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### $T_r$ -SPAN OF DIRECTED WHEEL GRAPHS

MARC BESSON AND BARRY TESMAN

Dickinson College Carlisle, PA, USA

e-mail: bessonm@dickinson.edu tesman@dickinson.edu

### Abstract

In this paper, we consider T-colorings of directed graphs. In particular, we consider as a T-set the set  $T_r = \{0, 1, 2, \ldots, r-1, r+1, \ldots\}$ . Exact values and bounds of the  $T_r$ -span of directed graphs whose underlying graph is a wheel graph are presented.

Keywords: T-coloring, digraph, wheel graph, span.

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### 1. INTRODUCTION

T-colorings are a generalization of proper vertex colorings of graphs. They were introduced by Hale [2] to model the frequency assignment problem. The problem in assigning frequencies to requesters comes from the need to assign them in a manner that minimizes the use of the frequency spectrum (e.g., AM radio or UHF television) while avoiding the interferences and separation constraints that can occur amongst transmitters. T-colorings have been widely studied for over three decades since Hale's paper appeared in 1980. Much of the research has optimized T-colorings for various classes of graphs, as well as classes of the separation constraints. For example, see Sivagami and Rajasingh [6], Juan *et al.* [4], and Janczewski [3]. Variants of T-colorings, in particular list T-colorings, have been studied by Junosza-Szaniawski and Rzążewski [5], Tesman [8], and Fiala *et al.* [1]. In this paper we are concerned with T-colorings of digraphs which were introduced by Tesman [7] to model the special case of unidirectional transmitters. A T-set is a set of nonnegative integers. Given a digraph D = (V, A) and a T-set T, a T-coloring of D is a function  $c : V(D) \to \mathbb{Z}^+$  such that if  $(x, y) \in A(D)$ , then  $c(x) - c(y) \notin T$ . The span of a T-coloring c of D is defined as:

span of 
$$c = \max_{x \in V(D)} c(x) - \min_{x \in V(D)} c(x).$$

The minimum span over all *T*-colorings of a digraph *D* for a fixed T-set *T* is called the *T*-span of *D* and denoted  $sp_T(D)$ . We will assume that  $0 \in T$  because a coloring must be proper and, without loss of generality, 1 will be the minimum color in any *T*-coloring. We will find bounds and the exact *T*-span for some special classes of directed wheel graphs.

The underlying graph of the digraphs studied in this paper is a wheel graph,  $W_n$ , i.e., a circuit for which every vertex on the circuit,  $b_1, b_2, \ldots, b_n$ , is connected to a single "hub" vertex, y.<sup>1</sup> (See Figure 1.) The chromatic number of a wheel graph,  $\chi(W_n)$ , will be used in Sections 3 and 4. Recall that  $\chi(W_n) = 3$  or 4 depending on whether or not the wheel graph's circuit is even or odd, respectively. These values provide lower bounds for the *T*-span of their respective directed wheel graphs since any *T*-coloring is also a proper vertex coloring because 0 is in every T-set.

The underlying wheel graph's circuit edges  $\{b_i, b_{i+1}\}$ , for i = 1, 2, ..., n - 1, and  $\{b_1, b_n\}$  will be directed clockwise, i.e.,  $(b_i, b_{i+1})$ , for i = 1, 2, ..., n - 1, and  $(b_n, b_1)$  will be the circuit arcs of the digraph. (Note that a counterclockwise orientation of the circuit edges will yield the same results that we prove in this paper; the two digraphs are merely mirror images of one another.) Arcs directed from a circuit vertex to the hub vertex will be called *introverted*. Circuit vertices incident with introverted arcs will be called *introverts*. Similarly, arcs directed from the hub vertex to a circuit vertex will be called *extroverted* and circuit vertices incident with extroverted arcs will be called *extroverts*.

We will denote our directed wheel graphs by  $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$  where, going clockwise and starting with "spoke"  $\{b_1, y\}$ , there are  $a_1$  introverted arcs, followed by  $a_2$  extroverted arcs, ..., and ending with  $a_{2j}$  extroverted arcs.<sup>2</sup> The notation refers to a digraph with n + 1 vertices where  $n = a_1 + a_2 + \cdots + a_{2j}$  vertices are on the circuit and there is one hub vertex. We will assume that n > 1, otherwise, the digraph has a loop and no *T*-coloring is possible. See Figure 2 for a specific example,  $CW_{2,5,1,3}$ . Note that for  $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$ , each  $a_i > 0$  except in the special case of a directed wheel graph with only extroverted arcs, i.e.,  $CW_{0,n}$ .

The T-sets that we will be considering in this paper will be denoted by  $T_r$ and are of the form

$$T_r = \{0, 1, 2, 3, \dots, r-1, r+1, r+2, \dots\}.$$

<sup>&</sup>lt;sup>1</sup>Our definition of  $W_n$  is slightly nonstandard to simplify the notation on the circuit vertices. <sup>2</sup>Note that  $CW_{a_1,a_2,a_3,\ldots,a_{2j}} \cong CW_{a_3,a_4,\ldots,a_{2j},a_1,a_2} \cong CW_{a_5,a_6,\ldots,a_{2j},a_1,a_2,a_3,a_4} \cong \cdots$  by rotational symmetry.



