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PAIR $L(2,1)$-LABELINGS OF INFINITE GRAPHS

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Abstract

An $L(2,1)$-labeling of a graph $G = (V,E)$ is an assignment of non-negative integers to V such that two adjacent vertices must receive numbers (labels) at least two apart and further, if two vertices are in distance 2 then they receive distinct labels. This article studies a generalization of the $L(2,1)$-labeling. We assign sets with at least one element to vertices of G under some conditions.

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1. Introduction

Inspired by a channel assignment problem proposed by Lanfear to Roberts [7] in 1988, Griggs and Yeh [6] formulated the $L(2,1)$-labeling problem for graphs. Since then, there are considerable amounts of articles studying this labeling and its generalizations or related problems. Readers can see [1] and [8] for a survey. Now we like to consider another generalization.

Let S be a finite set and A and B be two subsets of S. Define $\|A - B\| = \min\{|a-b| : a \in A, b \in B\}$. Denote the set $[k] = \{0, 1, \ldots, k\}$ and $\binom{[k]}{m}$ the collection of all m-subsets of $[k]$. Motivated by the article [3], we propose the following labeling on a graph.

Let $G = (V,E)$ be a graph and n be a positive integer. Given non-negative integers δ_1, δ_2, an $L^{(n)}(\delta_1, \delta_2)$-*labeling* is a function $f : V(G) \to \binom{[k]}{n}$ for some $k \geq 1$ such that $\|f(u) - f(v)\| \geq \delta_i$ whenever the distance between u and v in G is i, for $i = 1, 2$. (The minimum value and the maximum value of $\cup f(v)$ is 0 and

k, respectively.) The number k is called of the *span* of f. The smallest k so that there is an $L^{(n)}(\delta_1, \delta_2)$- labeling f with span k, is denoted by $\lambda^{(n)}(G; \delta_1, \delta_2)$ and called the $L^{(n)}(\delta_1, \delta_2)$-*labeling number* of G. An $L^{(n)}(\delta_1, \delta_2)$-labeling with span $\lambda^{(n)}(G; \delta_1, \delta_2)$ is called an *optimal* $L^{(n)}(\delta_1, \delta_2)$-labeling. If $n = 1$ then notations $L^{(1)}$ and $\lambda^{(1)}$ will be simplified as L and λ, respectively.

The elements in $[k]$ are called "numbers" and $f(u)$ is called the "label" of u. So a label is a set in this problem.

Using our notation, the labeling in [3] is the $L(\delta_1, 0)$-labeling for $\delta_1 \geq 1$.

In this article we will consider the case when $(\delta_1, \delta_2) = (2, 1)$ and the corresponding labeling number is denoted by $\lambda^{(n)}(G)$ (or just $\lambda^{(n)}$ if G is understood) for short.

If $n = 1$ then it is just the ordinary $L(2, 1)$-labeling (cf. [6]) and $\lambda^{(1)} = \lambda$. For $n = 2$, it is also called the *pair* $L(2, 1)$-*labeling* and $\lambda^{(2)}$ is called the *pair* $L(2, 1)$-*labeling number*. In the following, we first investigate properties of the $L^{(n)}(2, 1)$-labelings. Then, we consider the pair $L(2, 1)$-labelings on several classes of graphs. Finally, we present generalized results for $n \geq 2$ without giving proofs.

2. Preliminarily

Let G be a graph and n a positive integer. Now, we construct a new graph $G^{(n)}$ by replacing each vertex v in G by n vertices v_i, $1 \leq i \leq n$ and u_i is adjacent to v_j for all i, j, in $G^{(n)}$, whenever u and v is adjacent in G. (That is, u_i and v_j, for all i, j, induces a complete bipartite graph $K_{n,n}$. Note that $G^{(1)} = G$.

It is easy to verify that $\lambda^{(n)}(G) = \lambda(G^{(n)})$. Thus, for example, for $m \geq 2$, $\lambda^{(n)}(K_m) = \lambda(K_{n,n,\dots,n}) = nm + m - 2$, by previous results (cf. [6]) on complete m-partite graph $K_{n,n,\dots,n}$.

A vertex u with the maximum degree Δ in a graph G is called a *major vertex* of G. It is easy to see that for any G, $\lambda^{(n)}(G) \geq n(\Delta + 1)$. Also, similar to the $L(2, 1)$-labeling, we have $\lambda^{(n)}(H) \leq \lambda^{(n)}(G)$ for any subgraph H of G.

When we say "a vertex v is labeled by a set A and *rules out* i numbers for some vertices" means that once v is labeled by A then there will be i numbers in the base set not available for labeling these vertices. Usually, $i \geq |A| + 1$.

