# PAIR $L(2,1)$-LABELINGS OF INFINITE GRAPHS 

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#### Abstract

An $L(2,1)$-labeling of a graph $G=(V, E)$ is an assignment of nonnegative integers to $V$ such that two adjacent vertices must receive numbers (labels) at least two apart and further, if two vertices are in distance 2 then they receive distinct labels. This article studies a generalization of the $L(2,1)$-labeling. We assign sets with at least one element to vertices of $G$ under some conditions.


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## 1. Introduction

Inspired by a channel assignment problem proposed by Lanfear to Roberts [7] in 1988, Griggs and Yeh [6] formulated the $L(2,1)$-labeling problem for graphs. Since then, there are considerable amounts of articles studying this labeling and its generalizations or related problems. Readers can see [1] and [8] for a survey. Now we like to consider another generalization.

Let $S$ be a finite set and $A$ and $B$ be two subsets of $S$. Define $\|A-B\|=$ $\min \{|a-b|: a \in A, b \in B\}$. Denote the set $[k]=\{0,1, \ldots, k\}$ and $\binom{[k]}{m}$ the collection of all $m$-subsets of $[k]$. Motivated by the article [3], we propose the following labeling on a graph.

Let $G=(V, E)$ be a graph and $n$ be a positive integer. Given non-negative integers $\delta_{1}, \delta_{2}$, an $L^{(n)}\left(\delta_{1}, \delta_{2}\right)$-labeling is a function $f: V(G) \rightarrow\binom{[k]}{n}$ for some $k \geq 1$ such that $\|f(u)-f(v)\| \geq \delta_{i}$ whenever the distance between $u$ and $v$ in $G$ is $i$, for $i=1,2$. (The minimum value and the maximum value of $\cup f(v)$ is 0 and
$k$, respectively.) The number $k$ is called of the span of $f$. The smallest $k$ so that there is an $L^{(n)}\left(\delta_{1}, \delta_{2}\right)$ - labeling $f$ with span $k$, is denoted by $\lambda^{(n)}\left(G ; \delta_{1}, \delta_{2}\right)$ and called the $L^{(n)}\left(\delta_{1}, \delta_{2}\right)$-labeling number of $G$. An $L^{(n)}\left(\delta_{1}, \delta_{2}\right)$-labeling with span $\lambda^{(n)}\left(G ; \delta_{1}, \delta_{2}\right)$ is called an optimal $L^{(n)}\left(\delta_{1}, \delta_{2}\right)$-labeling. If $n=1$ then notations $L^{(1)}$ and $\lambda^{(1)}$ will be simplified as $L$ and $\lambda$, respectively.

The elements in $[k]$ are called "numbers" and $f(u)$ is called the "label" of $u$. So a label is a set in this problem.

Using our notation, the labeling in [3] is the $L\left(\delta_{1}, 0\right)$-labeling for $\delta_{1} \geq 1$.
In this article we will consider the case when $\left(\delta_{1}, \delta_{2}\right)=(2,1)$ and the corresponding labeling number is denoted by $\lambda^{(n)}(G)$ (or just $\lambda^{(n)}$ if $G$ is understood) for short.

If $n=1$ then it is just the ordinary $L(2,1)$-labeling (cf. [6]) and $\lambda^{(1)}=\lambda$. For $n=2$, it is also called the pair $L(2,1)$-labeling and $\lambda^{(2)}$ is called the pair $L(2,1)$-labeling number. In the following, we first investigate properties of the $L^{(n)}(2,1)$-labelings. Then, we consider the pair $L(2,1)$-labelings on several classes of graphs. Finally, we present generalized results for $n \geq 2$ without giving proofs.

## 2. Preliminarily

Let $G$ be a graph and $n$ a positive integer. Now, we construct a new graph $G^{(n)}$ by replacing each vertex $v$ in $G$ by $n$ vertices $v_{i}, 1 \leq i \leq n$ and $u_{i}$ is adjacent to $v_{j}$ for all $i, j$, in $G^{(n)}$, whenever $u$ and $v$ is adjacent in $G$. (That is, $u_{i}$ and $v_{j}$, for all $i, j$, induces a complete bipartite graph $K_{n, n}$. Note that $G^{(1)}=G$.

It is easy to verify that $\lambda^{(n)}(G)=\lambda\left(G^{(n)}\right)$. Thus, for example, for $m \geq 2$, $\lambda^{(n)}\left(K_{m}\right)=\lambda\left(K_{n, n, \ldots, n}\right)=n m+m-2$, by previous results (cf. [6]) on complete $m$-partite graph $K_{n, n, \ldots, n}$.

A vertex $u$ with the maximum degree $\Delta$ in a graph $G$ is called a major vertex of $G$. It is easy to see that for any $G, \lambda^{(n)}(G) \geq n(\Delta+1)$. Also, similar to the $L(2,1)$-labeling, we have $\lambda^{(n)}(H) \leq \lambda^{(n)}(G)$ for any subgraph $H$ of $G$.

