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ORIENTED INCIDENCE COLOURINGS OF DIGRAPHS

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Abstract

Brualdi and Quinn Massey [6] defined incidence colouring while studying the strong edge chromatic index of bipartite graphs. Here we introduce a similar concept for digraphs and define the oriented incidence chromatic number. Using digraph homomorphisms, we show that the oriented incidence chromatic number of a digraph is closely related to the chromatic number of the underlying simple graph. This motivates our study of the oriented incidence chromatic number of symmetric complete digraphs. We give upper and lower bounds for the oriented incidence chromatic number of these graphs, as well as digraphs arising from common graph constructions and decompositions. Additionally we construct, for all $k \geqslant 2$, a target digraph H_k for which oriented incidence k colouring is equivalent to homomorphism to H_k.

Keywords: digraph homomorpism, graph colouring, incidence colouring, computational complexity.

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1. Introduction and Preliminaries

Let G be a simple graph. We obtain an *orientation* of G by assigning to each of the edges of G a direction to form a digraph. If a digraph is an orientation of a simple graph we refer to it as an *oriented graph.* A digraph is called *semicomplete* if it has a spanning tournament. Herein we assume that digraphs do not contain loops or parallel arcs. Since we are dealing primarily with digraphs, we will omit the over arrow ($\overrightarrow{G}$ is written as G and $\overrightarrow{uv}$ is written as uv) unless there is the possibility for confusion. For all other notation we refer to [5].

Let G and H be digraphs. We say that G admits a *homomorphism* to H if there exists $\phi: V(G) \to V(H)$ such that if uv is an arc in G, then $\phi(u)\phi(v)$ is an arc in H. If G admits a homomorphism to H, we write $G \to H$. If ϕ is a homomorphism of G to H we write $\phi: G \to H$. Two directed graphs G and H are called *homomorphically equivalent* if there are homomorphisms $G \to H$ and $H \to G$. If G and H are homomorphically equivalent, then a given digraph D has a homomorphism to G if and only if it has a homomorphism to H. A directed graph is called a *core* if it is not homomorphically equivalent to any of its proper subgraphs. We note that every directed graph contains a unique core, up to isomorphism [8, 21], to which it is homomorphically equivalent. If $\mathcal{F}$ is a class of digraphs so that there exists a digraph H such that $F \to H$ for every $F \in \mathcal{F}$, then we say that H is a *universal target* for $\mathcal{F}$. The *oriented chromatic number* of the oriented graph G is the least k such that there exists an oriented graph H on k vertices such that $G \to H$. We use $\chi_o(G)$ to denote this parameter.

In recent years the study of homomorphisms of oriented graphs has received particular attention (for examples see [16, 12, 19]), as homomorphisms allow for a definition of vertex colouring of an oriented graph that respects the orientation of the arcs [7].

In this paper we study a colouring parameter for digraphs based on incidence. Two arcs are said to be *related* if the head vertex of one of the arcs is the tail vertex of the other arc. For every arc uv in a digraph, we define two *incidences*; the *tail incidence* of uv is the ordered pair (u, uv) and the *head incidence* of uv is the ordered pair (v, uv).

Two distinct incidences in a digraph G are *adjacent* if and only if they correspond to one the following four cases.

For every arc uv,

- the incidences (u, uv) and (v, uv) are adjacent.

For every two related arcs uv and vw,

- the incidences (v, uv) and (v, vw) are adjacent,
- the incidences (u, uv) and (v, vw) are adjacent,
- the incidences (v, uv) and (w, vw) are adjacent.

Let $\mathcal{I}_G$ be the simple graph such that every vertex corresponds to an incidence of G and every edge corresponds to two adjacent incidences. An *oriented incidence colouring* of G assigns a colour to every incidence of G such that adjacent incidences receive different colours. An oriented incidence colouring of G is thus a proper vertex colouring of $\mathcal{I}_G$. If $c\colon V(\mathcal{I}_G) \to \{1, 2, \ldots, k\}$ is an oriented incidence k-colouring of G, then the colour of (x_i, x_1x_2) is denoted $c(x_i, x_1x_2)$, instead of $c((x_i, x_1x_2))$ $(i \in \{1, 2\})$.