Lemma 1. *Let G be a graph with a major vertex u and f be an $L^{(n)}(2, 1)$-labeling of G.*

(1) *If the span of f is $n(\Delta + 1) + 1$, then $f(u)$ is allowed to rule out at most $n + 2$ numbers for its neighbors.*
(2) *If the span of f is $n(\Delta + 1)$, then $f(u)$ must be $\{0, 1, \dots, n - 1\}$ or $\{n\Delta + 1, n\Delta + 2, \dots, n\Delta + n(= n(\Delta + 1))\}$.*
(3) *If u is adjacent to two other major vertices in G, then $\lambda^{(n)}(G) \geq n(\Delta+1)+1$.*

Proof. (1) If $f(u)$ rules out more than $n+3$ numbers for its neighbors then at least $n+3$ numbers are not available for labeling u's neighbors. But, we need $n\Delta$ numbers for u's neighbors since they are distance 2 apart each other and hence there shall have at least $n\Delta + n + 3 = n(\Delta+1)+3$ numbers available. Thus the span of f is at least $n(\Delta+1)+2$. This contradicts to our assumption.

(2) Suppose the conclusion is wrong. Let $f(u)$ be of the form $\{x_1, x_2, \ldots, x_n\}$, where $1 \le x_1 < x_2 < \cdots < x_n \le n(\Delta+1)-1$. Then $f(u)$ rules out at least $n+2$ numbers for labels of its neighbors. Hence, totally, we need at least $n\Delta + n + 2 = n(\Delta+1)+2$ numbers for u and its neighbors. That is, the span of f is at least $n(\Delta+1)+1$. This is a contradiction. Thus, (2) is true.

(3) Suppose the span of f is $n(\Delta+1)$. Then by (2), $f(u) = \{0, 1, \ldots, n-1\}$ or $\{n\Delta+1, n\Delta+2, \ldots, n\Delta+n(=n(\Delta+1)\}$.

By assumption, there are three major vertices within distance at most 2. So, they must receive three disjoint sets to be labeled but we only get two. We have a contradiction here. Hence the result asserts. ■

In the end of this section, we present an observation on the relation between the labeling numbers for $n = 1$ and $n \ge 1$.

Proposition 2. $\lambda^{(n)}(G) \le \lambda(G; n+1, n) + n - 1$, $n \ge 1$.

Proof. Let $\lambda(G; n+1, n) = k$ and f an optimal $L(n+1, n)$-labeling. Define sets $L_i = \{i, i+1, \ldots, i+n-1\}$, $i = 0, 1, \ldots, k$ and function $g_f : V(G) \to \binom{[k+n-1]}{n}$ by $g_f(u) = L_i$ whenever $f(u) = i$ for $i = 0, 1, \ldots, k$.

Suppose u and v are adjacent in G, $f(u) = i$ and $f(v) = i+n'+1$ for $n' \ge n$. Then $g_f(u) = \{i, i+1, \ldots, i+n-1\}$ and $g_f(v) = \{i+n'+1, i+n'+2, \ldots, i+n'+n\}$. Hence $\|g_f(u) - g_f(v)\| = (i+n'+1) - (i+n-1) = n' - n + 2 \ge 2$. Suppose the distance between u and v is 2. Let $f(u) = i$ and $f(v) = i + n'$ for $n' \ge n$. Then $g_f(u) = \{i, i+1, \ldots, i+n-1\}$ and $g_f(v) = \{i+n', i+n'+1, \ldots, i+n'+n-1\}$. Hence $\|g_f(u) - g_f(v)\| = n' - n + 1 \ge 1$. Thus g_f is an $L^{(n)}(2,1)$-labeling with span $k+n-1$. Therefore $\lambda^{(n)}(G) \le \lambda(G; n+1, n) + n - 1$. ■

Corollary 3. $\lambda^{(2)}(G) \le \lambda(G; 3, 2) + 1 \le 2\lambda(G) + 1$.

Proof. The first inequality is the direct consequence of Proposition 2, for $n = 1$. By the previous result, $\lambda(G; 3, 2) + 1 \le \lambda(G; 4, 2) + 1 = 2\lambda(G) + 1$. ■

Next, we want to further look at the relation between $\lambda^{(2)}$ and λ since most people are much more familiar with the $L(2,1)$-labeling than the $L(3,2)$-labeling. Let f be an optimal $L(2,1)$-labeling on G with $\lambda(G) = k$, and let $h_f = k + 1 - |f(V(G))|$. (Note: h_f is the number of elements not used by f (called *holes*) in $[k]$.) Let $h = \max h_f$ over all optimal $L(2,1)$-labelings f. It is known that $h \le \lfloor k/2 \rfloor$ (cf. [2]).

Proposition 4. *Let $\lambda(G) = k$ and h be defined above. Then*

$$\lambda^{(n)}(G) \leq n(k+1) - 1 - h(n-1).$$

Proof. Let f be an optimal $L(2,1)$-labeling on G with $\lambda(G) = k$. Suppose $f(V(G)) = \{0 = \ell_0, \ell_1, \ldots, \ell_t = k\}$. Now we construct a labeling g_f on G as follows: Let $L_0 = \{0, 1, \ldots, n-1\}$. For $i \geq 1$, define set L_i with its minimum element being (1) $\max L_{i-1} + 1$ if $\ell_i - \ell_{i-1} = 1$ or (2) $\max L_{i-1} + 2$ if $\ell_i - \ell_{i-1} = 2$. Suppose $\min L_i = x$. Then $L_i = \{x, x+1, \ldots, x+n-1\}$. The process ends until L_t is determined. Let u be a vertex. Define $g_f(u) = L_i$ whenever $f(u) = \ell_i$. We can check that g_f is an $L^{(n)}(2,1)$-labeling of G with span M where $M = n(k+1) - 1 - h_f(n-1)$. Note that this is true for all optimal $L(2,1)$-labeling f of G. Therefore, $\lambda^{(n)}(G) \leq n(k+1) - 1 - h(n-1)$. ■

Take $G = K_m$ and $n = 2$. Then $\lambda(G) = 2(m-1)$ and $h = m-1$ (since we only can use $0, 2, 4, \ldots, 2(m-1)$ to label G). So, $n(k+1) - 1 - h(n-1) = 2\,(2(m-1)+1) - 1 - (m-1)(2-1) = 3m - 2 = \lambda^{(2)}(K_m)$.

