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# ON THE **b**-DOMATIC NUMBER OF GRAPHS

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### Abstract

A set of vertices S in a graph G = (V, E) is a *dominating set* if every vertex not in S is adjacent to at least one vertex in S. A *domatic partition* of graph G is a partition of its vertex-set V into dominating sets. A domatic partition  $\mathcal{P}$  of G is called *b-domatic* if no larger domatic partition of G can be obtained from  $\mathcal{P}$  by transferring some vertices of some classes of  $\mathcal{P}$  to form a new class. The minimum cardinality of a b-domatic partition of Gis called the *b-domatic number* and is denoted by bd(G). In this paper, we explain some properties of b-domatic partitions, and we determine the b-domatic number of some families of graphs.

**Keywords:** domatic partition, domatic number, b-domatic partition, b-domatic number.

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#### 1. INTRODUCTION

Let G = (V, E) be a finite, simple and undirected graph with vertex-set V and edge-set E. We call |V| the order of G and denote it by n. For any nonempty subset  $S \subset V$ , let G[S] denote the subgraph of G induced by S. For any vertex v of G, the open neighborhood of v is the set  $N_G(v) = \{u \in V(G) \mid (u, v) \in E\}$ and the closed neighborhood of v is the set  $N_G[v] = N_G(v) \cup \{v\}$ . The private neighborhood of a vertex  $v \in S$  with respect to S is the set  $pn[v, S] = \{u \in$  $V(G) : N_G[u] \cap S = \{v\}\}$ . Each vertex in pn[v, S] is called a private neighbor of v with respect to S. Remark that pn[v, S] is a set contained in  $\{v\} \cup (V \setminus S)$ . Let  $\Delta(G)$  (respectively,  $\delta(G)$ ) be the maximum (respectively, minimum) degree in G. Through this paper, the notations  $P_n, C_n$ , and  $K_n$  always denote a path, a cycle, and a complete graph of order n, respectively, while  $K_{p,q}$  ( $p \ge q$ ) denotes the complete bipartite graph with partite sets of sizes p, q. For further terminology on graphs we refer to the book by Berge [2].

Graph coloring and domination are two major areas in graph theory that have been extensively studied. These two concepts can be defined as a vertex partition into classes according to certain rules. By a *vertex partition* (partition for short), we will mean a partition of its vertex-set into disjoint subsets (classes). The cardinality of a partition is the number of its classes.

A set  $S \subseteq V(G)$  is called *independent* if no two vertices in S are adjacent. A partition  $\mathcal{P}$  in which each of its classes is an independent set is called a *proper* coloring of G. The smallest integer k such that G admits a proper coloring with k colors is called the *chromatic number* of G and is denoted by  $\chi(G)$ . In general, it is NP-complete to compute the chromatic number. For this reason, many heuristics have been developed for finding an approximate solution to this problem. One approach is to start with an arbitrary proper coloring and try to reduce the number of colors used by transferring all vertices from one color class to other classes. This technique is not possible if each color class contains a vertex having neighbors in all other color classes. A coloring satisfying such a property is called *b*-coloring. The *b*-chromatic number b(G) of a graph G is the largest integer k such that G admits a b-coloring with k colors. This concept was introduced by Irving and Manlove [7, 8].

A set  $S \subseteq V$  is called a *dominating set* if every vertex in  $V \setminus S$  is adjacent to some vertex in S. The minimum cardinality of a dominating set of G is called the *domination number* of G and is denoted by  $\gamma(G)$ . By analogy to the concept of chromatic partition, Cockayne and Hedetniemi [3] introduced the concept of *domatic partition* of a graph. A partition  $\mathcal{P}$  in which each of its classes is a dominating set is called a domatic partition of G. The *domatic number* d(G) is defined as the largest number of sets in a domatic partition.

In [5], Favaron introduced the *b*-domatic number by considering a new type of

domatic partition. As defined in [5], a domatic partition  $\mathcal{P}$  of G is *b*-domatic if no larger domatic partition of G can be obtained from  $\mathcal{P}$  by transferring some vertices of some classes of  $\mathcal{P}$  to form a new class. Formally, a domatic partition  $\mathcal{P} =$  $\{U_1, \ldots, U_k\}$  is called *b*-domatic if there do not exist k non-empty subsets  $\pi_i \subseteq U_i$ ,  $i \in \{1, \ldots, k\}$  with  $\bigcup_{i=1}^k (U_i \setminus \pi_i) \neq \emptyset$ , such that  $\{\pi_1, \ldots, \pi_k, \bigcup_{i=1}^k (U_i \setminus \pi_i)\}$  is a domatic partition of G. The minimum cardinality of a b-domatic partition of Gis called the *b*-domatic number and is denoted by bd(G).