For a digraph G, we define the *oriented incidence chromatic number* $\overrightarrow{\chi_i}(G)$ as the least k such that G has an oriented incidence k-colouring. For a class $\mathcal{C}$ of digraphs, we define $\overrightarrow{\chi_i}(\mathcal{C})$ as the least k such that $\overrightarrow{\chi_i}(G) \leqslant k$ for every $G \in \mathcal{C}$. Figures 1, 2, and 4 give examples of oriented incidence colourings of some digraphs with few vertices.

Incidence colouring arose in 1993 when Brualdi and Quinn Massey first defined the incidence chromatic number of a simple graph (then called the incidence colouring number), denoted $\chi_i(G)$ [6]. They gave upper and lower bounds for $\chi_i(G)$ based on the maximum degree of the graph and they used their results as a method to improve a bound for the strong chromatic index of bipartite graphs. Since then, bounds have been investigated for a variety of graph classes, including planar graphs, k-trees, k-regular graphs and k-degenerate graphs [11, 20, 22].

In this paper our main goal is to study the relationship between oriented incidence colouring and digraph homomorphisms. Using this relationship we find a connection between the oriented incidence chromatic number of a digraph and the chromatic number of its underlying simple graph. Subsequently, we find upper and lower bounds for the oriented incidence chromatic number of the symmetric complete loopless digraph $\overleftrightarrow{K}_k$ obtained by replacing every edge xy of the complete graph K_k by the arcs xy and yx.

Using an appropriate definition of *half arc*, oriented incidence colouring can be viewed as a colouring of half arcs. We can then examine this colouring parameter through the lens of arc colouring by first subdividing each of the arcs. By taking a directed line graph of this subdivided graph, an oriented incidence colouring can be viewed as a vertex colouring in which adjacent vertices and vertices at the end of a directed path of length two (a 2-*dipath*) must receive different colours. The oriented incidence chromatic number is the 2-dipath chromatic number of the digraph resulting from first subdividing each arc and then taking the directed line graph.

The study of 2-dipath colourings of oriented graphs in Sherk's (née Young) thesis [23] contains a result that provides an upper bound on the oriented chromatic number as a function of the 2-dipath chromatic number. We consider the possibility of a result relating the oriented chromatic number and the oriented incidence chromatic number. This idea is explored in Section 2.

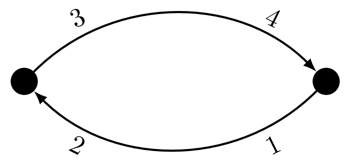

Figure 1. An oriented incidence colouring of the 2-cycle.

We begin our study of the oriented incidence chromatic number by relating the oriented incidence chromatic number of an oriented graph to the incidence chromatic number of the underlying simple graph. To do so, we observe that the set of incidences of an oriented graph is exactly equal to the set of incidences of the underlying graph, as defined in [6]. As $\chi_i(K_k) = k$ for every $k \geqslant 2$ [6], we thus obtain.

Proposition 1. *If $\overrightarrow{G}$ is an orientation of the graph G, then*

$$\vec{\chi_i}(\overrightarrow{G}) \leqslant \chi_i(G) \leqslant |V(G)|.$$

We improve this bound in Section 3 by observing that any oriented graph G is a subgraph of the symmetric complete digraph (without loops) on the same number of vertices and applying our bound for the oriented incidence chromatic number of these digraphs.

2. Oriented Incidence Colouring and Homomorphism

A useful observation regarding oriented incidence colouring is the following general result relating oriented incidence colouring and digraph homomorphism.