When we say "a vertex $v$ is labeled by a set $A$ and rules out $i$ numbers for some vertices" means that once $v$ is labeled by $A$ then there will be $i$ numbers in the base set not available for labeling these vertices. Usually, $i \geq|A|+1$.

Lemma 1. Let $G$ be a graph with a major vertex $u$ and $f$ be an $L^{(n)}(2,1)$-labeling of $G$.
(1) If the span of $f$ is $n(\Delta+1)+1$, then $f(u)$ is allowed to rule out at most $n+2$ numbers for its neighbors.
(2) If the span of $f$ is $n(\Delta+1)$, then $f(u)$ must be $\{0,1, \ldots, n-1\}$ or $\{n \Delta+$ $1, n \Delta+2, \ldots, n \Delta+n(=n(\Delta+1))\}$.
(3) If $u$ is adjacent to two other major vertices in $G$, then $\lambda^{(n)}(G) \geq n(\Delta+1)+1$.

Proof. (1) If $f(u)$ rules out more than $n+3$ numbers for its neighbors then at least $n+3$ numbers are not available for labeling $u$ 's neighbors. But, we need $n \Delta$ numbers for $u$ 's neighbors since they are distance 2 apart each other and hence there shall have at least $n \Delta+n+3=n(\Delta+1)+3$ numbers available. Thus the span of $f$ is at least $n(\Delta+1)+2$. This contradicts to our assumption.
(2) Suppose the conclusion is wrong. Let $f(u)$ be of the form $\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$, where $1 \leq x_{1}<x_{2}<\cdots<x_{n} \leq n(\Delta+1)-1$. Then $f(u)$ rules out at least $n+2$ numbers for labels of its neighbors. Hence, totally, we need at least $n \Delta+n+2=n(\Delta+1)+2$ numbers for $u$ and its neighbors. That is, the span of $f$ is at least $n(\Delta+1)+1$. This is a contradiction. Thus, (2) is true.
(3) Suppose the span of $f$ is $n(\Delta+1)$. Then by (2), $f(u)=\{0,1, \ldots, n-1\}$ or $\{n \Delta+1, n \Delta+2, \ldots, n \Delta+n(=n(\Delta+1)\}$.

By assumption, there are three major vertices within distance at most 2. So, they must receive three disjoint sets to be labeled but we only get two. We have a contradiction here. Hence the result asserts.

In the end of this section, we present an observation on the relation between the labeling numbers for $n=1$ and $n \geq 1$.
Proposition 2. $\lambda^{(n)}(G) \leq \lambda(G ; n+1, n)+n-1, n \geq 1$.
Proof. Let $\lambda(G ; n+1, n)=k$ and $f$ an optimal $L(n+1, n)$-labeling. Define sets $L_{i}=\{i, i+1, \ldots, i+n-1\}, i=0,1, \ldots, k$ and function $g_{f}: V(G) \rightarrow\binom{[k+n-1]}{n}$ by $g_{f}(u)=L_{i}$ whenever $f(u)=i$ for $i=0,1, \ldots, k$.

Suppose $u$ and $v$ are adjacent in $G, f(u)=i$ and $f(v)=i+n^{\prime}+1$ for $n^{\prime} \geq n$. Then $g_{f}(u)=\{i, i+1, \ldots, i+n-1\}$ and $g_{f}(v)=\left\{i+n^{\prime}+1, i+n^{\prime}+2, \ldots, i+n^{\prime}+n\right\}$. Hence $\left\|g_{f}(u)-g_{f}(v)\right\|=\left(i+n^{\prime}+1\right)-(i+n-1)=n^{\prime}-n+2 \geq 2$. Suppose the distance between $u$ and $v$ is 2. Let $f(u)=i$ and $f(v)=i+n^{\prime}$ for $n^{\prime} \geq n$. Then $g_{f}(u)=\{i, i+1, \ldots, i+n-1\}$ and $g_{f}(v)=\left\{i+n^{\prime}, i+n^{\prime}+1, \ldots, i+n^{\prime}+n-1\right\}$. Hence $\left\|g_{f}(u)-g_{f}(v)\right\|=n^{\prime}-n+1 \geq 1$. Thus $g_{f}$ is an $L^{(n)}(2,1)$-labeling with span $k+n-1$. Therefore $\lambda^{(n)}(G) \leq \lambda(G ; n+1, n)+n-1$.

Corollary 3. $\lambda^{(2)}(G) \leq \lambda(G ; 3,2)+1 \leq 2 \lambda(G)+1$.
Proof. The first inequality is the direct consequence of Proposition 2, for $n=1$. By the previous result, $\lambda(G ; 3,2)+1 \leq \lambda(G ; 4,2)+1=2 \lambda(G)+1$.