It was observed in [5] that if  $\delta(G) = 0$ , then  $\{V(G)\}$  is the unique domatic partition and so bd(G) = d(G) = 1. For this, all graphs considered in this paper are without isolated vertices. Many other properties of domatic and b-domatic partitions were given in [5]. In particular, it was observed that any graph G with minimum degree  $\delta(G) \ge 1$ , satisfies  $2 \le bd(G) \le d(G)$ . The same author [5] asked several questions, some of which we answer in this paper. We first investigate a new property of a b-domatic partition by giving a sufficient condition for which a given domatic partition of a graph G is b-domatic. We next present some classes of graphs for which bd(G) = 2 and  $bd(G) = \delta(G) + 1$ , and we determine the bdomatic number of some special bipartite graphs and block graphs. Other results are given for particular classes of graphs.

# 2. KNOWN RESULTS

The authors of [3] showed the following result.

**Proposition 1** [3]. Let G be a graph of order n and minimum degree  $\delta(G)$ . Then

(1) 
$$d(G) \le \min\left\{\frac{n}{\gamma(G)}, \, \delta(G) + 1\right\}.$$

For some other results on domatic partitions see [1, 4, 10].

**Proposition 2** [5]. Let G be a graph of minimum degree  $\delta(G)$ . If  $\delta(G) = 0$ , then bd(G) = d(G) = 1, otherwise  $2 \leq bd(G) \leq d(G)$ .

Hence, by Propositions 1 and 2, the next result follows immediately.

**Proposition 3** [5]. For any graph G of minimum degree  $\delta(G)$ , we have  $bd(G) \leq \delta(G) + 1$ .

The following results are proved by Favaron in [5].

**Theorem 4** [5]. Let  $G_1, \ldots, G_k$  be the components of a disconnected graph G without isolated vertices. Then  $bd(G) = \min\{bd(G_i) : 1 \le i \le k\}$ .

**Proposition 5** [5]. Every domatic partition such that each class is a minimal dominating set of G is b-domatic.

The same author has computed the b-domatic number for some particular classes of graphs.

**Proposition 6** [5].  $bd(K_n) = n$ ,  $bd(C_3) = 3$ ,  $bd(C_n) = 2$  for  $n \ge 4$ , and  $bd(K_{p,q}) = 2$ .

# 3. MAIN RESULTS

We start this section by giving a sufficient condition for which a given domatic partition of a graph G is b-domatic. Let  $\mathcal{P}$  be a domatic partition of a graph G. For a vertex  $v \in V(G)$ , let  $U_v$  denote the class of  $\mathcal{P}$  containing v.

**Theorem 7.** Let  $\mathcal{P}$  be a domatic partition of a graph G = (V, E). If G has a vertex v such that for each  $u \in N_G[v]$  the set  $pn[u, U_u]$  is not empty, then  $\mathcal{P}$  is b-domatic.

**Proof.** Let  $\mathcal{P} = \{U_1, \ldots, U_k\}$  be a domatic partition of a graph G and let  $v \in V(G)$  such that for each vertex  $u \in N_G[v]$ ,  $pn[u, U_u] \neq \emptyset$ . Suppose, to the contrary, that  $\mathcal{P}$  is not b-domatic. Then, there exist k non-empty subsets  $\pi_i \subseteq U_i$ ,  $i \in \{1, \ldots, k\}$  with  $\bigcup_{i=1}^k (U_i \setminus \pi_i) \neq \emptyset$  for which  $\pi = \{\pi_1, \ldots, \pi_k, \bigcup_{i=1}^k (U_i \setminus \pi_i)\}$  is a domatic partition of G. Let  $\pi_{k+1} = \bigcup_{i=1}^k (U_i \setminus \pi_i)$ . We claim that any vertex u in  $N_G[v]$  cannot be in  $\pi_{k+1}$ . Suppose, to the contrary, that  $u \in N_G[v] \cap \pi_{k+1}$ . Then, there is a class  $\pi_p \subset U_u$  that does not contain u for a some  $p \in \{1, \ldots, k\}$ . Therefore, either u is isolated in  $U_u$  in which case no vertex of  $\pi_p$  can dominate u for the partition  $\pi$ , or there exists a vertex  $z \in pn[u, U_u]$  in which case no vertex of  $\pi_p$  can dominate z for the partition  $\pi$ . In either case, we have a contradiction with the fact that  $\pi$  is a domatic partition of G.