Theorem 2. *If G and H are digraphs such that $G \to H$, then $\vec{\chi_i}(G) \leqslant \vec{\chi_i}(H)$.*

Proof. Let f be an oriented incidence colouring of H using $\vec{\chi_i}(H)$ colours and let ϕ be a homomorphism of G to H. We show that the mapping c such that

$c(x_i, x_1x_2) = f(\phi(x_i), \phi(x_1)\phi(x_2)), (i \in \{1,2\})$ for every $x_1x_2 \in A(G)$ is an oriented incidence colouring.

If $uv \in A(G)$, then

- $c(u,uv) = f(\phi(u), \phi(u)\phi(v)) \neq f(\phi(v), \phi(u)\phi(v)) = c(v,uv)$.

If $uv \in A(G)$ and $vw \in A(G)$, then

- $c(v,uv) = f(\phi(v), \phi(u)\phi(v)) \neq f(\phi(v), \phi(v)\phi(w)) = c(v,vw)$,
- $c(u,uv) = f(\phi(u), \phi(u)\phi(v)) \neq f(\phi(v), \phi(v)\phi(w)) = c(v,vw)$,
- $c(v,uv) = f(\phi(v), \phi(u)\phi(v)) \neq f(\phi(w), \phi(v)\phi(w)) = c(w,vw)$.

Therefore, c is an oriented incidence colouring of G using at most $\overrightarrow{\chi_i}(H)$ colours. ■

Corollary 3. *If G is an oriented graph, then $\overrightarrow{\chi_i}(G) \leqslant \chi_o(G)$.*

Proof. By the definition of $\chi_o(G)$, there exists an oriented graph H such that $G \to H$ and $\chi_o(G) = |V(H)|$. By Theorem 2 and Proposition 1, we have

$$\overrightarrow{\chi_i}(G) \leqslant \overrightarrow{\chi_i}(H) \leqslant |V(H)| = \chi_o(G),$$

as required. ■

The corollary allows us to give an upper bound for the oriented incidence chromatic number of any class of oriented graphs that has bounded oriented chromatic number.

Proposition 4. *If G is an oriented forest, then $\overrightarrow{\chi_i}(G) \leqslant 3$.*

Proof. Every oriented forest admits a homomorphism to the directed cycle on three vertices. The result now follows directly from Corollary 3. ■

Proposition 5. *If $\overrightarrow{G}$ is a digraph and G is the underlying simple graph of $\overrightarrow{G}$, then*

$$\overrightarrow{\chi_i}(\overrightarrow{G}) \leqslant \overrightarrow{\chi_i}(\overrightarrow{K}_{\chi(G)}).$$

Proof. Observe that if $\chi(G) = k$, then $\overrightarrow{G}$ admits a homomorphism to $\overrightarrow{K}_k$. Thus, we have $\overrightarrow{\chi_i}(\overrightarrow{G}) \leqslant \overrightarrow{\chi_i}(\overrightarrow{K}_k)$ by Theorem 2. ■

By viewing oriented incidence colouring as a problem on sets, we arrive at the following characterisation of the oriented incidence chromatic number. Let c be an oriented incidence colouring of a digraph G. For a vertex u, let $A_u = \bigcup_{uv \in A(G)} c(u,uv)$ and let $B_u = \bigcup_{vu \in A(G)} c(u,vu)$. Informally, A_u is the set of colours that are assigned to the tail incidences (u,uv) and B_u is the set of colours that are assigned to the head incidences (u,vu). We observe the following result.