Next, we want to further look at the relation between $\lambda^{(2)}$ and $\lambda$ since most people are much more familiar with the $L(2,1)$-labeling than the $L(3,2)$-labeling. Let $f$ be an optimal $L(2,1)$-labeling on $G$ with $\lambda(G)=k$, and let $h_{f}=k+1-$ $|f(V(G))|$. (Note: $h_{f}$ is the number of elements not used by $f$ (called holes) in $[k]$.) Let $h=\max h_{f}$ over all optimal $L(2,1)$-labelings $f$. It is known that $h \leq\lfloor k / 2\rfloor$ (cf. [2]).

Proposition 4. Let $\lambda(G)=k$ and $h$ be defined above. Then

$$
\lambda^{(n)}(G) \leq n(k+1)-1-h(n-1)
$$

Proof. Let $f$ be an optimal $L(2,1)$-labeling on $G$ with $\lambda(G)=k$. Suppose $f(V(G))=\left\{0=\ell_{0}, \ell_{1}, \ldots, \ell_{t}=k\right\}$. Now we construct a labeling $g_{f}$ on $G$ as follows: Let $L_{0}=\{0,1, \ldots, n-1\}$. For $i \geq 1$, define set $L_{i}$ with its minimum element being (1) max $L_{i-1}+1$ if $\ell_{i}-\ell_{i-1}=1$ or (2) max $L_{i-1}+2$ if $\ell_{i}-\ell_{i-1}=2$. Suppose $\min L_{i}=x$. Then $L_{i}=\{x, x+1, \ldots, x+n-1\}$. The process ends until $L_{t}$ is determined. Let $u$ be a vertex. Define $g_{f}(u)=L_{i}$ whenever $f(u)=\ell_{i}$. We can check that $g_{f}$ is an $L^{(n)}(2,1)$-labeling of $G$ with span $M$ where $M=$ $n(k+1)-1-h_{f}(n-1)$. Note that this is true for all optimal $L(2,1)$-labeling $f$ of $G$. Therefore, $\lambda^{(n)}(G) \leq n(k+1)-1-h(n-1)$.

Take $G=K_{m}$ and $n=2$. Then $\lambda(G)=2(m-1)$ and $h=m-1$ (since we only can use $0,2,4, \ldots, 2(m-1)$ to label $G)$. So, $n(k+1)-1-h(n-1)=$ $2(2(m-1)+1)-1-(m-1)(2-1)=3 m-2=\lambda^{(2)}\left(K_{m}\right)$.

## 3. Elementary Graphs

This section will study the pair $L(2,1)$-labelings on paths, cycles and wheels.
In the following, when we say "several consecutive vertices" in a path means that they are adjacent one by one in the path.

Proposition 5. Let $P_{n}$ be a path of order $n \geq 2$. Then
(1) $\lambda^{(2)}\left(P_{2}\right)=4$,
(2) $\lambda^{(2)}\left(P_{3}\right)=\lambda^{(2)}\left(P_{4}\right)=6$ and
(3) $\lambda^{(2)}\left(P_{n}\right)=7$ for $n \geq 5$.

Proof. (1) Since $P_{2}=K_{2}$, we have $\lambda^{(2)}\left(P_{2}\right)=4$.
(2) $\lambda^{(2)}\left(P_{3}\right) \geq 6$, since the maximum degree of $P_{3}$ is 2 . On the other hand, we can use $\{5,6\},\{0,1\},\{3,4\}$ to label these three consecutive vertices of $P_{3}$. For a $P_{4}$, we can label four consecutive vertices by $\{2,3\},\{5,6\},\{0,1\},\{3,4\}$. Further, $6=\lambda^{(2)}\left(P_{3}\right) \leq \lambda^{(2)}\left(P_{4}\right)$, since $P_{3}$ is a subgraph of $P_{4}$.
(3) Suppose $n \geq 5$. Since $n \geq 5$, we have three consecutive major vertices $\left(\Delta\left(P_{n}\right)=2\right)$, by Lemma $1(3), \lambda^{(2)}\left(P_{n}\right) \geq 2 \Delta+3=7$. On the other hand, we can repeatedly use $\{0,1\},\{3,4\},\{6,7\}$ to label consecutive vertices starting from one end-vertex of $P_{n}$ to the other end. Hence $\lambda^{(2)}\left(P_{n}\right) \leq 7$. The result then asserts.

Let $A$ be a subset of $[k]$. Define $A+i(\bmod m)=\{a+i(\bmod m):$ for all $a \in$ $A\}$, for some $i$. If $m>k$ then obviously $A$ and $A+i(\bmod m)$ have the same
cardinality. Let $\sigma=\left\langle A_{1}, A_{2}, \ldots, A_{t}\right\rangle$ be a sequence of sets. Denote by $\sigma^{(i)}$ the sequence formed by duplicating $\sigma i$ times. For example, $\langle\{1,2\},\{3,4\}\rangle^{(2)}=$ $\langle\{1,2\},\{3,4\},\{1,2\},\{3,4\}\rangle$.