For T-sets of this type, we will refer to their T-coloring as a  $T_r$ -coloring. Again, we will only consider  $T_r$  for  $r \ge 1$ . Note that

(1)  

$$c \text{ is a } T_r \text{-coloring of } CW_{a_1,a_2,a_3,\dots,a_{2j}}$$

$$if \text{ and only if}$$

$$(x,y) \in A\left(CW_{a_1,a_2,a_3,\dots,a_{2j}}\right) \Rightarrow c(x) - c(y) = r \text{ or } c(x) - c(y) < 0.$$

# 2. $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$

Our first two lemmas prove a relationship between the colors of most of the vertices on the circuit of the directed wheel graph  $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$  and the color of the hub vertex for any  $T_r$ -coloring. Let the vertices of  $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$  be labeled as in Figure 1.

**Lemma 1.** For any  $T_r$ -coloring c of digraph  $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$ ,  $c(b_i) < c(y)$  for all introverts  $b_i$  such that  $b_{i+1}$  is also an introvert where y is the hub vertex.

**Proof.** Note that  $c(b_i) \neq c(y)$  since  $(b_i, y) \in A(CW_{a_1, a_2, a_3, \dots, a_{2j}})$ . So, suppose that

(2)  $c(b_i) > c(y)$  for some introvert  $b_i$  such that  $b_{i+1}$  is also an introvert.

Since  $(b_i, y) \in A(CW_{a_1, a_2, a_3, \dots, a_{2j}})$  and applying (1),

$$(3) c(b_i) - c(y) = r$$

Now consider  $c(b_{i+1})$ . Since  $(b_{i+1}, y) \in A(CW_{a_1, a_2, a_3, \dots, a_{2j}}), c(b_{i+1}) \neq c(y)$ .

Case 1.  $c(b_{i+1}) < c(y)$ . By (2), we have  $c(b_{i+1}) < c(b_i)$ . Since  $(b_i, b_{i+1}) \in A(CW_{a_1, a_2, a_3, \dots, a_{2j}})$ , (4)  $c(b_i) - c(b_{i+1}) = r$ .

By equations (3) and (4),  $c(y) = c(b_{i+1})$ , which is a contradiction since  $(b_{i+1}, y) \in A(CW_{a_1,a_2,a_3,\ldots,a_{2j}})$ .

Case 2.  $c(b_{i+1}) > c(y)$ . Since  $(b_{i+1}, y) \in A(CW_{a_1, a_2, a_3, \dots, a_{2j}})$  and applying (1),

(5) 
$$c(b_{i+1}) - c(y) = r.$$

By equations (3) and (5),  $c(b_i) = c(b_{i+1})$  which is a contradiction since  $(b_i, b_{i+1}) \in A(CW_{a_1, a_2, a_3, \dots, a_{2j}})$ .

Lemma 1 proved that for any  $T_r$ -coloring of  $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$  and any maximal set of consecutive introverts,  $b_i, b_{i+1}, \ldots, b_{i+h}$ , the color assigned to each introvert, except possibly  $b_{i+h}$ , is less than the color assigned to the hub vertex y.

Similarly, Lemma 2 proves that for any  $T_r$ -coloring of  $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$  and any maximal set of consecutive extroverts,  $b_i, b_{i+1}, \ldots, b_{i+h}$ , the color assigned to each extrovert, except possibly  $b_i$ , is greater than the color assigned to the hub vertex y.

**Lemma 2.** For any  $T_r$ -coloring c of digraph  $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$ ,  $c(b_i) > c(y)$  for all extroverts  $b_i$  such that  $b_{i-1}$  is also an extrovert where y is the hub vertex.

**Proof.** Note that  $c(b_i) \neq c(y)$  since  $(y, b_i) \in A(CW_{a_1, a_2, a_3, \dots, a_{2j}})$ . So, suppose that

(6)  $c(b_i) < c(y)$  for some extrovert  $b_i$  such that  $b_{i-1}$  is also an extrovert.

Since  $(y, b_i) \in A(CW_{a_1, a_2, a_3, \dots, a_{2_i}})$  and applying (1),

(7) 
$$c(y) - c(b_i) = r.$$

Now consider  $c(b_{i-1})$ . Since  $(y, b_{i-1}) \in A(CW_{a_1, a_2, a_3, \dots, a_{2j}}), c(y) \neq c(b_{i-1})$ .

Case 1.  $c(b_{i-1}) > c(y)$ . By (6), we have  $c(b_{i-1}) > c(b_i)$ . Since  $(b_{i-1}, b_i) \in A(CW_{a_1,a_2,a_3,\dots,a_{2j}})$ , (8)  $c(b_{i-1}) - c(b_i) = r$ .

By (7) and (8), we have  $c(y) = c(b_{i-1})$ , which is a contradiction since  $(y, b_{i-1}) \in (CW_{a_1,a_2,a_3,\ldots,a_{2j}})$ .

Case 2. 
$$c(b_{i-1}) < c(y)$$
. Since  $(y, b_{i-1}) \in A(CW_{a_1, a_2, a_3, \dots, a_{2j}})$ ,

(9) 
$$c(y) - c(b_{i-1}) = r$$

By (7) and (9), we have  $c(b_i) = c(b_{i-1})$  which is a contradiction since  $(b_{i-1}, b_i) \in A(CW_{a_1,a_2,a_3,\ldots,a_{2j}})$ .

Consider the special case when our directed wheel graph only has one set of introverts and one set of extroverts. We will use the notation  $CW_{k,l}$  for such a digraph and use the vertex labeling as in Figure 3. The next corollary and lemma consider the relationship between  $c(v_l)$  and  $c(u_1)$ , as well as  $c(u_k)$  and  $c(v_1)$ , i.e., the colors of the last/first introvert and first/last extrovert, for any  $T_r$ -coloring c of  $CW_{k,l}$ .



Figure 3.  $CW_{k,l}$ .