Theorem 6. *For a digraph G with n vertices, $\overrightarrow{\chi_i}(G)$ is the least k such that there exist sets $X_{u_1}, X_{u_2}, \ldots, X_{u_n} \subseteq \{1,2,3,\ldots,k\}$ and sets $Y_{u_1}, Y_{u_2}, \ldots, Y_{u_n} \subseteq \{1,2,3,\ldots,k\}$ so that each of the following hold.*

(1) *For every vertex v, $X_v \cap Y_v = \emptyset$.*

(2) *For every arc uv, $X_u \setminus X_v \neq \emptyset$ and $Y_v \setminus Y_u \neq \emptyset$.*

(3) *For every arc uv, if $X_u \setminus X_v = Y_v \setminus Y_u$, then $|X_u \setminus X_v| \neq 1$.*

Proof. It is easily checked that if c is an oriented incidence colouring of G, then (1), (2) and (3) are satisfied by setting $X_u = A_u$ and $Y_u = B_u$ for every $u \in V(G)$.

Assume now that there exist sets $X_{u_1}, X_{u_2}, \ldots, X_{u_n} \subseteq \{1,2,3,\ldots,k\}$ and sets $Y_{u_1}, Y_{u_2}, \ldots, Y_{u_n} \subseteq \{1,2,3,\ldots,k\}$ that satisfy the hypotheses. We construct an oriented incidence colouring c by assigning to each incidence (u, uv) a colour from the set $X_u \setminus X_v$ and to each incidence (v, uv) a colour from the set $Y_v \setminus Y_u$ such that $c(u, uv) \neq c(v, uv)$. ■

Since homomorphism to the complete digraph is useful in finding an upper bound on the oriented incidence chromatic number, we study the oriented incidence chromatic number of a complete digraph in the next section.

3. Symmetric Complete Digraphs

In this section, we give upper and lower bounds for $\overrightarrow{\chi_i}(\overrightarrow{K}_n)$.

Table 1 gives the oriented incidence chromatic number of $\overrightarrow{K}_n$ for $0 \leqslant n \leqslant 8$. These values are found by computer search and are best possible. Figures 1 and 2 give oriented incidence colourings of $\overrightarrow{K}_2$ and $\overrightarrow{K}_3$, respectively, using the fewest number of colours. Appendix gives the explicit colourings that verify Table 1 for $n = 4, 5, 6, 7, 8$.

n	0	1	2	3	4	5	6	7	8
$\overrightarrow{\chi_i}\left(\overrightarrow{K}_n\right)$	0	0	4	4	5	5	6	6	6

Table 1. Oriented incidence chromatic number of complete symmetric digraphs for $n \leq 8$.

Theorem 7. *If k and n are integers such that $n > \binom{k}{\lfloor k/2 \rfloor}$, then $\overrightarrow{\chi_i}(\overrightarrow{K}_n) > k$.*

Proof. We prove the contrapositive, so we suppose that $\overrightarrow{K}_n$ admits an oriented incidence k-colouring c.

It follows directly from Theorem 6 that the sets $A_1, A_2, \ldots, A_n$ (with respect to c) form an antichain in the inclusion order for subsets of $\{1,2,\ldots,k\}$. By

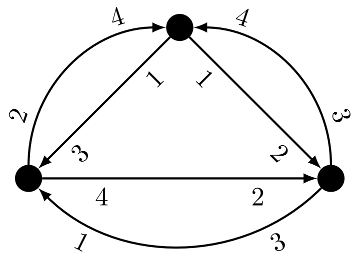

Figure 2. An oriented incidence colouring of the complete digraph on three vertices $\overrightarrow{K}_3$.

Sperner's Theorem, the size of such an antichain is at most $\binom{k}{\lfloor k/2 \rfloor}$ [5]. This implies that $n \leqslant \binom{k}{\lfloor k/2 \rfloor}$. ■

The *Johnson graph* $J(r,s)$ is the simple graph whose vertices are the s-element subsets of an r-element set such that two vertices are adjacent if and only if their intersection has $s-1$ elements. In coding theory, the independence number of $J(r,s)$ is known as the maximum size, $A(r,4,s)$, of a set of binary words of length r and Hamming weight s such that the Hamming distance between every two distinct words is at least 4.