Note that the converse is not true in general. For example, the domatic partition  $\mathcal{P}_0 = \{\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}\}$  of the graph  $H_0$  in Figure 1 is b-domatic but there is no vertex of  $H_0$  which satisfies the sufficient condition of Theorem 7 for  $\mathcal{P}_0$ . Remark that, as  $\mathcal{P}_0$  is a b-domatic partition of  $H_0$  of cardinality 2, the lower bound of Proposition 2 implies that  $bd(H_0) = 2$ .

We next show that for any integer  $k \ge 6$ , there exists a graph  $G_k$  of order k that contains  $H_0$  as an induced subgraph and has b-domatic number equal to 2. Recall that, as proved in [9], if G = (V, E) is a graph with no isolated vertices, then the complement  $V \setminus S$  of every minimal dominating set S is a dominating set.

**Theorem 8.** For any integer  $k \ge 6$ , there exists a graph  $G_k$  of order k containing  $H_0$  as an induced subgraph, such that  $bd(G_k) = 2$ .

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Figure 1. Graph  $H_0$ .

**Proof.** Let  $v_1, v_2, v_3, v_4, v_5, v_6$  be the vertices of  $H_0$  as shown in Figure 1. Let  $V(H_0) = A_1 \cup A_2$  such that  $A_1 = \{v_1, v_2, v_3\}$  and  $A_2 = \{v_4, v_5, v_6\}$ . It is not difficult to see that  $\{A_1, A_2\}$  is a domatic partition of  $H_0$ . Let  $G_k$   $(k \ge 6)$  be a graph of order k, having no isolated vertices and with vertex-set  $V(G_k) =$  $A_1 \cup A_2 \cup A_3$  where  $G[A_1 \cup A_2] = H_0$  and  $A_3$  is a set of extra vertices (may be empty) such that there is no edge between  $A_1$  and  $A_3$ . Note that if k = 6, then  $A_3$  is an empty set and therefore  $G_6 = H_0$ . By the remark before the Theorem 8,  $bd(G_6) = 2$ . Assume now that  $k \ge 7$ , so  $|A_3| \ge 1$ . Let  $H_1 = G[A_3]$ , and let S be a minimal dominating set of  $H_1$ . Then, as proved in [9],  $A_3 \setminus S$  is a dominating set of  $H_1$  implying that  $\{S, A_3 \setminus S\}$  is a domatic partition of  $H_1$ . Let  $U_1 = A_1 \cup S$ and  $U_2 = A_2 \cup (A_3 \setminus S)$ . Clearly  $V(G_k) = U_1 \cup U_2$ , further  $\{U_1, U_2\}$  is a domatic partition of  $G_k$ . In addition, it is a routine exercise to show that  $U_1$  is a minimal dominating set of  $G_k$ . We shall show that  $\{U_1, U_2\}$  is a b-domatic partition of  $G_k$ . Suppose not. Then there exist two subsets  $\pi_i \subseteq U_i$  (i = 1, 2) such that  $(U_1 \setminus \pi_1) \cup (U_2 \setminus \pi_2) \neq \emptyset$  and  $\pi = \{\pi_1, \pi_2, (U_1 \setminus \pi_1) \cup (U_2 \setminus \pi_2)\}$  is a domatic partition of  $G_k$ . Let  $\pi_3 = (U_1 \setminus \pi_1) \cup (U_2 \setminus \pi_2)$ . Since  $U_1$  is a minimal dominating set of  $G_k$ , then no vertex of  $U_1$  can be in  $\pi_3$ , so  $\pi_1 = U_1$ . Likewise, no vertex of  $A_2$ belongs to  $\pi_3$  because if not, there is a vertex in  $A_1 (\subset \pi_1)$  that is not dominated by  $\pi_3$  (or by  $\pi_2$ ), a contradiction. For example, if  $v_4$  is the only vertex of  $A_2$  in  $\pi_3$ , then  $v_3$  will have no neighbors in  $\pi_3$ , and if  $A_2$  has at least two vertices, say  $v_4, v_5$ in  $\pi_3$ , then  $v_2$  will have no neighbors in  $\pi_2$ ; in each case, we have a contradiction with the fact that  $\pi$  is a domatic partition of  $G_k$ . Thus, no vertex of  $A_2$  can be in  $\pi_3$ , which means that  $A_2 \subset \pi_2$ . Therefore, since there is no edge between  $A_1$ and  $A_3$ , no vertex of  $A_1$  is dominated by  $\pi_3$ , this contradicts again that  $\pi$  is a domatic partition of  $G_k$ . Thus  $\{U_1, U_2\}$  is a b-domatic partition of  $G_k$ , and so  $bd(G_k) \leq 2$ . The lower bound of Proposition 2 implies that  $bd(G_k) = 2$ .