**Corollary 3.** In any  $T_r$ -coloring c of  $CW_{k,l}$  where  $k, l \geq 2$ ,

$$c(v_l) > c(u_1).$$

**Proof.** Since  $k, l \ge 2$ ,  $c(u_1) < c(y) < c(v_l)$  by Lemmas 1 and 2.

**Lemma 4.** In any  $T_r$ -coloring c of  $CW_{k,l}$ ,  $c(v_1) > c(u_k)$ .

**Proof.** Since  $(u_k, v_1) \in A(CW_{k,l}), c(u_k) \neq c(v_1)$ . So, suppose that  $c(u_k) > c(v_1)$ . Since  $(u_k, v_1) \in A(CW_{k,l}), c(v_1) = c(u_k) - r$ . Since  $(u_k, y) \in A(CW_{k,l})$ ,

(a) 
$$c(y) = c(u_k) - r$$
 or (b)  $c(y) > c(u_k)$ .

Also, since  $(y, v_1) \in A(CW_{k,l})$ ,

$$(c) c(y) = (c(u_k) - r) + r = c(u_k) \text{ or } (d) c(y) < c(u_k) - r.$$

We reach a contradiction since no pair of these conditions, (a)–(c), (a)–(d), (b)–(c), or (b)–(d), can occur simultaneously.

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**Corollary 5.** In any  $T_r$ -coloring c of  $CW_{k,l}$ , it is never the case that both  $c(u_k) > c(y)$  and  $c(v_1) < c(y)$ .

Our next lemma considers those vertices of  $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$  which cannot be colored 1, i.e., the minimum color of any *T*-coloring.

**Lemma 6.** For any  $T_r$ -coloring c of digraph  $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$ , c(x) > 1 for any extrovert  $x \in V(CW_{a_1,a_2,a_3,\ldots,a_{2j}})$ .

**Proof.** We prove this lemma for the digraph  $CW_{a_1,a_2}$ , i.e.,  $CW_{k,l}$ . The general case for  $CW_{a_1,a_2,a_3,\ldots,a_{2j}}$  follows immediately. Let the vertices of  $CW_{k,l}$  be labeled as in Figure 3.

Case 1. consider  $v_i \in V(CW_{k,l})$  for i = 2, 3, ..., l. By Lemma 2,  $c(v_i) > c(y)$ Thus,  $c(v_i) > 1$ , for i = 2, 3, ..., l.

Case 2. consider  $v_1 \in V(CW_{k,l})$  and suppose that  $c(v_1) = 1$ . Then, c(y) = r + 1 since  $(y, v_1) \in A(CW_{k,l})$ . Also,  $c(u_k) = r + 1$  since  $(u_k, v_1) \in A(CW_{k,l})$ . Therefore,  $c(y) = c(u_k)$  which is a contradiction since  $(u_k, y) \in A(CW_{k,l})$ . Thus,  $c(v_1) > 1$ .

We end this section by showing that the *T*-span of  $CW_{a_1,a_2,\ldots,a_{2j}}$  is not entirely dependent on the order of the  $a_i$ s. Note that we will use the notation  $\overleftarrow{D}$  to denote the digraph obtained by reversing all of the arcs of digraph D, i.e., the *reverse digraph* of D.

Lemma 7. 
$$sp_{T_r}(CW_{k,l}) = sp_{T_r}\left(\overleftarrow{CW}_{k,l}\right)$$

**Proof.** Let c be a  $T_r$ -coloring of  $CW_{k,l}$  and let  $M_c = \max c(x)$  for  $x \in V(CW_{k,l})$ . Then define a new coloring d of  $CW_{k,l}$  by

$$d(x) = M_c - c(x) + 1$$
, for  $x \in V(CW_{k,l})$ .

Suppose that  $(x, z) \in \overleftarrow{CW}_{k,l}$ , i.e.,  $(z, x) \in CW_{k,l}$ . Then  $d(x) - d(z) = (M_c - c(x) + 1) - (M_c - c(z) + 1) = c(z) - c(x) < 0$  or equals r by (1). Therefore, d is a  $T_r$ -coloring of  $\overleftarrow{CW}_{k,l}$  and not a  $T_r$ -coloring of  $CW_{k,l}$ .

The maximum and minimum values of c are M and 1, respectively. Then, max d(x) = M - 1 + 1 = M and min d(x) = M - M + 1 = 1 since d(x) is a strictly decreasing linear function of c(x). Therefore, the span of d is less than or equal to the span of c which implies  $sp_{T_r}\left(\overleftarrow{CW}_{k,l}\right) \leq sp_{T_r}(CW_{k,l})$ .

A similar proof shows that  $sp_{T_r}(CW_{k,l}) \leq sp_{T_r}(\overline{CW}_{k,l})$  using the fact that  $\overleftarrow{CW}_{k,l} = CW_{k,l}$ .

Therefore, 
$$sp_{T_r}(CW_{k,l}) = sp_{T_r}\left(\overleftarrow{CW}_{k,l}\right).$$

# **Theorem 8.** $sp_{T_r}(CW_{k,l}) = sp_{T_r}(CW_{l,k}).$

**Proof.** Note that  $\overleftarrow{CW}_{k,l}$  has a consecutive set of k extroverted arcs and a consecutive set of l introverted arcs since  $CW_{k,l}$  has a consecutive set of k introverted arcs and a consecutive set of l extroverted arcs which are being reversed. Then, using a mirror image of  $CW_{k,l}$ , it is easy to see that  $\overleftarrow{CW}_{k,l}$  is isomorphic to  $CW_{l,k}$ . Therefore, by Lemma 7, the theorem follows.