Theorem 8. *If k, n are integers such that $n \leqslant A(k,4,\lfloor k/2 \rfloor)$, then $\overrightarrow{\chi_i}(\overrightarrow{K}_n) \leqslant k$.*

Proof. If I_k is an independent set of $J(k,\lfloor k/2 \rfloor)$ and $n \leqslant |I_k|$, then we assign a set $S_i \in I_k$ to every vertex $v_i \in V(\overrightarrow{K}_n)$. For every arc $v_iv_j \in A(\overrightarrow{K}_n)$, we have that $|S_i \setminus S_j| \geqslant 2$. So we assign two distinct colours from $S_i \setminus S_j$ to (v_i, v_iv_j) and (v_j, v_iv_j). This way, the four incidences of two related arcs of $\overrightarrow{K}_n$ get distinct colours and we have an oriented incidence k-colouring of $\overrightarrow{K}_n$. ■

Theorems 7 and 8 imply the following bounds on $\overrightarrow{\chi_i}(\overrightarrow{K}_n)$.

Corollary 9. *If $n \geqslant 8$, then*

$$\log_2(n) + \frac{1}{2}\log_2(\log_2(n)) \leqslant \overrightarrow{\chi_i}(\overrightarrow{K}_n) \leqslant \log_2(n) + \frac{3}{2}\log_2(\log_2(n)) + 2.$$

Proof. Theorem 7 gives $\overrightarrow{\chi_i}(\overrightarrow{K}_n) \geqslant \log_2(n) + \frac{1}{2}\log_2(\log_2(n))$. By Theorem 8, a lower bound on $A(k,4,\lfloor k/2 \rfloor)$ provides an upper bound on $\overrightarrow{\chi_i}(\overrightarrow{K}_n)$. Graham and Sloane [9] obtained that $A(k,4,r) \geqslant \frac{\binom{k}{r}}{k}$, so we have $A(k,4,\lfloor k/2 \rfloor) \geqslant \frac{\binom{k}{\lfloor k/2 \rfloor}}{k}$. Using this and the better lower bounds on $A(k,4,\lfloor k/2 \rfloor)$ for $k \leqslant 30$ gathered at `win.tue.nl/ aeb/codes/Andw.html#d4`, we obtain $\overrightarrow{\chi_i}(\overrightarrow{K}_n) \leqslant \log_2(n) + \frac{3}{2}\log_2(\log_2(n)) + 2$. ■

This result confirms the computational results given in Table 3 for $n = 8$. For the case $n = 9$, this result implies $\overrightarrow{\chi_i}(\overrightarrow{K}_9) = 6$. Using Theorem 2, we obtain upper bounds for other digraphs.

Corollary 10. *If G is a digraph, then $\overrightarrow{\chi_i}(G) \leqslant (1 + o(1)) \log_2 (\chi(G))$.*

Proof. Since $G \to \overrightarrow{K}_{\chi(G)}$, by Theorem 2 we have $\overrightarrow{\chi_i}(G) \leqslant \overrightarrow{\chi_i}(\overrightarrow{K}_{\chi(G)})$. The result follows by observing

$$\frac{\frac{3}{2}\log_2 (\log_2(\chi(G))) + 2}{\log_2 (\chi(G))} \to 0 \text{ as } n \to \infty.$$

■

Corollary 11. *If T is a tournament on n vertices, then $\overrightarrow{\chi_i}(T) \leqslant (1+o(1)) \log_2(n)$.*

Proof. This follows similarly to the proof of Corollary 10 by noting that $T \to \overrightarrow{K}_n$ and applying Theorem 2. ■

4. Constructions and Decompositions

In this section, we consider oriented incidence colourings of digraph decompositions and products. We begin with upper bounds for digraphs that can be realized as the union of digraphs.