We now give other classes of graphs for which the b-domatic number is equal to 2.

**Theorem 9.** If G has a vertex such that its neighbors form an independent set, then bd(G) = 2.

**Proof.** Let H be a connected component of G (possibly H = G). Let v be a vertex of H such that  $N_H(v)$  is an independent set. Let  $U_1$  be the set of vertices of H whose distance from v is even, and let  $U_2$  be the set of vertices of H whose distance from v is odd. Observe that  $v \in U_1$  and each neighbor of v is in  $U_2$ . Clearly,  $\{U_1, U_2\}$  is a domatic partition of H. In view of Theorem 7,  $\{U_1, U_2\}$  is a b-domatic partition of H because each neighbor of v is isolated in  $U_2$  and v is isolated in  $U_1$ . Therefore  $bd(H) \leq 2$ , and so Theorem 4 and Proposition 2 yield bd(G) = 2.

**Corollary 10.** If G is triangle-free, then bd(G) = 2.

Consider a graph H with vertex-set V(H). For any permutation  $\pi$  of V(H), the *prism* of H with respect to  $\pi$  is the graph obtained by taking two disjoint copies of H, denoted by  $H_1$  and  $H_2$ , and joining every  $u \in V(H_1)$  with  $\pi(u) \in V(H_2)$ . The complementary prism of H is the graph formed from the disjoint union of H and its complementary graph  $\overline{H}$  by adding the edges of a perfect matching between the corresponding vertices of H and  $\overline{H}$ .

**Proposition 11.** Let H be a graph. If G is a prism of H or a complementary prism of H, then bd(G) = 2.

**Proof.** Let  $H_1, H_2$  be two disjoint copies of H and  $\mathcal{P} = \{V(H_1), V(H_2)\}$  be a partition of V(G). It is not difficult to see that  $\mathcal{P}$  is a domatic partition of G, further, for i = 1, 2, each vertex of  $V(H_i)$  has a private neighbor with respect to  $V(H_i)$ . Therefore, in view of Theorem 7,  $\mathcal{P}$  is b-domatic of G, and hence by Proposition 2, bd(G) = 2. The same proof still holds if G is the complementary prism of H, by substituting  $H_2$  with the complementary graph  $\overline{H}$ .

**Theorem 12.** Let G = (V, E) be an r-regular graph and  $\mu = \max\{|S_v| : v \in V(G) \text{ and } S_v \text{ is a maximum independent set in } G[N(v)]\}$ . If d(G) = r + 1, then  $bd(G) \leq r - \mu + 2$ .

**Proof.** Let  $\mathcal{P} = \{U_1, \ldots, U_{r+1}\}$  be a domatic partition of G of cardinality r+1. We can easily observe that for  $i \in \{1, \ldots, r+1\}$ ,

(2)  $U_i$  is an independent set of G, and each vertex in  $U_i$  has exactly one neighbor in each other class  $U_j, j \neq i$ .