Although we have no need for a stronger result, it can also be shown that

$$sp_{T_r}\left(CW_{a_1,a_2,\ldots,a_{2j}}\right) = sp_{T_r}\left(CW_{a_{i+1},a_{i+2},\ldots,a_{2j-1},a_{2j},a_1,a_2,\ldots,a_i}\right)$$

using a similar proof as above and the second footnote on page 2.

An example of Lemma 7 and Theorem 8 for a  $T_5$ -coloring of  $CW_{5,3}$  is shown in Figure 4.



Figure 4. Equal span  $T_5$ -colorings of  $CW_{5,3}$ ,  $\overleftarrow{CW}_{5,3}$ ,  $CW_{3,5}$ .

# 3. $sp_{T_r}(CW_{k,l}), r > 1$

In this section and the next, we consider the special case of  $CW_{k,l}$ , i.e., where the directed wheel graph's spokes only consist of a consecutive set of introverted arcs and a consecutive set of extroverted arcs. We assume that r > 1; the special case of r = 1 is addressed in Section 4. To prove our main result, i.e., the calculation of  $sp_{T_r}(CW_{k,l})$ , we consider different conditions on k and l.

First, we consider  $CW_k$  and  $CW_{0,l}$ , i.e., the directed wheel graphs with only introverted arcs and only extroverted arcs. Recall that we only consider  $CW_k$  for  $k \geq 3$ , and  $CW_{0,l}$ , for  $l \geq 3$ .

Theorem 9. For r > 1,

$$sp_{T_r}(CW_k) = sp_{T_r}(CW_{0,l}) = r + 1.$$

M. Besson and B. Tesman



Figure 5. Optimal  $T_r$ -colorings of  $CW_k$ .

**Proof.** Applying Theorem 8, it is sufficient to show that  $sp_{T_r}(CW_k) = r + 1$ . Figure 5 provides  $T_r$ -colorings of  $CW_k$ , depending on the parity of k, whose span is r + 1. Next we show that the span of any  $T_r$ -coloring of  $CW_k$  is greater than or equal to r + 1.

Let the vertices of  $CW_k$  be labeled as in Figure 1 and c be a  $T_r$ -coloring of  $CW_k$ . By Lemma 1,  $c(b_i) < c(y)$  for i = 1, 2, ..., k. Without loss of generality, let  $c(b_1) = 1$ . Then  $c(b_k) = r+1$  since  $(b_k, b_1) \in A(CW_k)$ . Therefore,  $c(y) \ge r+2$  which implies that the span of the  $T_r$ -coloring c of  $CW_k$  is greater than or equal to r+1. Therefore,  $sp_{T_r}(CW_k) = r+1$ .

Next, we prove our main result for  $CW_{k,l}$ . Note that the exceptions to the theorem are treated in Section 6.

**Theorem 10.** *For* r > 1*,* 

**Proof.** Case 1.  $k + l \leq r$  and  $\min(k, l) > 0$ . Let  $CW_{k,l}$  be labeled as in Figure 3 and c be the following coloring:

$$c(u_s) = s$$
, when  $s = 1, 2, ..., k$ ;

$$c(y) = k + 1;$$
  
 $c(v_t) = r - (l - 1) + t$ , when  $t = 1, 2, ..., l$ 

See Figure 6. Note that the minimum color of the extroverts,  $c(v_1) = r - l + 2$ , is greater than c(y) = k + 1 since  $k + l \leq r$ . Thus, the sequence of colors  $c(u_1)$ ,  $c(u_2), \ldots, c(u_k), c(v_1), c(v_2), \ldots, c(v_l)$  is strictly increasing and  $c(v_l) > c(u_1) \Rightarrow$  $c(v_1) - c(v_1) = r$ . Also,  $c(y) > c(u_i)$  for  $i = 1, 2, \ldots, k$  and  $c(y) < c(v_j)$  for  $j = 1, 2, \ldots, l$ . Thus, by (1), c is a  $T_r$ -coloring of  $CW_{k,l}$  and the span of c is r.



Figure 6. Optimal  $T_r$ -coloring of  $CW_{k,l}$ ,  $k+l \leq r$ .

Next, we show that the span of any  $T_r$ -coloring of  $CW_{k,l}$  is greater than or equal to r which implies  $sp_{T_r}(CW_{k,l}) = r$ . Let d be a  $T_r$ -coloring of  $CW_{k,l}$  and let  $CW_{k,l}$  be labeled as in Figure 1. The sequence of colors on the non-hub vertices,  $d(b_1), d(b_2), \ldots, d(b_n), d(b_1)$ , can not be strictly increasing since the vertices form a cycle. When the sequence decreases, it must do so by exactly r. Therefore, the span of  $T_r$ -coloring d of  $CW_{k,l}$  is greater than or equal to r.

Case 2. k + l > r and  $\min(k, l) = 1$ . Applying Theorem 8, it is sufficient to only consider  $\min(k, l) = k = 1$ , i.e.,  $CW_{1,l}$ . Figure 7 provides  $T_r$ -colorings with span r + 1, i.e., whose maximum color is r + 2, for  $CW_{1,l}$  except when r = 2 and l = 3. This special case, as well as when r = 2, k = 3, and l = 1, are addressed in Section 6.

Next we show that if c is any  $T_r$ -coloring of  $CW_{1,l}$ , k+l > r and  $\min(k,l) = k = 1$ , then  $\max c(x) \ge r+2$  for  $x \in V(CW_{1,l})$  which implies  $sp_{T_r}(CW_{1,l}) = r+1$ , for all but the special case. Let the vertices of  $CW_{1,l}$  be labeled as in Figure 3 and c be a  $T_r$ -coloring of  $CW_{1,l}$ . By Lemma 6, the minimum color 1 must be assigned to an introvert or the hub vertex.