Proposition 6. Let $C_{m}(m \geq 3)$ be a cycle of order $m$. Then

$$
\lambda^{(2)}\left(C_{m}\right)=\left\{\begin{array}{ll}
9, & m=5, \\
7, & m=3 \text { or } m \geq 6 \\
8, & m=4,7,10,13
\end{array} \text { but } m \neq 7,10,13,\right.
$$

Proof. It is easy to verify results for $m=3,4,5$. Suppose $m \geq 6$.
Since $m \geq 6$, the cycle $C_{m}$ satisfies the condition in Lemma $1(3), \lambda^{(2)}\left(C_{m}\right) \geq$ $2 \Delta+3=7$. To prove that the equality holds, it suffices to construct a pair $L(2,1)-$ labeling for $C_{m}$ with span 7 .

Suppose $m \equiv 0(\bmod 3)$. Then we use $\{0,1\},\{3,4\},\{6,7\}$ repeatedly to label vertices of $C_{m}$. We see that it is a pair $L(2,1)$-labeling with span 7 . Thus $\lambda^{(2)}\left(C_{m}\right)=7$ in this case.

Suppose $m \equiv 2(\bmod 3)$. Let $\sigma_{1}=\langle\{0,1\},\{5,6\},\{2,3\},\{0,7\},\{4,5\},\{1,2\}$, $\{6,7\},\{3,4\}\rangle$ and $\sigma_{2}=\langle\{0,1\},\{7,6\},\{3,4\}\rangle$. Then we use $\left\langle\sigma_{1}, \sigma_{2}^{(p)}\right\rangle$ where $p=(m-8) / 3\left(\right.$ since $m \equiv 2(\bmod 3), p$ is an integer) to label $C_{m}$. We see that this is a pair $L(2,1)$-labeling with span 7 of $C_{m}$. Hence $\lambda^{(2)}\left(C_{m}\right)=7$.

Suppose $m \equiv 1(\bmod 3)$ and $m \geq 16$. That is $m \neq 7,10,13$. Then we use $\left\langle\sigma_{1}^{(2)}, \sigma_{2}^{(p)}\right\rangle$ where $p=(m-16) / 3$ (also an integer) to label $C_{m}$ with the span 7. Again, this is a pair $L(2,1)$-labeling of $C_{m}$. Hence, $\lambda^{(2)}\left(C_{m}\right)=7$ in this case.

Suppose $m=7,10$ or 13 . Set $\sigma_{3}=\langle\{0,1\},\{5,6\},\{2,3\},\{7,8\}\rangle$ and $\sigma_{4}=$ $\langle\{0,1\},\{3,4\},\{7,8\}\rangle$.

For $m=7,10,13$, we use $\left\langle\sigma_{3}, \sigma_{4}\right\rangle,\left\langle\sigma_{3}, \sigma_{4}^{(2)}\right\rangle$ and $\left\langle\sigma_{3}, \sigma_{4}^{(3)}\right\rangle$, respectively, to label $C_{m}$. We see that they are pair $L(2,1)$-labelings for each $m$. Hence $\lambda^{(2}\left(C_{m}\right) \leq 8$ in each case. It remains to show that 8 is the best possible.

Assume there is a pair $L(2,1)$-labeling $f$ on $C_{m}$ with span 7. Then there must be a vertex, say $v$, labeled by $\{0, x\}$ for $1 \leq x \leq 7$. By Lemma $1(1), x$ can only be 1,2 or 7 . Also we know that $f(v)$ contains consecutive numbers if it does not contain 0 , by Lemma 1(1).

Whenever $f$ exists, it means that there is a sequence of length $m+1$ starting with $\{0, x\}$ and end at $\{0, x\}$ again. Moreover, if there is a sequence $\sigma$ (with first term $\{0, x\}$ ) so that we can use the sequence $\sigma^{(p)}$ for some $p \geq 1$, to proper label vertices of $C_{m}$ consecutively. By considering all possibilities, we have the following results. (This is not difficult since we assume the span of $f$ is 7.)
$x=1$. The are two sequences $\sigma_{1}=\langle\{0,1\},\{3,4\},\{6,7\}\rangle$ and $\sigma_{2}=\langle\{0,1\}$, $\{5,6\},\{2,3\},\{0,7\},\{4,5\},\{1,2\},\{6,7\},\{3,4\}\rangle$.
$x=7$. There is only one sequence $\sigma_{3}=\langle\{0,7\},\{4,5\},\{1,2\},\{6,7\},\{3,4\}$, $\{0,1\},\{5,6\},\{2,3\}\rangle$.
$x=2$. There is no such sequence.
We find that when $m=7,10$ or 13 , these sequences are not proper for them. Hence their $\lambda^{(2)}$-labeling number is greater than or equal to 8 . Therefore we obtain the exact labeling numbers.

A wheel $W_{m}$ is the graph formed by joining a vertex to each vertices of the cycle $C_{m}$ for $m \geq 3$. In fact, $W_{3}=K_{4}$. The following proposition is easy to derive and we shall omit the proof.