Proposition 12. *If G is a digraph such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \emptyset$, then*

$$\overrightarrow{\chi_i}(G) = \max\{\overrightarrow{\chi_i}(G_1), \overrightarrow{\chi_i}(G_2)\}.$$

Proposition 13. *If G is a digraph such that $G = G_1 \cup G_2$ and $V(G_1) = V(G_2)$, then*

$$\overrightarrow{\chi_i}(G) \leqslant \overrightarrow{\chi_i}(G_1) + \overrightarrow{\chi_i}(G_2).$$

Proof. Using disjoint sets of colours on G_1 and G_2 ensures that two incidences that do not belong to the same G_i have distinct colours. ■

We consider now a graph operation that arises in the study of oriented colourings and oriented cliques (for an example see [18, 4]). Let G and H be digraphs on disjoint vertex sets. We define the digraph $G \star H$ as follows.

$V(G \star H) = V(G) \cup V(H) \cup \{z\}$,

$A(G \star H) = A(G) \cup A(H) \cup \{uz \mid u \in V(G)\} \cup \{zv \mid v \in V(H)\}$.

Theorem 14. *Let G and H be digraphs such that $A(G) \cup A(H) \neq \emptyset$ and let $k = \max\{\overrightarrow{\chi_i}(G), \overrightarrow{\chi_i}(H)\}$. Then $k \leqslant \overrightarrow{\chi_i}(G \star H) \leqslant k + 2$.*

Proof. The condition $A(G) \cup A(H) \neq \emptyset$ implies $k \geqslant 2$. Thus $\{1, 2\} \cap \{k + 1, k + 2\} = \emptyset$. We construct an oriented incidence colouring c of $G \star H$ using the colours $\{1, 2, \dots, k + 2\}$ as follows.

- We colour the incidences of G using the colours $\{1, 2, \ldots, k\}$.
- We colour the incidences of H using the colours $\{3, 4, \ldots, k+2\}$.
- $c(u, uz) = k+1$ and $c(z, uz) = k+2$ for every $u \in V(G)$.
- $c(v, zv) = 1$ and $c(v, vz) = 2$ for every $v \in V(H)$.

■

The upper bound in Theorem 14 is not always achieved with equality.

For example, $\overrightarrow{\chi_i}(\overrightarrow{P_2}) = 2$ and $\overrightarrow{\chi_i}(\overrightarrow{P_2} \star \overrightarrow{P_2}) = 3$. We also have $\overrightarrow{\chi_i}(\overrightarrow{P_3}) = 3$ and $\overrightarrow{\chi_i}(\overrightarrow{P_3} \star \overrightarrow{P_3}) = 4$.

Finally we consider the oriented incidence chromatic number of the symmetric join of digraphs. Let G and H be digraphs. The *join* of G and H, denoted $G + H$, is the digraph with

$V(G+H) = V(G) \cup V(H)$,

$A(G+H) = A(G) \cup A(H) \cup \{u_G v_H \mid u_G \in V(G), v_H \in V(H)\}$

$\cup \{u_H v_G \mid u_H \in V(H), v_G \in V(G)\}$.

Informally, the join of digraphs is the disjoint union of the digraphs together with all possible arcs between vertices of different digraphs. We give a pair of bounds for the oriented incidence chromatic number of the join of digraphs.

Theorem 15. *If G_1, G_2, and G_3 are digraphs, then*

$$\overrightarrow{\chi_i}(G_1 + G_2 + G_3) \leqslant \max_{1 \leqslant j \leqslant 3} \{\overrightarrow{\chi_i}(G_j)\} + 4.$$

Proof. We split $A(G_1 + G_2 + G_3)$ into the set X of arcs in the disjoint union of G_1, G_2, and G_3 and the set $Y = A(G_1 + G_2 + G_3) \setminus X$. Using Proposition 12, we colour the incidences of X with $\max_{1 \leqslant j \leqslant 3} \{\overrightarrow{\chi_i}(G_j)\}$ colours. Since the graph induced by Y is 3-colourable, we can colour the incidences of Y with 4 colours by Proposition 5. Then we obtain

$$\overrightarrow{\chi_i}(G_1 + G_2 + G_3) \leqslant \max_{1 \leqslant j \leqslant 3} \{\overrightarrow{\chi_i}(G_j)\} + 4,$$

by Proposition 13. ■

In the case that each vertex of G has both an in-neighbour and an out-neighbour, this bound may be slightly improved.