Case 2.a. c(y) = 1. Each  $c(b_i) \neq 1$  since  $(b_i, y)$  or  $(y, b_i) \in A(CW_{1,l})$ , for i = 1, 2, ..., n. Let  $m = \min c(b_i)$ , for i = 1, 2, ..., n. Then  $m \ge 2$  and the color on the circuit vertex preceding it in the sequence  $b_1, b_2, ..., b_n, b_1$  must be colored  $r + m \ge r + 2$ .



Figure 7. Optimal  $T_r$ -colorings of  $CW_{1,l}$ .

Case 2.b.  $c(u_1) = 1$ . By Lemma 6,  $c(v_i) > 1$ , for i = 1, 2, ..., l and c(y) > 1since  $(u_1, y) \in A(CW_{1,l})$ . Note that  $k + l > r \Rightarrow 1 + l > r \Rightarrow l > r - 1 \Rightarrow l \ge r$ . Assume max  $c(x) \le r + 1$ , for  $x \in V(CW_{1,l})$ . Thus there are r available colors, 2, 3, ..., r + 1, to color  $l + 1 \ge r + 1$  vertices,  $y, v_1, v_2, ..., v_l$ . By the pigeonhole principle, at least two of these vertices must be colored the same and neither of them can be y since  $(y, v_i) \in A(CW_{1,l})$  for all i. Thus, the sequence of colors  $c(v_1), c(v_2), ..., c(v_l)$  is not strictly increasing. Suppose that  $c(v_i) > c(v_{i+1})$ . Then,  $c(v_i) - c(v_{i+1}) = r \Rightarrow c(v_i) = c(v_{i+1}) + r \ge 2 + r$  which is a contradiction. Therefore,  $sp_{T_r}(CW_{1,l}) = r + 1$  (for all but the special cases).

Therefore,  $Sp_{I_r}(CW_{1,l}) = l + 1$  (for an but the special cases).

Case 3. k + l > r and  $1 < \min(k, l) = k \le r$ . Applying Theorem 8, it is sufficient to only consider  $\min(k, l) = k$ . Figures 8, 9 and 10 provide  $T_r$ -colorings with span k + r, i.e., whose maximum color is k + r + 1, for  $CW_{k,l}$  except for the following six cases: l = 4 and k = r - 1 for r = 3, 4, 5; l = 3 and k = r for r = 2, 3; and l = 2 and k = r - 1 for r = 3, as well as their corresponding reverse digraphs. All of these special cases are addressed in Section 6.



Figure 8. Optimal  $T_r$ -coloring of  $CW_{k,l}$ , k + l > r and  $1 < \min(k, l) = k \le r$ .



Figure 9. Optimal  $T_r$ -colorings of  $CW_{k,l}$ , k+l > r and  $1 < \min(k,l) = k = r$ .



Figure 10. Optimal  $T_r$ -colorings of  $CW_{k,l}$ , k + l > r and  $1 < \min(k, l) = k < r$ .

Next we show that if c is any  $T_r$ -coloring of  $CW_{k,l}$ , k + l > r and  $1 < \min(k,l) = k \leq r$ , then  $\max c(x) \geq k + r + 1$  for  $x \in V(CW_{k,l})$  which implies  $sp_{T_r}(CW_{k,l}) = k + r$ , for all of the cases addressed in Figures 8, 9 and 10. Let the vertices of  $CW_{k,l}$  be labeled as in Figure 3 and c be a  $T_r$ -coloring of  $CW_{k,l}$ . By Lemma 6, the minimum color 1 must be assigned to an introvert or the hub vertex.

Case 3.a. Suppose  $c(u_i) = 1$ , for  $i \neq 1$ . Then  $c(u_{i-1}) = r+1$  since  $(u_{i-1}, u_i) \in A(CW_{k,l})$ . Also, by Lemma 1,  $c(y) \geq r+2$  and, by Lemma 2,  $c(v_i) > c(y) \geq r+2$ , for  $i = 2, 3, \ldots, l$ . Assume max  $c(x) \leq k + r$ , for  $x \in V(CW_{k,l})$ . Thus there are k-1 available colors,  $r+2, r+3, \ldots, r+k$ , to color  $l \geq k$  vertices,  $y, v_2, v_3, \ldots, v_l$ . By the pigeonhole principle, at least two of these vertices must be colored the same and neither of these same-colored vertices can be y since  $(y, v_i) \in A(CW_{k,l})$  for all i. Thus, the sequence of colors  $c(v_1), c(v_2), \ldots, c(v_l)$  is not strictly increasing. Suppose that  $c(v_i) > c(v_{i+1})$ . Then,  $c(v_i) - c(v_{i+1}) = r \Rightarrow c(v_i) = c(v_{i+1}) + r \geq (r+2) + r \geq k + r + 2$ , since  $k \leq r$ , which is a contradiction.

Thus, if  $c(u_i) = 1$ , for  $i \neq 1$ , then  $\max c(x) \ge k + r + 1$ , for  $x \in V(CW_{k,l})$ .

Case 3.b. Suppose that  $c(u_1) = 1$ . Then  $c(u_i) \neq 1$  for  $i = 2, 3, \ldots, k$  and  $c(v_l) = r + 1$  since  $(v_l, u_1) \in A(CW_{k,l})$ .

Case 3.b.i.  $c(u_k) < c(y)$  and  $c(v_1) < c(y)$ . Since  $c(v_l) = r + 1$  then, by Lemma 2,  $c(y) \le r$ . Also since  $(y, v_1) \in A(CW_{k,l})$  and  $c(y) > c(v_1)$  then  $c(y) - c(v_1) = r \Rightarrow c(v_1) = c(y) - r \le 0$  which is a contradiction.