Let v be a vertex of G such that  $\mu = |S_v|$ . Clearly  $r \ge \mu \ge 1$ . Let  $v_1, \ldots, v_r$  be the neighbors of v. By (2), we may assume that  $v \in U_1$  and  $v_i \in U_{i+1}$  for each  $i \in \{1, \ldots, r\}$ . Without loss of generality, assume also that  $S_v = \{v_1, \ldots, v_\mu\}$ . Set  $q = r - \mu + 2$  and let  $\pi = \{\pi_1, \ldots, \pi_q\}$  be a partition of G of cardinality q obtained from  $\mathcal{P}$  as follows.  $\pi_1 = \{v\} \cup ((\bigcup_{i=1}^{\mu} U_{i+1}) \setminus S_v), \pi_2 = S_v \cup (U_1 \setminus \{v\})$  and  $\pi_i = U_{i+\mu-1}$  for  $i \in \{3, \ldots, q\}$ . Now, we shall show that  $\pi$  is a domatic partition of G. For  $i \in \{3, \ldots, q\}, \pi_i$  is a dominating set of G because  $U_{i+\mu-1}$  is a dominating set of G. So, it suffices to show that  $\pi_1$  and  $\pi_2$  are dominating sets of G. Observe first that (2) implies that each vertex in  $U_{i+1} \setminus \{v_i\}, i = 1, \dots, \mu$ , has at least one neighbor in  $U_1 \setminus \{v\}$  and vice versa. Therefore, each vertex of  $\pi_2$  is dominated by  $\pi_1$  and vice versa. On the other hand, as  $S_v$  is a maximum independent set of  $G[N_G(v)]$ , each vertex in  $\{v_{\mu+1},\ldots,v_r\}$  has at least one neighbor in  $S_v$  and so in  $\pi_2$ . Notice that v has no neighbor in  $\bigcup_{i=1}^r (U_{i+1} \setminus \{v_i\})$ . Therefore, since  $U_1$  is a dominating set of G, each vertex in  $U_{i+1} \setminus \{v_i\}$   $(i \ge 1)$  has at least one neighbor in  $U_1 \setminus \{v\}$  and so in  $\pi_2$ . Hence, each vertex in  $\pi_i$ ,  $3 \leq i \leq q$  has a neighbor in  $\pi_2$ . This means that  $\pi_2$  is a dominating set of G. Thus, it remains to show that  $\pi_1$  is a dominating set of G. To this end, we show that each vertex  $u \in U_{j+1}, (j \ge \mu + 1)$  is dominated by  $\pi_1$ . Remember that  $v_j$  is the neighbor of v in  $U_{j+1}$ . Clearly, if  $u = v_j$   $(j \ge \mu + 1)$ , then u is adjacent to v and so u is dominated by  $\pi_1$ . Suppose now that  $u \neq v_j$ . Then u cannot be adjacent to all vertices of  $S_v$ , otherwise, since u and  $v_j$  are in the same class  $U_{j+1}$ , the second part of Observation 2 implies that  $v_i$  cannot be adjacent to any vertex of  $S_v$ and so  $S_v \cup \{v_i\}$  is an independent set of  $G[N_G(v)]$ , a contradiction. Hence u is non-adjacent to at least one vertex in  $S_v$ . Therefore, u must be adjacent to at least one vertex of  $\bigcup_{i=1}^{\mu} (U_{i+1} \setminus \{v_i\})$  and hence u is dominated by  $\pi_1$ . Thus each vertex of  $\bigcup_{i=3}^{q} \pi_i$  is dominated by  $\pi_1$ . Hence,  $\pi_1$  is a dominating set of G. Consequently,  $\pi$  is a domatic partition of cardinality q for which each vertex of  $N_G[v]$  is isolated in its class. Therefore, Theorem 7 implies that  $\pi$  is b-domatic of G, which means that  $bd(G) \leq q = r - \mu + 2$ .

This bound is achieved, for example, by a complete bipartite graph minus a perfect matching G of order 2p, with partite sets of the same size p, in which  $\delta(G) = \mu = p - 1$  and d(G) = p, while, by Theorem 9, bd(G) = 2.

**Theorem 13.** Let r be a positive integer and let G be a r-regular graph. Then bd(G) = r + 1 if and only if  $G = pK_{r+1}$  for some positive integer p.

**Proof.** Using Theorem 4 and Proposition 6, we can easily verify that the statement is true when  $G = pK_{r+1}$ . So, let us prove the converse. As bd(G) = r+1 = d(G), Theorem 12 implies that  $r+1 \leq r-\mu+2$ . So, since  $\mu \geq 1$ , it follows that  $r+1 \leq r-\mu+2 \leq r+1$ . Hence  $\mu = 1$  implying that the neighborhood of any vertex of G induces a complete subgraph. This means that G is the union of  $p \geq 1$  copies of complete graphs of order r+1.

A vertex v in a graph G is universal if it is adjacent to every other vertex in G. Recall that if G has no universal vertex, then  $\gamma(G) \ge 2$ . So, the next result follows immediately by applying Propositions 1 and 2.

**Observation 14.** If G is a graph of order n without universal vertices, then  $bd(G) \leq \frac{n}{2}$ .

This bound is achieved, for example, by (n-2)-regular graphs of order n.