## Proposition 7.

$$
\lambda^{(2)}\left(W_{m}\right)= \begin{cases}10, & \text { if } m=3 \\ 11, & \text { if } m=4 \\ 2 m+2, & \text { if } m \geq 5\end{cases}
$$

## 4. Infinite Graphs

In this section, we consider three infinite graphs. Let $\mathbb{Z}$ be the set of integers. Define the graph $P_{\infty}$ by letting $V\left(P_{\infty}\right)=\mathbb{Z}$ and $E\left(P_{\infty}\right)=\{i j:|i-j|=1, i, j \in \mathbb{Z}\}$. $P_{\infty}$ is a path of infinite order. Denote $T_{\infty}(\Delta)$ the $\Delta(\geq 2)$-regular infinite tree. That is, a tree with infinite many vertices and each vertex having degree $\Delta$. If $\Delta=2$ then it is just the $P_{\infty}$.

Theorem 8. $\lambda^{(2)}\left(T_{\infty}(\Delta)\right)=2 \Delta+3$, for $\Delta \geq 2$.
Proof. Since $T_{\infty}$ is a $\Delta$-regular graph with at least three vertices, by Lemma 1, $\lambda^{(2)}\left(T_{\infty}\right) \geq 2 \Delta+3$. To prove that the equality holds, it suffices to construct a proper labeling of $T_{\infty}$. In the following, we define an labeling by a greedy algorithm and then show that it is proper.

Let $v$ be any vertex of $T_{\infty}$. First, label it by $\{0,1\}$ and then label its neighbors by $\{3,4\},\{5,6\}, \ldots,\{2 \Delta+1,2 \Delta+2\}$ in any order. Now, pick a neighbor then label its $\Delta-1$ unlabeled neighbors greedily by selecting numbers from $[2 \Delta+3]$. Next, consider another neighbor of $v$. Use the same manner to label its unlabeled neighbors and so on.

Now we have to make sure that we can run this process as long as we like since $T_{\infty}(\Delta)$ has infinite order and the process can provide a proper labeling.

For any label $\{a, b\}$, we assume $a<b$. Let $u$ be a vertex labeled by $\{x, y\}$ and $w$ be its labeled neighbor with label $\{s, t\}$. Now we are going to label $u$ 's $\Delta-1$ unlabeled neighbors. By our process, $w$ is been labeled first, so $\{x, y\} \subseteq$ $\{t+2, t+3, \ldots, s-2\}$. For these unlabeled neighbors must receive numbers from $\{y+2, y+3, \ldots, x-2\}$. Note that $[2 \Delta+3]$ has even cardinality. So
$|\{y+2, y+3, \ldots, x-2\}|$ is even as well. Also $A=\{y+2, \ldots, s-1\}$ and $B=\{t+1, \ldots, x-2\}$ both have even cardinality. Thus we divide $A$ and $B$ into pairs. These pairs together with $\{s, t\}$ are proper labels for all neighbors of $u$ where $\{s, t\}$ has been used in advanced. Therefore, we can keep going run this process. A pair $L(2,1)$-labeling of $T_{\infty}(\Delta)$ is then obtained.

The direct consequence of Theorem 8 is the following.
Corollary 9. Let $T$ be a tree with the maximum degree $\Delta$. Then

$$
2 \Delta+2 \leq \lambda^{(2)}(T) \leq 2 \Delta+3 .
$$

Proof. Since $T$ is a subtree of $T_{\infty}(\Delta), \lambda^{(2)}(T) \leq \lambda^{(2)}\left(T_{\infty}(\Delta)\right)=2 \Delta+3$.
By the observation in Section $1, \lambda^{(2)}\left(K_{1, \Delta}\right)=\lambda\left(K_{2,2 \Delta}\right)=2 \Delta+2 . T$ has the maximum degree $\Delta$, so $T$ contains a $K_{1, \Delta}$ as a subtree, $\lambda^{(2)}(T) \geq \lambda^{(2)}\left(K_{1, \Delta}\right)=$ $2 \Delta+2$. Therefore we have the corollary.

Let $\Gamma_{S}=P_{\infty} \square P_{\infty}$ be the Cartesian product of $P_{\infty}$ and $P_{\infty}$. In particular, $V\left(\Gamma_{S}\right)=\mathbb{Z} \times \mathbb{Z}$ and $E\left(\Gamma_{S}\right)=\left\{\left(i_{1}, i_{2}\right)\left(j_{1}, j_{2}\right): i_{1}=j_{1}\right.$ and $\left|i_{2}-j_{2}\right|=1$ or $j_{2}=$ $j_{2}$ and $\left.\left|i_{1}-j_{1}\right|=1\right\}$. The graph $\Gamma_{S}$ is called the square lattice. See Figure 1.