Case 3.b.ii.  $c(u_k) < c(y)$  and  $c(v_1) > c(y)$ . Since  $c(u_k) < c(y)$ , then the color of each of the k introverts is less than c(y). If the introverts' colors are distinct then  $c(y) \ge k + 1$ . If the introverts' colors are not distinct then there exists a  $u_i$ such that  $c(u_i) > c(u_{i+1}) \Rightarrow c(u_i) = r + c(u_{i+1}) \ge r + 2$  since  $c(u_{i+1}) > 1$ . Thus,  $c(y) > c(u_i) \ge r + 2 \ge k + 2 \Rightarrow c(y) > k + 2$ .

Thus,  $c(y) \ge k+3$ .

Case 3.b.ii.a.  $c(v_i) > c(v_{i+1})$  for some  $i \in \{1, 2, ..., l-1\}$ . Thus,  $c(v_i) = c(v_{i+1}) + r$ . Since  $c(v_{i+1}) > c(y) \ge k+3$ , then  $c(v_i) = c(v_{i+1}) + r > k+3+r$ . Thus,  $\max c(x) \ge k+r+1$  for  $x \in V(CW_{k,l})$ .

Case 3.b.ii. $\beta$ .  $c(v_i) < c(v_{i+1})$  for all  $i \in \{1, 2, \ldots, l-1\}$ . Thus, the extroverts' colors are all distinct and all greater than c(y). They are also strictly increasing from  $v_1$  to  $v_l$  and they are all less than or equal to r+1 since  $c(v_l) = r+1$ . Since  $c(v_1) > c(y) \ge k+3$ , then  $c(v_l) \ge k+3+l > r+3$  which contradicts  $c(v_l) = r+1$ .

Case 3.b.iii.  $c(u_k) > c(y)$ . Then  $c(u_k) = c(y) + r$  since  $(u_k, y) \in A(CW_{k,l})$ .

Case 3.b.iii. $\alpha$ .  $c(v_1) < c(u_k)$ . Thus, we reach a contradiction by Lemma 4.

Case 3.b.iii. $\beta$ .  $c(v_1) > c(u_k)$ . Thus,  $c(v_1) > r + c(y)$ . Since  $c(y) \ge k$  by Lemma 1, then  $c(v_1) \ge k + r + 1$ , i.e., max  $c(x) \ge k + r + 1$  for any  $x \in V(CW_{k,l})$ .

Case 3.c. c(y) = 1. By Lemma 1, it must be the case that k = 1 otherwise there will be an introvert's color which is less than 1. This contradicts a condition of this case.

Therefore, for all but the special cases,  $\max c(x) \ge k + r + 1$  for any  $x \in V(CW_{k,l})$  and our  $T_r$ -colorings in Figures 8, 9 and 10 are optimal with respect to span.

Case 4. k + l > r and  $\min(k, l) > r$ . Note that Figure 11 provides a  $T_r$ coloring of  $CW_{k,l}$  whose span is 2r + 2. Next, we show that the span of any  $T_r$ -coloring, c, of  $CW_{k,l}$  is greater than or equal to 2r + 2.

Again, let the vertices of  $CW_{k,l}$  be labeled as in Figure 3. By Lemmas 1 and 2,  $c(u_i) < c(y)$ , for all  $1 \le i \le k - 1$ , and  $c(v_j) > c(y)$ , for all  $2 \le j \le l$ . We consider four cases:

(a)  $c(u_k) > c(y)$  and  $c(v_1) < c(y)$ ; (b)  $c(u_k) < c(y)$  and  $c(v_1) < c(y)$ ; (c)  $c(u_k) < c(y)$  and  $c(v_1) > c(y)$ ; (d)  $c(u_k) > c(y)$  and  $c(v_1) > c(y)$ .



Figure 11. Optimal  $T_r$ -coloring of  $CW_{k,l}$ , k + l > r and  $\min(k, l) > r$ .

Case 4.a.  $c(u_k) > c(y)$  and  $c(v_1) < c(y)$ . This case is not possible by Corollary 5.

Case 4.b.  $c(u_k) < c(y)$  and  $c(v_1) > c(y)$ . Since  $c(u_k) < c(y)$ , then  $c(u_i) < c(y)$  for i = 1, 2, ..., k. If the  $c(u_i)$ s are all distinct then  $c(y) \ge r+2$  since k > r. If the  $c(u_i)$ s are not all distinct then  $c(u_j) < c(u_{j-1})$  for at least one j since colors on consecutive  $u_i$ s can not be equal. Given that  $c(u_j) \ge 1$  then  $c(u_{j-1}) \ge r+1$  since  $(u_{j-1}, u_j) \in A(CW_{k,l})$ . So, in general,  $c(y) \ge r+2$ .

Since  $c(v_1) > c(y)$ , then  $c(v_j) > c(y)$  for j = 1, 2, ..., l. If the  $c(v_j)$ s are all distinct and since l > r then  $\max c(v_j) \ge c(y) + l \ge c(y) + (r+1) \ge (r+2) + (r+1) = 2r+3$ . If the  $c(v_j)$ s are not all distinct then  $c(v_i) < c(v_{i-1})$  for at least one *i* since colors on consecutive  $v_i$ s can not be equal. Since  $c(v_i) > c(y)$  and  $c(y) \ge r+2$  then  $c(v_i) \ge r+3$ , for all *i*. Thus,  $c(v_{i-1}) \ge (r+3) + r = 2r+3$  since  $(v_{i-1}, v_i) \in A(CW_{k,l})$ . So, in general,  $\max c(v_j) \ge 2r+3$ . Therefore, in this case, the span of any  $T_r$ -coloring of  $CW_{k,l}$  is greater than or equal to 2r+2.