**Proposition 15.** If G is an (n-2)-regular graph of order n, then  $bd(G) = \frac{n}{2}$ .

**Proof.** Let G be an r-regular graph of order n = r+2. Obviously, n is even, every vertex of G has exactly n-2 neighbors and one non-neighbor, and G is without isolated vertices. Hence, according to the Observation 14, we have  $bd(G) \leq \frac{n}{2}$ . Suppose to the contrary that  $bd(G) = k < \frac{n}{2}$ , and consider a b-domatic partition  $\mathcal{P} = \{U_1, \ldots, U_k\}$  of G of cardinality k. Therefore, since any set of two vertices of G dominates G, there are two classes  $U_i, U_j$   $(i \neq j)$  of  $\mathcal{P}$  such that both together contain at least 6 vertices. In this case, we can split  $U_i \cup U_j$  into three dominating sets  $U'_i, U'_j, U_{k+1}$  each of them of size at least two such that  $U'_i \subseteq U_i, U'_j \subseteq U_j$  and  $U_{k+1} \subseteq U_i \cup U_j$ . It is easy to check that  $(\mathcal{P} \setminus \{U_i, U_j\}) \cup (\{U'_i, U'_j, U_{k+1}\})$  is a domatic partition of G of cardinality k + 1, a contradiction. So  $bd(G) = \frac{n}{2}$ .

It was shown in [4] that if v is a universal vertex, then  $d(G) = d(G \setminus v) + 1$ . We give here a similar result for the b-domatic number.

**Proposition 16.** If v is a universal vertex in G, then  $bd(G) = bd(G \setminus v) + 1$ .

**Proof.** Let v be a universal vertex in G. Set  $k = bd(G \setminus v)$  and let  $\{U_1, \ldots, U_k\}$  be a b-domatic partition of  $G \setminus v$ . Clearly,  $\{U_1, \ldots, U_k, \{v\}\}$  is a b-domatic partition of G, so  $bd(G) \leq bd(G \setminus v) + 1$ . Now set t = bd(G) and let  $\{\pi_1, \ldots, \pi_t\}$  be a b-domatic partition of G. Assume that  $v \in \pi_1$ . Observe that  $\{(\pi_1 \cup \pi_2) \setminus \{v\}, \pi_3, \ldots, \pi_t\}$  is a b-domatic partition of  $G \setminus v$ . Thus  $bd(G \setminus v) \leq bd(G) - 1$  which gives the desired result.

A threshold graph is a graph that can be constructed from the one-vertex graph by repeatedly adding an isolated vertex or a universal vertex. As a consequence of Proposition 16, we determine the b-domatic number of threshold graph and its complementary graph.

**Corollary 17.** Let  $G_n$  be a threshold graph of order n. Then

$$bd(G_1) = 1$$
 and for  $n \ge 2$ ,  $bd(G_n) = 1 + \sum_{j=2}^n \alpha_n \cdots \alpha_j$ ,

where  $\alpha_j = \begin{cases} 1 & \text{if the added vertex to } G_{j-1} & \text{is an universal vertex,} \\ 0 & \text{if the added vertex to } G_{j-1} & \text{is an isolated vertex.} \end{cases}$ 

**Proof.** Proposition 2 and Proposition 16 imply that  $bd(G_1) = 1$  and  $bd(G_n) = \alpha_n \cdot bd(G_{n-1}) + 1$  for  $n \ge 2$ . This recurrence relation admits the unique solution  $bd(G_n) = 1 + \sum_{j=2}^n \alpha_n \cdots \alpha_j$ .

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**Corollary 18.** If  $\overline{G_n}$  is the complementary graph of a threshold graph  $G_n$ , then

$$bd(\overline{G_n}) = 1 + \sum_{j=2}^n (1 - \alpha_n) \cdots (1 - \alpha_j).$$

A *block* of a graph is a maximal connected subgraph that has no cut-vertex. A block is trivial if it has only one edge. A *block graph* is a connected graph in which each block induces a complete subgraph. In the next theorem, we determine the b-domatic number of any block graph G such that each of its blocks contains at least one vertex that is non-cut for G.

**Theorem 19.** Let G be a block graph and  $B_1, \ldots, B_r$   $(r \ge 2)$  be the blocks of G. For  $i \in \{1, \ldots, r\}$ , let  $|V(B_i)| = n_i$ , and let  $k_i$  denote the number of cut vertices in  $B_i$ . If  $l = \min\{n_i - k_i : 1 \le i \le r\} \ge 1$ , then bd(G) = l + 1.