Figure 1. The square lattice $\Gamma_{S}$.
Theorem 10. $\lambda^{(2)}\left(\Gamma_{S}\right)=11$.
Proof. By Lemma 1, $\lambda^{(2)}\left(\Gamma_{S}\right) \geq 2 \Delta+3=11$ since $\Delta\left(\Gamma_{S}\right)=4$. On the other hand, we define a proper labeling $f$ on $V\left(\Gamma_{S}\right)$ with span 11 .

Define $f(0,0)=\{0,1\}$. For any $i, j$, define $f(i, j)=f(0,0)+(3 i+5 j)$ $(\bmod 12)=\{3 i+5 j, 3 i+5 j+1\}(\bmod 12)$.

Suppose $\left(i_{1}, j_{1}\right)$ and $\left(i_{2}, j_{2}\right)$ are adjacent. Then either $i_{1}=i_{2}$ and $\left|j_{1}-j_{2}\right|=1$ or $\left|i_{1}-i_{2}\right|=1$ and $j_{1}=j_{2}$. Hence $\left\|f\left(i_{1}, j_{1}\right)-f\left(i_{2}, j_{2}\right)\right\|$ is either 3 or 5 . Suppose the distance between $\left(i_{1}, j_{1}\right)$ and $\left(i_{2}, j_{2}\right)$ is 2 .

Then either $\left|i_{1}-i_{2}\right|=2, j_{1}=j_{2}$ or $\left|j_{1}-j_{2}\right|=2, i_{1}=i_{2}$ or $\left|i_{1}-i_{2}\right|=1$, $\left|j_{1}-j_{2}\right|=1$. So the possible label difference are $\pm 6, \pm 10, \pm 8(\bmod 12)$. Hence
they are $6,10,2,8$ and 4 after taking modulo 12 . Therefore, $f$ is a proper labeling with span 11. That is, $\lambda^{(2)}\left(\Gamma_{S}\right) \leq 11$. The theorem then asserts.

Now we consider induced subgraphs of $\Gamma_{S}$. Denote by $P(m, n)=P_{m} \square P_{n}$ the Cartesian product of path $P_{m}$ and $P_{n}$ where $2 \leq n \leq m$. Notice that $P_{m} \square P_{n}=P_{n} \square P_{m}$. By definition, $P(m, n)$ is an induced subgraph of $\Gamma_{S}$. For convenience, we denote $V(P(m, n))=\{(i, j): 0 \leq i \leq m-1,0 \leq j \leq n-1\}$. The adjacency of vertices is same as $\Gamma_{S}$.

## Proposition 11.

$$
\lambda^{(2)}(P(m, n))=\left\{\begin{aligned}
8, & \text { if } m=n=2 \\
9, & \text { if } m \geq 3 \text { and } n=2 \\
10, & \text { if } m=n=3 \\
11, & \text { otherwise }
\end{aligned}\right.
$$

Proof. Let $n=2=m$. Since $P(2,2)=C_{4}, \lambda^{(2)}(P(2,2))=8=\lambda^{(2)}\left(C_{4}\right)$.
Let $m \geq 3$ and $n=2$. Define $f$ on $V(P(m, n))$ by $f(0,0)=\{5,6\}$ and $f(0,1)=\{0,1\}$. Further, let $f(i, 0)=f(0,0)+7(\bmod 10)$ and $f(i, 1)=f(0,1)+7$ $(\bmod 10)$, for $1 \leq i \leq n-1$. We see that $f$ is an $L^{(2)}(2,1)$-labeling with span 9 . Hence $\lambda^{(2)}(P(m, n)) \leq 9$. To prove that the equality holds, we first consider the subgraph $P(3,2)$.

Suppose there exists an $L^{(2)}(2,1)$-labeling $f$ in $P(3,2)$ with the base set [8]. There are two adjacent major vertices in $P(3,2)$. They are $(1,0)$ and $(1,1)$. By Lemma $1(2)$, they shall be labeled by $\{0,1\}$ or $\{7,8\}$. By the symmetry of $P(3,2)$, say $f(1,1)=\{0,1\}$ and $f(1,0)=\{7,8\}$. Then $f(0,1) \subset\{3,4,5,6\}$ and $f(0,0) \subset\{2,3,4,5\}$. Since $(0,0)$ and $(0,1)$ are adjacent, there is only one possible, that is, $f(0,0)=\{2,3\} f(0,1)=\{5,6\}$. And then $f(2,1)=\{3,4\}$ and $f(2,0)=\{4,5\}$. But this is impossible. Hence $\lambda^{(2)}(P(3,2)) \geq 9 . P(3,2)$ is a subgraph of $P(m, 2)$ for $m \geq 3,9=\lambda^{(2)}(P(3,2)) \leq \lambda^{(2)}(P(m, 2))$. This proves the result.