Case 4.c.  $c(u_k) < c(y)$  and  $c(v_1) < c(y)$ . As in Case 4.b, since  $c(u_k) < c(y)$  and thus  $c(u_i) < c(y)$  for all i = 1, 2, ..., k, we know that  $c(y) \ge r + 2$ .

Case 4.c.i.  $c(v_i) > c(v_{i+1})$  for some i = 2, 3, ..., l-1. Thus,  $c(v_i) - c(v_{i+1}) = r \Rightarrow c(v_i) = c(v_{i+1}) + r$  since  $(v_i, v_{i+1}) \in A(CW_{k,l})$ . Since  $c(v_{i+1}) > c(y)$ , then  $c(v_{i+1}) \ge r+3$ . Therefore,  $c(v_i) \ge (r+3) + r = 2r+3$ .

Case 4.c.ii.  $c(v_i) < c(v_{i+1})$ , for all i = 2, 3, ..., l-1. Since  $c(v_i) > c(y)$ , for i = 2, 3, ..., l, and  $c(y) \ge r+2$ , then  $c(v_l) \ge c(y) + (l-1) \ge c(y) + r$  since l > r. Thus,  $c(v_l) \ge c(y) + l - 1 \ge c(y) + r$ . However,  $c(u_1) \le c(y) - 1$ . We reach a contradiction since  $(v_l, u_1) \in A(CW_{k,l})$  but  $c(v_l) - c(u_1) \ge c(y) + r - (c(y) - 1) = r + 1 \notin T_r$ .

So, it must be the case that  $\max c(v_j) \ge 2r + 3$ . Therefore, in this case, the span of any  $T_r$ -coloring of  $CW_{k,l}$  is greater than or equal to 2r + 2.

Case 4.d.  $c(u_k) > c(y)$  and  $c(v_1) > c(y)$ .

Case 4.d.i.  $c(u_1) = 1$ . Given that  $c(u_k) > c(y)$ , then  $c(u_i) < c(y)$  for i = 1, 2, ..., k - 1. If these  $c(u_i)$ s are all distinct then  $c(y) \ge r + 1$  since k > r. However, by Lemma 2,  $c(v_l) > c(y) \Rightarrow c(v_l) \ge r + 2$ . We reach a contradiction since  $(v_l, u_1) \in A(CW_{k,l})$  but  $c(v_l) - c(u_1) \ge (r+2) - 1 = r + 1 \notin T_r$ . If  $c(u_i)$ , for i = 1, 2, ..., k - 1, are not all distinct then  $c(u_j) < c(u_{j-1})$  for at least one j since colors on consecutive  $u_i$ s can not be equal. Since  $c(u_j) \ge 1$ , then  $c(u_{j-1}) \ge r + 1$  since  $(u_{j-1}, u_j) \in A(CW_{k,l})$ . So, in general,  $c(y) \ge r + 2$ . Now that we know  $c(y) \ge r + 2$ , then, as in Case 4.b, since  $c(v_j) > c(y)$  for all j = 1, 2, ..., l, max  $c(v_j) \ge 2r + 3$ .

Case 4.d.ii.  $c(u_1) \neq 1$ . By Lemma 1, Lemma 6, and given that  $c(u_k) > c(y)$ , then  $c(y) \neq 1$  and  $c(u_i) = 1$  for some  $i = 2, 3, \ldots, k - 1$ . This implies  $c(u_{i-1}) = r+1 \Rightarrow c(y) \ge r+2$ . Then, as in Case 4.b, since  $c(v_j) > c(y)$  for all  $j = 1, 2, \ldots, l$ , max  $c(v_j) \ge 2r+3$ .

Thus, in this final case, the span of any  $T_r$ -coloring of  $CW_{k,l}$  is again greater than or equal to 2r + 2.

Therefore,  $sp_{T_r}(CW_{k,l}) = 2r + 2$  for k + l > r and  $\min(k, l) > r$ .

4. 
$$CW_{k,l}, r = 1$$

In this section, we consider the case of  $T_r$ -coloring  $CW_{k,l}$  where  $k, l \ge 0$  and r = 1. First, we consider the "all-introvert" case,  $CW_n$ , and the "all-extrovert" case,  $CW_{0,n}$ .

**Theorem 11.** For 
$$n \ge 2$$
,  $sp_{T_1}(CW_n) = sp_{T_1}(CW_{0,n}) = \begin{cases} 2 & \text{for } n \text{ even,} \\ 3 & \text{for } n \text{ odd.} \end{cases}$ 

**Proof.** Applying Theorem 8, it is sufficient to prove this theorem for only  $CW_n$ .

Case 1. n even. The same proof as in Theorem 9 holds here for  $CW_n$ .

Case 2. n odd. Figure 12 provides a  $T_1$ -coloring of  $CW_n$  whose span is 3. Next we show that the span of any  $T_1$ -coloring of  $CW_n$  is greater than or equal to 3.

Let c be any  $T_1$ -coloring of  $CW_n$ . At least four distinct colors must be used to  $T_1$ -color the vertices of  $CW_n$  since its underlying graph is a wheel graph with an odd cycle and  $0 \in T_1$ . Therefore, the span of c is greater than or equal to 4-1=3.

The next theorem considers the case of  $CW_{k,l}$  where r = 1 and  $\min(k, l) = 1$ .



Figure 12. Optimal  $T_1$ -coloring of  $CW_n$ , n odd.