**Proof.** Let  $r \ge 2$ . For  $i \in \{1, \ldots, r\}$ , denote by  $\delta_i$  the minimum degree in  $B_i$  and let  $l_i = n_i - k_i$ . As  $l = \min\{l_i : 1 \le i \le r\} \ge 1$ , it follows that  $\delta = \min \delta_i \ge 1$  and  $1 \le l_i \le \delta_i$ . Thus,  $l \le \delta_i$  and in particular, we have

$$(3) l \le \delta.$$

If r = 2, then G has exactly one cut vertex, say w. Hence,

(4) 
$$l_i = n_i - 1 = \delta_i \text{ for each } i \text{ in } \{1, 2\}.$$

Observe that w is a universal vertex in G, and  $G \setminus w = K_{\delta_1} \cup K_{\delta_2}$  is the union of two complete subgraphs of G. Hence, by Theorem 4,  $bd(G \setminus w) = \min\{bd(K_{\delta_1}), bd(K_{\delta_2})\} = \min\{\delta_1, \delta_2\}$ , and by (4), we get  $bd(G \setminus w) = \min\{l_1, l_2\} = l$ . Therefore, Proposition 16 implies that  $bd(G) = bd(G \setminus w) + 1 = l + 1$ . Hence, the statement is true. Assume now that  $r \geq 3$ . Denote by

$$V(B_i) = \left\{ v_1^i, v_2^i, \dots, v_{l_i}^i, u_1^i, u_2^i, \dots, u_{k_i}^i \right\},\$$

the set of the vertices of the block  $B_i$  such that  $v_1^i, v_2^i, \ldots, v_{l_i}^i$  and  $u_1^i, u_2^i, \ldots, u_{k_i}^i$  are respectively the non-cut vertices and the cut vertices of G in  $B_i$ .

We first show that  $bd(G) \ge l+1$ . Let k = bd(G) and suppose to the contrary that  $k \le l$ . Hence, by (3), we obtain

(5) 
$$k \le l \le \delta$$
.

Let  $\mathcal{P} = \{U_1, U_2, \dots, U_k\}$  be a b-domatic partition of G of cardinality k. Notice that  $|B_i| = \delta_i + 1 \ge \delta + 1$  for each  $i \in \{1, \dots, r\}$ ; so by (5), we have

(6) 
$$|B_i| \ge l+1 \ge k+1.$$

As each block  $B_i$  contains at least l vertices that are non-cut for G, (6) implies that for each  $i \in \{1, \ldots, r\}$  there is a class of  $\mathcal{P}$  that intersects  $B_i$  in at least two vertices such that at least one of them is a non-cut-vertex, say  $v_1^i$ . Let  $X = \{v_1^1, v_1^2, \ldots, v_1^r\}$  and consider a partition  $\pi$  of V(G) of cardinality k + 1obtained from  $\mathcal{P}$  by collecting vertices  $v_1^1, v_1^2, \ldots, v_1^r$  of some classes of  $\mathcal{P}$  to form a new class. Partition  $\pi$  with classes  $\pi_1, \pi_2, \ldots, \pi_k, \pi_{k+1}$  is constructed as follows. For each  $i \in \{1, \ldots, r\}$ , let  $\pi_i = U_i \setminus X_i$  where  $X_i = X \cap U_i$  ( $X_i$  may be empty for some integers i), and  $\pi_{k+1} = X$ . It is a routine exercise to verify that  $\pi =$  $\{\pi_1, \pi_2, \ldots, \pi_k, \pi_{k+1}\}$  is a domatic partition of G, contradicting the assumption that  $\mathcal{P}$  is a b-domatic partition of G. Thus

(7) 
$$bd(G) \ge l+1$$

Now, we shall show that  $bd(G) = k \leq l + 1$ . When  $l = \delta$ , the last inequality is clearly true, and therefore by (7), we have  $k = l + 1 = \delta + 1$ . Now, assume that  $l \leq \delta - 1$ , and suppose without loss of generality that  $B_1$  contains the smallest number of non-cut vertices in G. Then

(8) 
$$l_1 = l \text{ and so } V(B_1) = \left\{ v_1^1, v_2^1, \dots, v_l^1, u_1^1, u_2^1, \dots, u_{k_1}^1 \right\},$$

where  $u_1^1, u_2^1, \ldots, u_{k_1}^1$  are the cut vertices of  $B_1$ . It is known that a vertex is a cut vertex if and only if it belongs to at least two blocks. Hence, without loss of generality, we may suppose that

for 
$$i \in \{1, \ldots, k_1\}, u_i^1 \in V(B_1) \cap V(B_{i+1})$$
.