Let $n=3=m$. Then we use the following matrix of labels to label $P(3,3)$. (Note: $(0,0)$ is labeled by $(2,3)$ and so on.)

$$
\left(\begin{array}{ccc}
\{9,5\} & \{9,10\} & \{6,7\} \\
\{7,8\} & \{0,1\} & \{3,4\} \\
\{2,3\} & \{5,6\} & \{8,9\}
\end{array}\right)
$$

It is easy to verify that this matrix represents an $L^{(2)}(2,1)$-labeling of $P(3,3)$ with span 10. So $\lambda^{(2)}(P(3,3)) \leq 10$. On the other hand, since the maximum degree is 4 , by observation above, $\lambda^{(2)}(P(3,3)) \geq 2(\Delta+1)=2(4+1)=10$. Hence we have the equality.

Let $n=3$ and $m=4$. Suppose $\lambda^{(2)}(P(4,3)) \leq 10$. By Lemma $1(2)$, a major vertex must receive label $\{0,1\}$ or $\{9,10\}$. Since there are two major vertices in
$P(4,3)$, they are $(1,1)$ and $(2,1)$. By the symmetry of the graph, say, $(1,1)$ is labeled by $\{0,1\}$ and $(2,1)$ is labeled by $\{9,10\}$. Then we check possible labels of their neighbors and then the other vertices by brutal force method. This is not difficult for vertices and available labels are not so many. We skip the detail here. The conclusion is that there is no proper labeling for $P(4,3)$. Thus the graph has the labeling number greater than or equal 11.

Let $n \geq 3$ and $m \geq 4$. By (4), $11 \leq \lambda^{(2)}(P(4,3)) \leq \lambda^{(2)}(P(m, n)) \leq$ $\lambda^{(2)}\left(\Gamma_{S}\right)=11$. Since $P(4,3) \subseteq P(m, n) \subset \Gamma_{S}$, the equality holds.

The hexagonal lattice $\Gamma_{H}$ is defined as $V\left(\Gamma_{H}\right)=\{(i, j): i, j \in \mathbb{Z}\}$ where $(i, j)$ is adjacent to $(i+1, j)$ and $(i, j)$ is adjacent to $(i, j+1)$ whenever $i \not \equiv j(\bmod 2)$. See Figure 2.


Figure 2. $\Gamma_{H}$.

Theorem 12. $\lambda^{(2)}\left(\Gamma_{H}\right)=9$.
Proof. Since the hexagonal lattice is a 3-regular graph, by Lemma 1(3), $\lambda^{(2)}\left(\Gamma_{H}\right)$ $\geq 2(\Delta+1)+1=2(3+1)+1=9$. To prove that the equality hold, we will present a pair $L(2,1)$-labeling on $\Gamma_{H}$ using numbers less than or equal to 9 .

Let

We repeatedly use $A$ to label $\Gamma_{H}$ where vertex $(0,0)$ is labeled by $\{0,1\}$, $(1,0)$ by $\{3,4\}$ and $(0,1)$ by $\{5,6\}$. That is, we define an assignment $f$ with $f(1,0)=\{0,1\}, f(i+1, j)=f(i, j)+3(\bmod 10)$ and $f(i, j+1)=f(i, j)+5$ $(\bmod 10)$ for all $i, j$. It is easy to verify that we do produce a pair labeling with span 9 . Hence, we obtain the equality.

The hexagonal lattice can also be drawn as in Figure 3(a).


Figure 3


Figure 4. $\Gamma_{\Delta}$ with coordinate.

Since $\Gamma_{H}$ is a plane graph, we can take the dual of it. Let $\Gamma_{\Delta}$ be the dual of $\Gamma_{H}$. Then $\Gamma_{\Delta}$, the triangular lattice, is an infinite 3-regular plane graph. Every region of $\Gamma_{\Delta}$ is a triangle. See Figure 3(b). Similar to $\Gamma_{S}$ and $\Gamma_{H}$, we can associate
a coordinate $(i, j)$ to each vertex. That is, let $V\left(\Gamma_{\Delta}\right)=\mathbb{Z} \times \mathbb{Z}$. A vertex $(i, j)$ is adjacent to $(i, j+1,(i+1, j)$ and $(i+1, j-1)$ for all $i, j$. See Figure 4.

Theorem 13. $\lambda^{(2)}\left(\Gamma_{\Delta}\right)=16$.
Proof. Define a function on $V\left(\Gamma_{\Delta}\right)$ as follows: $f(0,0)=\{0,2\}$ and recurrently define
(1) $f(i+1, j)=f(i, j)+4(\bmod 17)$ and
(2) $f(i, j+1)=f(i, j)+9(\bmod 17)$ for all $i, j \in \mathbb{Z}$.

