbitrary prescribed edges in which $\sigma(F) \geq$ 5.*

Proof. We costruct these graphs like in Figure 4.

The graph G is cubic and 3-edge-connected. By symmetry, it is easy to see that G does not have an even factor F containing e and f such that $\sigma(F) \geq 5$. ■

3. Even Factors of 2-Edge-Connected Graphs

Now, there are some results in 2-edge-connected bipartite graphs with minimum degree at least 3.

Lemma 8 [6]**.** *Let $r \geq 2$ be an integer. Then every connected r-regular bipartite graph is* 2*-edge-connected.*

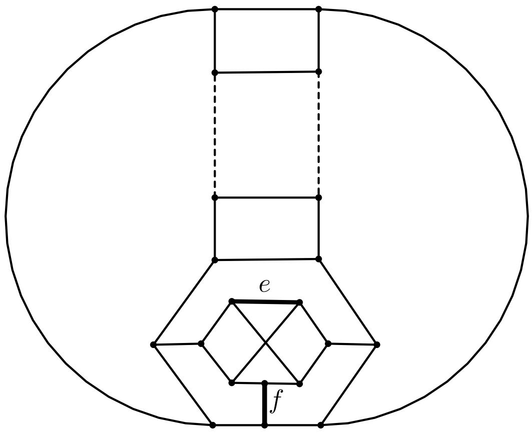

Figure 4. The 3-edge-connected graph G.

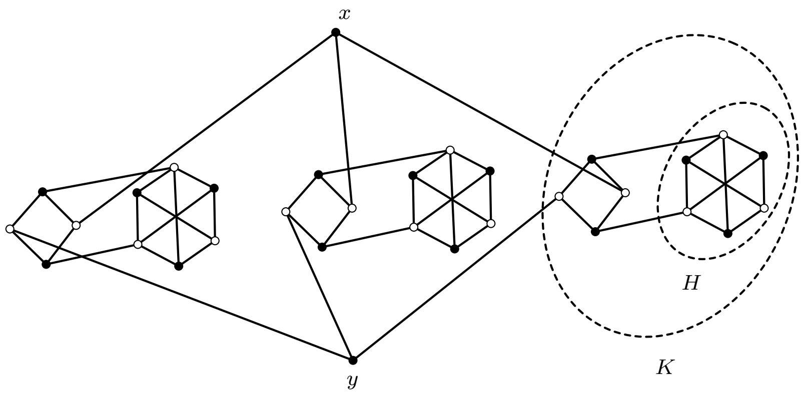

Figure 5. G.

Theorem 9 [6]**.** *Every connected cubic bipartite graph has a* $\{C_n|n \geq 6\}$*-factor.*

By König's theorem [1], the edges of an r-regular bipartite graph can be decomposed into 1-factors. By combining three 1-factors of an r-regular graph, we obtain a cubic bipartite graph and we have the following corollary.

Corollary 10. *Every* r*-regular bipartite graph has a* $\{C_n|n \geq 6\}$*-factor.*

Also, by Theorem 2, it is obvious that every 3-edge-connected bipartite graph has an even factor in which the order of its components is at least 6. But, we have the following theorem.

Theorem 11. *There are infinitely many* 2*-edge-connected bipartite graphs with minimum degree at least* 3 *having no even factor* F *in which* $\sigma(F) \geq 6$.

Proof. Consider the graph G depicted in Figure 5. By the symmetry of three components $G-\{x,y\}$, if G has an even factor F with $\sigma(F)\geq 6$, then K has an even factor F' with $\sigma(F')\geq 6$. The graph K does not have an even factor such that every component has order at least 6. Hence, G does not have a desired even factor. Now, if we put each 3-edge-connected graph instead of the subgraph H of G, then we can construct infinitely many such graphs. ■

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