3. Elementary Graphs

This section will study the pair $L(2,1)$-labelings on paths, cycles and wheels.

In the following, when we say "several consecutive vertices" in a path means that they are adjacent one by one in the path.

Proposition 5. *Let P_n be a path of order $n \geq 2$. Then*

(1) $\lambda^{(2)}(P_2) = 4$,

(2) $\lambda^{(2)}(P_3) = \lambda^{(2)}(P_4) = 6$ *and*

(3) $\lambda^{(2)}(P_n) = 7$ *for* $n \geq 5$.

Proof. (1) Since $P_2 = K_2$, we have $\lambda^{(2)}(P_2) = 4$.

(2) $\lambda^{(2)}(P_3) \geq 6$, since the maximum degree of P_3 is 2. On the other hand, we can use $\{5,6\}, \{0,1\}, \{3,4\}$ to label these three consecutive vertices of P_3. For a P_4, we can label four consecutive vertices by $\{2,3\}, \{5,6\}, \{0,1\}, \{3,4\}$. Further, $6 = \lambda^{(2)}(P_3) \leq \lambda^{(2)}(P_4)$, since P_3 is a subgraph of P_4.

(3) Suppose $n \geq 5$. Since $n \geq 5$, we have three consecutive major vertices ($\Delta(P_n) = 2$), by Lemma 1 (3), $\lambda^{(2)}(P_n) \geq 2\Delta + 3 = 7$. On the other hand, we can repeatedly use $\{0,1\}, \{3,4\}, \{6,7\}$ to label consecutive vertices starting from one end-vertex of P_n to the other end. Hence $\lambda^{(2)}(P_n) \leq 7$. The result then asserts. ■

Let A be a subset of $[k]$. Define $A + i \,(\mathrm{mod}\ m) = \{a + i \,(\mathrm{mod}\ m) : \text{ for all } a \in A\}$, for some i. If $m > k$ then obviously A and $A + i \,(\mathrm{mod}\ m)$ have the same cardinality. Let $\sigma = \langle A_1, A_2, \dots, A_t \rangle$ be a sequence of sets. Denote by $\sigma^{(i)}$ the sequence formed by duplicating σ i times. For example, $\langle \{1,2\},\{3,4\}\rangle^{(2)} = \langle \{1,2\},\{3,4\},\{1,2\},\{3,4\}\rangle$.

Proposition 6. *Let C_m ($m \geq 3$) be a cycle of order m. Then*

$$\lambda^{(2)}(C_m) = \begin{cases} 9, & m = 5, \\ 7, & m = 3 \text{ or } m \geq 6 \text{ but } m \neq 7, 10, 13, \\ 8, & m = 4, 7, 10, 13. \end{cases}$$

Proof. It is easy to verify results for $m = 3, 4, 5$. Suppose $m \geq 6$.

Since $m \geq 6$, the cycle C_m satisfies the condition in Lemma 1 (3), $\lambda^{(2)}(C_m) \geq 2\Delta + 3 = 7$. To prove that the equality holds, it suffices to construct a pair $L(2,1)$-labeling for C_m with span 7.

Suppose $m \equiv 0 \pmod 3$. Then we use $\{0,1\},\{3,4\},\{6,7\}$ repeatedly to label vertices of C_m. We see that it is a pair $L(2,1)$-labeling with span 7. Thus $\lambda^{(2)}(C_m) = 7$ in this case.

Suppose $m \equiv 2 \pmod 3$. Let $\sigma_1 = \langle \{0,1\},\{5,6\},\{2,3\},\{0,7\},\{4,5\},\{1,2\},\{6,7\},\{3,4\} \rangle$ and $\sigma_2 = \langle \{0,1\},\{7,6\},\{3,4\}\rangle$. Then we use $\left\langle \sigma_1, \sigma_2^{(p)} \right\rangle$ where $p = (m-8)/3$ (since $m \equiv 2 \pmod 3$, p is an integer) to label C_m. We see that this is a pair $L(2,1)$-labeling with span 7 of C_m. Hence $\lambda^{(2)}(C_m) = 7$.

Suppose $m \equiv 1 \pmod 3$ and $m \geq 16$. That is $m \neq 7, 10, 13$. Then we use $\left\langle \sigma_1^{(2)}, \sigma_2^{(p)} \right\rangle$ where $p = (m-16)/3$ (also an integer) to label C_m with the span 7. Again, this is a pair $L(2,1)$-labeling of C_m. Hence, $\lambda^{(2)}(C_m) = 7$ in this case.