Let  $s \ge 0$  be the number of blocks of G, that do not intersect  $B_1$ . Clearly s < r and, if s = 0, each block  $B_j$   $(j \ne 1)$  of G intersect  $B_1$ . If  $s \ge 1$  we may suppose, without loss of generality, that

$$V(B_1) \cap V(B_j) = \emptyset$$
 for  $j \in \{r - s + 1, \dots, r\}$ .

Let  $\mathcal{P} = \{U_1, U_2, \dots, U_{l+1}\}$  be a partition of G of cardinality  $l+1 \leq \delta$  defined according to the value of l as follows.

Case 1.  $l \geq 2$ .

• If  $s \ge 1$ , then  $U_{l+1} = \{u_i^1 : 1 \le i \le k_1\} \cup \{v_1^j : r - s + 1 \le j \le r\}$ ; otherwise  $U_{l+1} = \{u_i^1 : 1 \le i \le k_1\}$ . •  $U_i = \{v_i^j : 1 \le j \le r\}, 2 \le i \le l$ . •  $U_1 = V(G) \setminus \left(\bigcup_{i=2}^{l+1} U_i\right)$ . Case 2. l = 1.

• If 
$$s \ge 1$$
, then  $U_2 = \{u_i^1 : 1 \le i \le k_1\} \cup \{v_1^j : r - s + 1 \le j \le r\}$ ; otherwise  $U_2 = \{u_i^1 : 1 \le i \le k_1\}.$ 

•  $U_1 = V(G) \setminus U_2$ .

Remark that in either cases, each class of  $\mathcal{P}$  intersect each block of G in at least one vertex. This means that  $\mathcal{P}$  is a domatic partition of G. Observe also that for  $i \in \{1, \ldots, k_1\}$ ,  $v_l^{i+1}$  is a private neighbor of  $u_i^1$  with respect to  $U_{l+1}$ . In addition, each class  $U_i$   $(i = 1, \ldots, l)$  has a vertex  $v_i^1$  that is adjacent to no vertex of its own class and to exactly one vertex from each of the classes  $U_j$ ,  $j \in \{1, \ldots, l\} \setminus \{i\}$ . So, we conclude that  $v_1^1$  is an isolated vertex in  $U_1$  such that each of its neighbor is either isolated in its class or has a private neighbor with respect to  $U_{l+1}$ . Thus, in view of Theorem 7,  $\mathcal{P}$  is a b-domatic partition of G, which means that  $k \leq l + 1$ . Hence, by (7), we get k = l + 1.

A cactus graph is a connected graph in which each block is either an edge or a cycle. A friendship graph  $F_n$   $(n \ge 2)$  is a cactus graph of order 2n + 1 in which any two vertices have exactly one common neighbor.

In the following proposition, we prove that the b-domatic number of a cactus graph G in which every block has at least one vertex that is non-cut for G is equal to 2, except for  $K_3$  and  $F_n$   $(n \ge 2)$ .

**Proposition 20.** Let G be a cactus graph such that each block has at least one vertex that is non-cut for G. Then bd(G) = 2 unless G is  $K_3$  or  $F_n$   $(n \ge 2)$ . In these cases,  $bd(K_3) = bd(F_n) = 3$ .

**Proof.** Clearly  $\delta(G) \leq 2$ . Thus, by Proposition 3, bd(G) = 2 when  $\delta(G) = 1$ . So, assume that  $\delta(G) = 2$ . If G has a cycle of length at least 4, then G contains a non-cut vertex of degree 2 such that its neighbors form an independent set. Therefore, Theorem 9 yields bd(G) = 2. Now, assume that any cycle of G has length 3. Let l be as defined in Theorem 19 and let  $r \geq 1$  be the number of blocks of G. If r = 1, then  $G = K_3$  and so  $bd(K_3) = 3$  by Proposition 6. Assume that  $r \geq 2$ . If  $G = F_n$ , then l = 2 and so  $bd(F_n) = 3$  by Theorem 19. Otherwise l = 1implying that bd(G) = 2 by Theorem 19 again.

#### 4. Conclusion

In this paper, we have formulated and proved a sufficient condition for a given domatic partition of a graph to be b-domatic, however, we have shown that the converse is not true. Therefore, the necessary condition remains still an open problem.

We have also presented some infinite classes of graphs having b-domatic number equal to two and  $\delta + 1$ . In particular, we have determined the b-domatic number of block graph (cactus graph) G in which every block has at least one vertex that is non-cut for G. So it would be interesting to determine the b-domatic number for cacti and block graphs that contain at least one block whose vertices are all cut-vertices for G.

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