**Theorem 12.**  $sp_{T_1}(CW_{1,m}) = sp_{T_1}(CW_{m,1}) = \begin{cases} 2 & for \ m \ odd, \\ 3 & for \ m \ even. \end{cases}$ 

**Proof.** Applying Theorem 8, it is sufficient to prove this theorem for only  $CW_{1,m}$ .

Case 1. *m* odd. Figure 13 provides a  $T_1$ -coloring of  $CW_{1,m}$  whose span is 2. Next, let *c* be any  $T_1$ -coloring of  $CW_{1,m}$ . At least three distinct colors must be used to  $T_1$ -color the vertices of  $CW_{1,m}$  since its underlying graph is a wheel graph with an even cycle and  $0 \in T_1$ . Therefore, the span of *c* is greater than or equal to 3-1=2.



Figure 13. Optimal  $T_1$ -colorings of  $CW_{1,m}$ .

Case 2. *m* even. Figure 13(b) provides a  $T_1$ -coloring of  $CW_{1,m}$  whose span is 3. Next, let *c* be any  $T_1$ -coloring of  $CW_{1,m}$ . At least four distinct colors must

be used to  $T_1$ -color the vertices of  $CW_{1,m}$  since its underlying graph is a wheel graph with an odd cycle and  $0 \in T_1$ . Therefore, the span of c is greater than or equal to 4 - 1 = 3.

The case r = 1 is unique in the sense that it is the only case where it is not possible to *T*-color some directed wheel graphs  $CW_{k,l}$ .

**Theorem 13.** If  $k, l \geq 2$ , then it is impossible to  $T_1$ -color  $CW_{k,l}$ .

**Proof.** Let  $CW_{k,l}$  be labeled as in Figure 3. By Lemmas 1 and 2,  $c(v_l) > c(y) > c(u_1)$ . Since  $(v_l, u_1) \in A(CW_{k,l})$ ,  $c(v_l) - c(u_1) = r = 1 \Rightarrow c(v_l) = c(u_1) + 1$ . We reach a contradiction because c(y) must be strictly between two consecutive colors:  $c(u_1)$  and  $c(v_l)$ .

5. 
$$sp_{T_r}(CW_{a_1,a_2,a_3,\ldots,a_{2j}})$$

We conclude this paper with bounds on  $sp_{T_r}(CW_{a_1,a_2,a_3,\ldots,a_{2i}})$ .

Theorem 14.  $r \leq sp_{T_r} (CW_{a_1, a_2, \dots, a_{2j}}) \leq 2r + 2.$ 

**Proof.** For  $CW_{a_1,a_2,\ldots,a_{2j}}$ , let the vertices be labeled as in Figure 1 and c be any  $T_r$ -coloring. Consider the sequence of colors of the non-hub vertices,  $c(b_1), c(b_2), \ldots, c(b_n), c(b_1)$ . This sequence of colors can not be strictly increasing since the vertices form a circuit. When the sequence decreases, by (1), it must do so by exactly r. Therefore, the span of c is greater than or equal to r which implies  $sp_{T_r}(CW_{a_1,a_2,\ldots,a_{2j}}) \geq r$ .

Next, for each consecutive set of  $a_1, a_3, a_5, \ldots, a_{2j-1}$  introverted arcs, we color their corresponding introverts,  $x_1, x_2, \ldots, x_{a_i}$ , as follows:

if 
$$r > 2$$
:  $3, r + 1, 1, r + 1, 1, \dots$ ; or if  $r = 2$ :  $3, 1, 3, 1, \dots$ 

For each consecutive set of  $a_2, a_4, a_6, \ldots, a_{2j}$  extroverted arcs, we color their corresponding extroverts,  $x_{a_j}, x_{a_{j-1}}, \ldots, x_1$ , as follows:

$$r+3, 2r+3, r+3, 2r+3, \ldots$$

Finally, for the hub vertex: c(y) = r + 2.

Using the labeling of Figure 1, note that  $c(b_n) - c(b_1) = r$  and  $c(b_i) - c(b_{i+1})$ , for each i = 1, 2, ..., n - 1, equals r or is negative. Also,  $c(b_i) - c(y) < 0$  for every introvert  $b_i$  and  $c(y) - c(b_j) < 0$  for every extrovert  $b_j$ . Therefore, c is a  $T_r$ -coloring of  $CW_{a_1,a_2,...,a_{2j}}$  and  $sp_{T_r}(CW_{a_1,a_2,...,a_{2j}}) \leq 2r + 2$ .

#### 6. Special Cases

This section considers the eleven special cases which cannot be  $T_r$ -colored using the optimal colorings from Theorem 10. In fact, each of their T-spans disagrees



Figure 14. Optimal  $T_r$ -colorings of the special cases.

with the value that Theorem 10 would have given (see Table 1). Optimal  $T_r$ colorings for seven of the special cases are given in Figure 14. The remaining four
special cases follow from Theorem 8. Proof of their optimality is left to the reader.

Digraph	r	Theorem 10 predicted span	Actual $T_r$ -span	Figure 14
$CW_{4,4}$	5	9	10	(a)
$CW_{3,4}$	4	7	8	(b)
$CW_{4,3}$	4	7	8	(0)
$CW_{2,4}$	3	5	6	(c)
$CW_{4,2}$	3	5	6	(C)
$CW_{3,3}$	3	6	8	(d)
$CW_{2,2}$	3	5	6	(e)
$CW_{2,3}$	2	4	6	(f)
$CW_{3,2}$	2	4	6	(1)
$CW_{1,3}$	2	3	4	(g)
$CW_{3,1}$	2	3	4	(g)

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