Suppose $m = 7, 10$ or 13. Set $\sigma_3 = \langle \{0,1\},\{5,6\},\{2,3\},\{7,8\} \rangle$ and $\sigma_4 = \langle \{0,1\},\{3,4\},\{7,8\} \rangle$.

For $m = 7, 10, 13$, we use $\langle \sigma_3, \sigma_4 \rangle$, $\left\langle \sigma_3, \sigma_4^{(2)} \right\rangle$ and $\left\langle \sigma_3, \sigma_4^{(3)} \right\rangle$, respectively, to label C_m. We see that they are pair $L(2,1)$-labelings for each m. Hence $\lambda^{(2}(C_m) \leq 8$ in each case. It remains to show that 8 is the best possible.

Assume there is a pair $L(2,1)$-labeling f on C_m with span 7. Then there must be a vertex, say v, labeled by $\{0,x\}$ for $1 \leq x \leq 7$. By Lemma 1(1), x can only be 1, 2 or 7. Also we know that $f(v)$ contains consecutive numbers if it does not contain 0, by Lemma 1(1).

Whenever f exists, it means that there is a sequence of length $m+1$ starting with $\{0,x\}$ and end at $\{0,x\}$ again. Moreover, if there is a sequence σ (with first term $\{0,x\}$) so that we can use the sequence $\sigma^{(p)}$ for some $p \geq 1$, to proper label vertices of C_m consecutively. By considering all possibilities, we have the following results. (This is not difficult since we assume the span of f is 7.)

$x = 1$. The are two sequences $\sigma_1 = \langle \{0,1\},\{3,4\},\{6,7\}\rangle$ and $\sigma_2 = \langle \{0,1\},\{5,6\},\{2,3\},\{0,7\},\{4,5\},\{1,2\},\{6,7\},\{3,4\}\rangle$.

$x = 7$. There is only one sequence $\sigma_3 = \langle\, \{0,7\}, \{4,5\}, \{1,2\}, \{6,7\}, \{3,4\}, \{0,1\}, \{5,6\}, \{2,3\}\,\rangle$.

$x = 2$. There is no such sequence.

We find that when $m = 7, 10$ or 13, these sequences are not proper for them. Hence their $\lambda^{(2)}$-labeling number is greater than or equal to 8. Therefore we obtain the exact labeling numbers. ■

A *wheel* W_m is the graph formed by joining a vertex to each vertices of the cycle C_m for $m \geq 3$. In fact, $W_3 = K_4$. The following proposition is easy to derive and we shall omit the proof.

Proposition 7.

$$\lambda^{(2)}(W_m) = \begin{cases} 10, & \text{if } m = 3, \\ 11, & \text{if } m = 4, \\ 2m + 2, & \text{if } m \geq 5. \end{cases}$$

4. Infinite Graphs

In this section, we consider three infinite graphs. Let $\mathbb{Z}$ be the set of integers. Define the graph P_∞ by letting $V(P_\infty) = \mathbb{Z}$ and $E(P_\infty) = \{ij : |i - j| = 1, i, j \in \mathbb{Z}\}$. P_∞ is a path of infinite order. Denote $T_\infty(\Delta)$ the $\Delta(\geq 2)$-regular infinite tree. That is, a tree with infinite many vertices and each vertex having degree Δ. If $\Delta = 2$ then it is just the P_∞.

Theorem 8. $\lambda^{(2)}(T_\infty(\Delta)) = 2\Delta + 3$*, for* $\Delta \geq 2$.

Proof. Since T_∞ is a Δ-regular graph with at least three vertices, by Lemma 1, $\lambda^{(2)}(T_\infty) \geq 2\Delta + 3$. To prove that the equality holds, it suffices to construct a proper labeling of T_∞. In the following, we define an labeling by a greedy algorithm and then show that it is proper.

Let v be any vertex of T_∞. First, label it by $\{0,1\}$ and then label its neighbors by $\{3,4\}, \{5,6\}, \ldots, \{2\Delta+1, 2\Delta+2\}$ in any order. Now, pick a neighbor then label its $\Delta - 1$ unlabeled neighbors greedily by selecting numbers from $[2\Delta + 3]$. Next, consider another neighbor of v . Use the same manner to label its unlabeled neighbors and so on.

Now we have to make sure that we can run this process as long as we like since $T_\infty(\Delta)$ has infinite order and the process can provide a proper labeling.

For any label $\{a, b\}$, we assume $a < b$. Let u be a vertex labeled by $\{x, y\}$ and w be its labeled neighbor with label $\{s, t\}$. Now we are going to label u's $\Delta - 1$ unlabeled neighbors. By our process, w is been labeled first, so $\{x, y\} \subseteq \{t + 2, t + 3, \ldots, s - 2\}$. For these unlabeled neighbors must receive numbers from $\{y + 2, y + 3, \ldots, x - 2\}$. Note that $[2\Delta + 3]$ has even cardinality. So

$|\{y+2, y+3, \ldots, x-2\}|$ is even as well. Also $A = \{y+2, \ldots, s-1\}$ and $B = \{t+1, \ldots, x-2\}$ both have even cardinality. Thus we divide A and B into pairs. These pairs together with $\{s,t\}$ are proper labels for all neighbors of u where $\{s,t\}$ has been used in advanced. Therefore, we can keep going run this process. A pair $L(2,1)$-labeling of $T_\infty(\Delta)$ is then obtained. ■

The direct consequence of Theorem 8 is the following.