In the following, all operations are taking modulo 17. By the definition of $f$, we know that each label is of the form $\{x, x+2\}$. Let $(i, j)$ be a vertex labeled by $\{x, x+2\}$. Now we need to check all vertices within distance 2 from $(i, j)$. However, it suffices to consider vertices (1) distance 1 vertices: $(i+1, j),(i, j+1)$ and $(i+1, j-1)$ and (2) distance 2 vertices: $(i+2, j),(i+1, j+1),(i, j+2)(i+$ $2, j-1)$ and $(i+2, j-2)$.
(1) (a) $f(i+1, j)=\{x, x+2\}+4=\{x+4, x+6\}$, (b) $f(i, j+1)=$ $\{x, x+2\}+9=\{x+9, x+11\}$, (c) $f(i+1, j-1)=\{x, x+2\}+4-9=$ $\{x, x+2\}-5=\{x, x+2\}+12=\{x+12, x+14\}$.
(2) (a) $f(i+2, j)=f(i, j)+8=\{x, x+2\}+8=\{x+8, x+10\},(b) f(i+$ $1, j+1)=f(i, j)+4+9=f(i, j)+13=\{x, x+2\}+13=\{x+13, x+15\}$, (c) $f(i, j+2)=f(i, j)+18=f(i, j)+1=\{x+1, x+3\}$, (d) $f(i+2, j-1)=$ $f(i, j)+8-9=f(i, j)-1=f(i, j)+16=\{x+16 \cdot x+18\}=\{x+16, x+1\},(\mathrm{e})$ $f(i+2, j-2)=f(i, j)+8-18=f(i, j)-10=f(i, j)+7=\{x, x+2\}+7=$ $\{x+7, x+9\}$.

After checking the differences, we confirm that $f$ is a pair $L(2,1)$-labeling with span 16. Hence $\lambda^{(2)}\left(\Gamma_{\Delta}\right) \leq 16$.

Suppose there is a pair $L(2,1)$-labeling with span 16 . There must be a vertex, say $u$, with label of the form $\{0, x\}$. Then by Lemma $1(1), x$ can only be 1,2 , and 15 .


Figure 5. $W$.
$x=1$. Then one of its neighbor, say $v$, must be labeled by $\{3,4\}$ or $\{4,5\}$, for otherwise there is no way to label neighbors of $u$ using $3,4, \ldots, 15$.
$x=2$. If $u$ is labeled by $\{0,2\}$ then one of its neighbor, say $v$ must be labeled by $\{4,5\}$.
$x=15$. If $u$ is labeled by $\{0,15\}$, then one of its neighbor, say $v$, must be labeled by $\{2,3\}$.

By examining all possible labelings on the subgraph $W$ in Figure 5, we find that there is no proper labeling. Therefore, $\lambda^{(2)}\left(\Gamma_{\Delta}\right) \geq 16$.

## 5. Concluding Remarks

For further work, we shall extend our results from $n=2$ to $n \geq 3$. However, we have already obtained the generalized versions of Theorem 8 and 10 as stated below.

Theorem 14. (1) $\lambda^{(n)}\left(T_{\infty}(\Delta)\right)=n(\Delta+1)+1$ for $\Delta \geq 2$. (2) $\lambda^{(n)}\left(\Gamma_{S}\right)=5 n+1$.

Motivations for studying paths and cycles are: (1) A typical m-cell linear highway cellular system along a highway (with the base-stations/transmitters in the center of each cell) can be modeled by a path $P_{m}$. (2) A loop cellular system around a big city, due to the high buildings, can be modeled by a cycle $C_{m}$ (cf. [4]).

Further, some wireless communication networks can be modeled by the square lattice $\Gamma_{S}$ and the triangular lattice $\Gamma_{\Delta}$ or their subgraphs. However, we are not aware of the hexagonal lattice $\Gamma_{H}$ being used in real life for wireless networks, but it is mentioned in the engineering literature (cf. [5]).

In the end, we propose some open problems.
(1) By Corollary $3, \lambda^{(2)}(G) \leq \lambda(G ; 3,2)+1$. It is known that $\lambda\left(\Gamma_{S} ; 3,2\right)=$ $11=\lambda^{(2)}\left(\Gamma_{S}\right), \lambda\left(\Gamma_{H} ; 3,2\right)=9=\lambda^{(2)}\left(\Gamma_{H}\right)$ and $\lambda\left(\Gamma_{\Delta} ; 3,2\right)=16=\lambda^{(2)}\left(\Gamma_{\Delta}\right)$ (cf. [5]). The equality holds for a complete graph, path and cycle (cf. [4] ). Base on these results, we conjecture that $\lambda^{(2)}(G) \in\{\lambda(G ; 3,2), \lambda(G ; 3,2)+1\}$.
(2) Let $k \geq 2$. Consider all graphs with $\lambda=k$. Is $\left\{\lambda^{(2)}(G): \lambda(G)=k\right\} \subseteq$ $\{2 k+1-\lfloor k / 2\rfloor, \ldots, 2 k+1\}$ ? We know that it is true for $k=2,3$ and 4 where the equality holds for $k=4$. This question is motivated by a question in [3].
(3) We show that $\lambda^{(2)}(T)$ is either $2 \Delta+2$ or $2 \Delta+3$. But can we characterize these trees with the pair labeling number $2 \Delta+2$ or can we find an algorithm to evaluate $\lambda^{(2)}(T)$ ?

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