Corollary 9. *Let T be a tree with the maximum degree Δ. Then*
$$2\Delta + 2 \le \lambda^{(2)}(T) \le 2\Delta + 3.$$

Proof. Since T is a subtree of $T_\infty(\Delta)$, $\lambda^{(2)}(T) \le \lambda^{(2)}(T_\infty(\Delta)) = 2\Delta + 3$.

By the observation in Section 1, $\lambda^{(2)}(K_{1,\Delta}) = \lambda(K_{2,2\Delta}) = 2\Delta + 2$. T has the maximum degree Δ, so T contains a $K_{1,\Delta}$ as a subtree, $\lambda^{(2)}(T) \ge \lambda^{(2)}(K_{1,\Delta}) = 2\Delta + 2$. Therefore we have the corollary. ■

Let $\Gamma_S = P_\infty \Box P_\infty$ be the Cartesian product of P_∞ and P_∞. In particular, $V(\Gamma_S) = \mathbb{Z} \times \mathbb{Z}$ and $E(\Gamma_S) = \{(i_1, i_2)(j_1, j_2) : i_1 = j_1 \text{ and } |i_2 - j_2| = 1 \text{ or } j_2 = j_2 \text{ and } |i_1 - j_1| = 1\}$. The graph Γ_S is called the *square lattice*. See Figure 1.

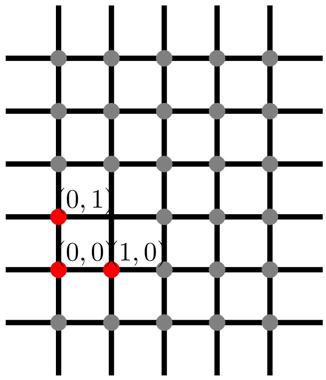


Figure 1. The square lattice Γ_S.

Theorem 10. $\lambda^{(2)}(\Gamma_S) = 11$.

Proof. By Lemma 1, $\lambda^{(2)}(\Gamma_S) \ge 2\Delta + 3 = 11$ since $\Delta(\Gamma_S) = 4$. On the other hand, we define a proper labeling f on $V(\Gamma_S)$ with span 11.

Define $f(0,0) = \{0,1\}$. For any i,j, define $f(i,j) = f(0,0) + (3i + 5j) \pmod{12} = \{3i + 5j, 3i + 5j + 1\} \pmod{12}$.

Suppose (i_1, j_1) and (i_2, j_2) are adjacent. Then either $i_1 = i_2$ and $|j_1 - j_2| = 1$ or $|i_1 - i_2| = 1$ and $j_1 = j_2$. Hence $\|f(i_1, j_1) - f(i_2, j_2)\|$ is either 3 or 5. Suppose the distance between (i_1, j_1) and (i_2, j_2) is 2.

Then either $|i_1 - i_2| = 2$, $j_1 = j_2$ or $|j_1 - j_2| = 2$, $i_1 = i_2$ or $|i_1 - i_2| = 1$, $|j_1 - j_2| = 1$. So the possible label difference are $\pm 6, \pm 10, \pm 8 \pmod{12}$. Hence they are 6, 10, 2, 8 and 4 after taking modulo 12. Therefore, f is a proper labeling with span 11. That is, $\lambda^{(2)}(\Gamma_S) \leq 11$. The theorem then asserts. ■

Now we consider induced subgraphs of Γ_S. Denote by $P(m,n) = P_m \Box P_n$ the Cartesian product of path P_m and P_n where $2 \leq n \leq m$. Notice that $P_m \Box P_n = P_n \Box P_m$. By definition, $P(m,n)$ is an induced subgraph of Γ_S. For convenience, we denote $V(P(m,n)) = \{(i,j) : 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$. The adjacency of vertices is same as Γ_S.

Proposition 11.

$$\lambda^{(2)}(P(m,n)) = \begin{cases} 8, & \text{if } m = n = 2, \\ 9, & \text{if } m \geq 3 \text{ and } n = 2, \\ 10, & \text{if } m = n = 3, \\ 11, & \text{otherwise.} \end{cases}$$

Proof. Let $n = 2 = m$. Since $P(2,2) = C_4$, $\lambda^{(2)}(P(2,2)) = 8 = \lambda^{(2)}(C_4)$.

Let $m \geq 3$ and $n = 2$. Define f on $V(P(m,n))$ by $f(0,0) = \{5,6\}$ and $f(0,1) = \{0,1\}$. Further, let $f(i,0) = f(0,0)+7 \pmod{10}$ and $f(i,1) = f(0,1)+7 \pmod{10}$, for $1 \leq i \leq n-1$. We see that f is an $L^{(2)}(2,1)$-labeling with span 9. Hence $\lambda^{(2)}(P(m,n)) \leq 9$. To prove that the equality holds, we first consider the subgraph $P(3,2)$.

Suppose there exists an $L^{(2)}(2,1)$-labeling f in $P(3,2)$ with the base set $[8]$. There are two adjacent major vertices in $P(3,2)$. They are $(1,0)$ and $(1,1)$. By Lemma 1(2), they shall be labeled by $\{0,1\}$ or $\{7,8\}$. By the symmetry of $P(3,2)$, say $f(1,1) = \{0,1\}$ and $f(1,0) = \{7,8\}$. Then $f(0,1) \subset \{3,4,5,6\}$ and $f(0,0) \subset \{2,3,4,5\}$. Since $(0,0)$ and $(0,1)$ are adjacent, there is only one possible, that is, $f(0,0) = \{2,3\}$ $f(0,1) = \{5,6\}$. And then $f(2,1) = \{3,4\}$ and $f(2,0) = \{4,5\}$. But this is impossible. Hence $\lambda^{(2)}(P(3,2)) \geq 9$. $P(3,2)$ is a subgraph of $P(m,2)$ for $m \geq 3$, $9 = \lambda^{(2)}(P(3,2)) \leq \lambda^{(2)}(P(m,2))$. This proves the result.

Let $n = 3 = m$. Then we use the following matrix of labels to label $P(3,3)$. (Note: (0,0) is labeled by $(2,3)$ and so on.)

$$\begin{pmatrix} \{9,5\} & \{9,10\} & \{6,7\} \\ \{7,8\} & \{0,1\} & \{3,4\} \\ \{2,3\} & \{5,6\} & \{8,9\} \end{pmatrix}$$

It is easy to verify that this matrix represents an $L^{(2)}(2,1)$-labeling of $P(3,3)$ with span 10. So $\lambda^{(2)}(P(3,3)) \leq 10$. On the other hand, since the maximum degree is 4, by observation above, $\lambda^{(2)}(P(3,3)) \geq 2(\Delta+1) = 2(4+1) = 10$. Hence we have the equality.

Let $n = 3$ and $m = 4$. Suppose $\lambda^{(2)}(P(4,3)) \leq 10$. By Lemma 1(2), a major vertex must receive label $\{0,1\}$ or $\{9,10\}$. Since there are two major vertices in

$P(4,3)$, they are $(1,1)$ and $(2,1)$. By the symmetry of the graph, say, $(1,1)$ is labeled by $\{0,1\}$ and $(2,1)$ is labeled by $\{9,10\}$. Then we check possible labels of their neighbors and then the other vertices by brutal force method. This is not difficult for vertices and available labels are not so many. We skip the detail here. The conclusion is that there is no proper labeling for $P(4,3)$. Thus the graph has the labeling number greater than or equal 11.

Let $n \geq 3$ and $m \geq 4$. By (4), $11 \leq \lambda^{(2)}(P(4,3)) \leq \lambda^{(2)}(P(m,n)) \leq \lambda^{(2)}(\Gamma_S) = 11$. Since $P(4,3) \subseteq P(m,n) \subset \Gamma_S$, the equality holds. ∎

The *hexagonal lattice* Γ_H is defined as $V(\Gamma_H) = \{(i,j) : i,j \in \mathbb{Z}\}$ where (i,j) is adjacent to $(i+1,j)$ and (i,j) is adjacent to $(i,j+1)$ whenever $i \not\equiv j \pmod 2$. See Figure 2.

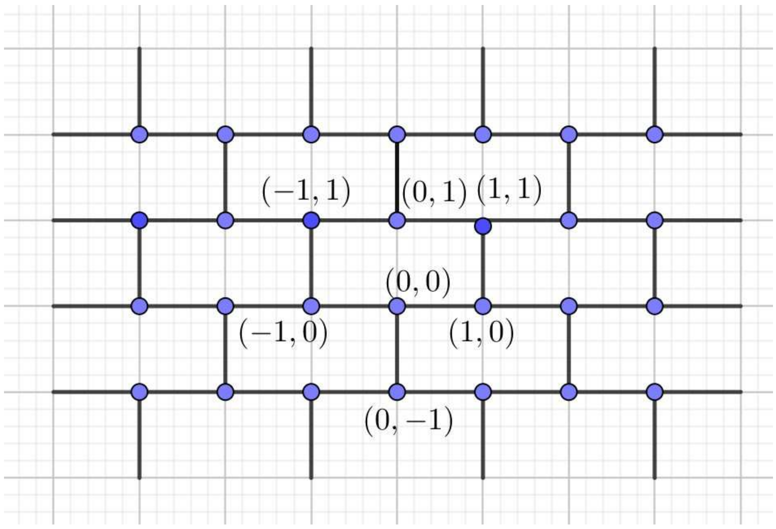


Figure 2. Γ_H.

Theorem 12. $\lambda^{(2)}(\Gamma_H) = 9$.

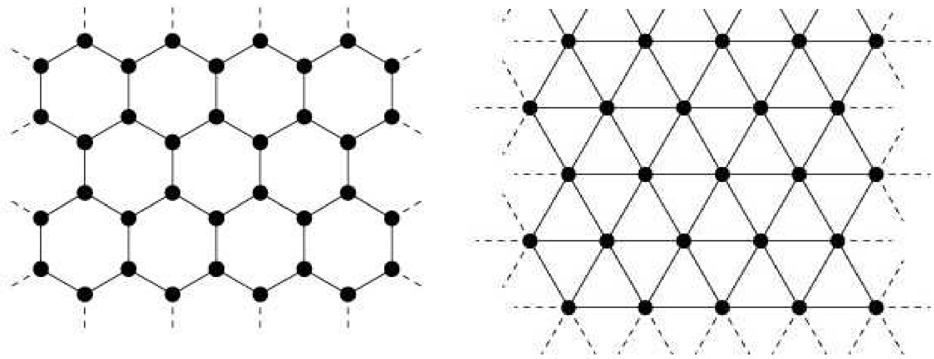
Proof. Since the hexagonal lattice is a 3-regular graph, by Lemma 1(3), $\lambda^{(2)}(\Gamma_H) \geq 2(\Delta+1)+1 = 2(3+1)+1 = 9$. To prove that the equality hold, we will present a pair $L(2,1)$-labeling on Γ_H using numbers less than or equal to 9.

Let

$$A = \begin{bmatrix} \begin{Bmatrix}5\\6\end{Bmatrix} & \begin{Bmatrix}8\\9\end{Bmatrix} & \begin{Bmatrix}1\\2\end{Bmatrix} & \begin{Bmatrix}4\\5\end{Bmatrix} & \begin{Bmatrix}7\\8\end{Bmatrix} & \begin{Bmatrix}0\\1\end{Bmatrix} & \begin{Bmatrix}3\\4\end{Bmatrix} & \begin{Bmatrix}6\\7\end{Bmatrix} & \begin{Bmatrix}9\\0\end{Bmatrix} & \begin{Bmatrix}2\\3\end{Bmatrix} \\ \\ \begin{Bmatrix}0\\1\end{Bmatrix} & \begin{Bmatrix}3\\4\end{Bmatrix} & \begin{Bmatrix}6\\7\end{Bmatrix} & \begin{Bmatrix}9\\0\end{Bmatrix} & \begin{Bmatrix}2\\3\end{Bmatrix} & \begin{Bmatrix}5\\6\end{Bmatrix} & \begin{Bmatrix}8\\9\end{Bmatrix} & \begin{Bmatrix}1\\2\end{Bmatrix} & \begin{Bmatrix}4\\5\end{Bmatrix} & \begin{Bmatrix}7\\8\end{Bmatrix} \end{bmatrix}.$$

We repeatedly use A to label Γ_H where vertex $(0,0)$ is labeled by $\{0,1\}$, $(1,0)$ by $\{3,4\}$ and $(0,1)$ by $\{5,6\}$. That is, we define an assignment f with $f(1,0) = \{0,1\}$, $f(i+1,j) = f(i,j) + 3 \pmod{10}$ and $f(i,j+1) = f(i,j) + 5 \pmod{10}$ for all i, j. It is easy to verify that we do produce a pair labeling with span 9. Hence, we obtain the equality. ∎

The hexagonal lattice can also be drawn as in Figure 3(a).



(a) Hexagonal lattice Γ_H. (b) Triangular lattice Γ_Δ.

Figure 3

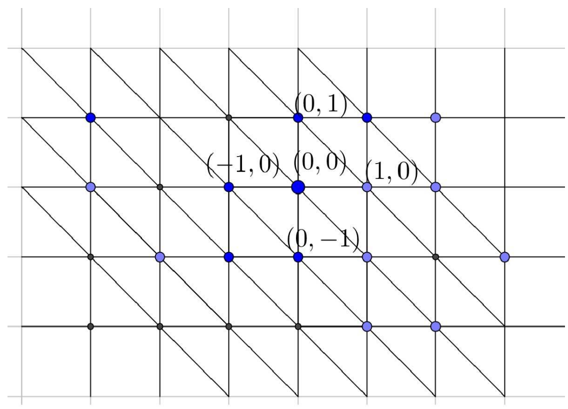


Figure 4. Γ_Δ with coordinate.

Since Γ_H is a plane graph, we can take the *dual* of it. Let Γ_Δ be the dual of Γ_H. Then Γ_Δ, the *triangular lattice*, is an infinite 3-regular plane graph. Every region of Γ_Δ is a triangle. See Figure 3(b). Similar to Γ_S and Γ_H, we can associate a coordinate (i,j) to each vertex. That is, let $V(\Gamma_\Delta)=\mathbb{Z}\times\mathbb{Z}$. A vertex (i,j) is adjacent to $(i,j+1$, $(i+1,j)$ and $(i+1,j-1)$ for all i,j. See Figure 4.

Theorem 13. $\lambda^{(2)}(\Gamma_\Delta)=16$.

Proof. Define a function on $V(\Gamma_\Delta)$ as follows: $f(0,0)=\{0,2\}$ and recurrently define

(1) $f(i+1,j)=f(i,j)+4\,(\text{mod } 17)$ and

(2) $f(i,j+1)=f(i,j)+9\,(\text{mod } 17)$ for all $i,j\in\mathbb{Z}$.

In the following, all operations are taking modulo 17. By the definition of f, we know that each label is of the form $\{x,x+2\}$. Let (i,j) be a vertex labeled by $\{x,x+2\}$. Now we need to check all vertices within distance 2 from (i,j). However, it suffices to consider vertices (1) distance 1 vertices: $(i+1,j),(i,j+1)$ and $(i+1,j-1)$ and (2) distance 2 vertices: $(i+2,j),(i+1,j+1),(i,j+2)(i+2,j-1)$ and $(i+2,j-2)$.

(1) (a) $f(i+1,j)=\{x,x+2\}+4=\{x+4,x+6\}$, (b) $f(i,j+1)=\{x,x+2\}+9=\{x+9,x+11\}$, (c) $f(i+1,j-1)=\{x,x+2\}+4-9=\{x,x+2\}-5=\{x,x+2\}+12=\{x+12,x+14\}$.

(2) (a) $f(i+2,j)=f(i,j)+8=\{x,x+2\}+8=\{x+8,x+10\}$, (b)$f(i+1,j+1)=f(i,j)+4+9=f(i,j)+13=\{x,x+2\}+13=\{x+13,x+15\}$, (c) $f(i,j+2)=f(i,j)+18=f(i,j)+1=\{x+1,x+3\}$, (d) $f(i+2,j-1)=f(i,j)+8-9=f(i,j)-1=f(i,j)+16=\{x+16.x+18\}=\{x+16,x+1\}$, (e) $f(i+2,j-2)=f(i,j)+8-18=f(i,j)-10=f(i,j)+7=\{x,x+2\}+7=\{x+7,x+9\}$.

After checking the differences, we confirm that f is a pair $L(2,1)$-labeling with span 16. Hence $\lambda^{(2)}(\Gamma_\Delta)\le 16$.

Suppose there is a pair $L(2,1)$-labeling with span 16. There must be a vertex, say u, with label of the form $\{0,x\}$. Then by Lemma 1(1), x can only be 1, 2, and 15.

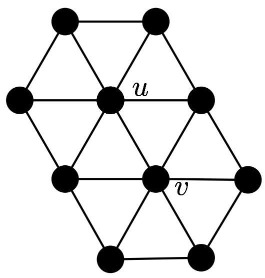


Figure 5. W.

$x=1$. Then one of its neighbor, say v, must be labeled by $\{3,4\}$ or $\{4,5\}$, for otherwise there is no way to label neighbors of u using $3,4,\ldots,15$.

$x = 2$. If u is labeled by $\{0, 2\}$ then one of its neighbor, say v must be labeled by $\{4, 5\}$.

$x = 15$. If u is labeled by $\{0, 15\}$, then one of its neighbor, say v, must be labeled by $\{2, 3\}$.

By examining all possible labelings on the subgraph W in Figure 5, we find that there is no proper labeling. Therefore, $\lambda^{(2)}(\Gamma_\Delta) \geq 16$. ■

5. Concluding Remarks

For further work, we shall extend our results from $n = 2$ to $n \geq 3$. However, we have already obtained the generalized versions of Theorem 8 and 10 as stated below.

Theorem 14. (1) $\lambda^{(n)}(T_\infty(\Delta)) = n(\Delta + 1) + 1$ *for* $\Delta \geq 2$.
(2) $\lambda^{(n)}(\Gamma_S) = 5n + 1$.

Motivations for studying paths and cycles are: (1) A typical m-cell linear highway cellular system along a highway (with the base-stations/transmitters in the center of each cell) can be modeled by a path P_m. (2) A loop cellular system around a big city, due to the high buildings, can be modeled by a cycle C_m (cf. [4]).

Further, some wireless communication networks can be modeled by the square lattice Γ_S and the triangular lattice Γ_Δ or their subgraphs. However, we are not aware of the hexagonal lattice Γ_H being used in real life for wireless networks, but it is mentioned in the engineering literature (cf. [5]).

In the end, we propose some open problems.

(1) By Corollary 3, $\lambda^{(2)}(G) \leq \lambda(G; 3, 2) + 1$. It is known that $\lambda(\Gamma_S; 3, 2) = 11 = \lambda^{(2)}(\Gamma_S)$, $\lambda(\Gamma_H; 3, 2) = 9 = \lambda^{(2)}(\Gamma_H)$ and $\lambda(\Gamma_\Delta; 3, 2) = 16 = \lambda^{(2)}(\Gamma_\Delta)$ (cf. [5]). The equality holds for a complete graph, path and cycle (cf. [4]). Base on these results, we conjecture that $\lambda^{(2)}(G) \in \{\lambda(G; 3, 2), \lambda(G; 3, 2) + 1\}$.

(2) Let $k \geq 2$. Consider all graphs with $\lambda = k$. Is $\{\lambda^{(2)}(G) : \lambda(G) = k\} \subseteq \{2k + 1 - \lfloor k/2 \rfloor, \ldots, 2k + 1\}$? We know that it is true for $k = 2, 3$ and 4 where the equality holds for $k = 4$. This question is motivated by a question in [3].

(3) We show that $\lambda^{(2)}(T)$ is either $2\Delta + 2$ or $2\Delta + 3$. But can we characterize these trees with the pair labeling number $2\Delta + 2$ or can we find an algorithm to evaluate $\lambda^{(2)}(T)$?

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