Theorem 16. *If G_1 and G_2 are digraphs with minimum out-degree at least one and minimum in-degree at least one, then $\overrightarrow{\chi_i}(G_1 + G_2) \leqslant \overrightarrow{\chi_i}(G_1) + \overrightarrow{\chi_i}(G_2)$.*

Proof. We consider an oriented incidence colouring c of the disjoint union G_1 and G_2 using $\overrightarrow{\chi_i}(G_1) + \overrightarrow{\chi_i}(G_2)$ colours and such that no colour appears both in G_1 and G_2. Then we extend c to the uncoloured arcs of $G_1 + G_2$ as follows. For every arc uv such that $u \in V(G_j)$ and $v \in V(G_{3-j})$, we set $c(u, uv) = c(u, uw)$ and $c(v, uv) = c(w, uw)$, where $uw \in A(G_j)$. ■

5. Homomorphisms and Complexity

In her Masters thesis [23] (more recently published as [14]), Sherk explores the relationship between oriented graph homomorphism and 2-dipath colouring. One of the main results of this work is to define a family of graphs, G_k $(k \geq 1)$, with the property that an oriented graph H has a 2-dipath colouring using k colours if and only if H admits a homomorphism to G_k. Here we consider the possibility of a similarly-styled result for the oriented incidence chromatic number. For the case $\overrightarrow{\chi_i}(G) = 2$ a fairly straightforward characterisation exists.

Theorem 17. *Let G be a digraph with at least one arc, then $\overrightarrow{\chi_i}(G) = 2$ if and only if G admits a homomorphism to $\overrightarrow{P_2}$.*

To construct a homomorphism model we utilize Theorem 6. Consider the directed graph, H_k with vertex set

$$V(H_k) = \{(X, Y) |\, X, Y \in \mathcal{P}(\{1, 2, 3, \ldots, k\}), X \cap Y = \emptyset\}.$$

Using the characterisation of oriented incidence colouring given in Theorem 6 we construct $A(H_k)$. For $(X_u, Y_u) \neq (X_v, Y_v) \in V(H_K)$, $(X_u, Y_u)(X_v, Y_v) \in A(H_k)$ provided the following conditions are met:

1. $X_u \setminus Y_v \neq \emptyset$;
2. $Y_v \setminus X_u \neq \emptyset$;
3. if $X_u \setminus X_v = Y_v \setminus Y_u$, then $|X_u \setminus X_v| \neq 1$.

It follows directly from Theorem 6 that $\overrightarrow{\chi_i}(H_k) = k$.

Theorem 18. *A digraph G has $\overrightarrow{\chi_i}(G) \leqslant k$ if and only if $G \to H_k$.*

Proof. Let G be a digraph such that $G \to H_k$. Since $\overrightarrow{\chi_i}(H_k) = k$, by Theorem 2 $\overrightarrow{\chi_i}(G) \leqslant k$. Assume now that $\overrightarrow{\chi_i}(G) \leqslant k$. The mapping $\phi: V(G) \to V(H_k)$ given by $\phi(u) = (A_u, B_u)$ is a homomorphism. ■

For the case $k = 3$ the core of H_3 is the tournament, T_5, given in Figure 3. From this we arrive at the following result.

Proposition 19. *If G is an oriented graph with $\overrightarrow{\chi_i}(G) \leqslant 3$, then $\chi_o(G) \leqslant 5$.*

This bound is tight since $\overrightarrow{\chi_i}(T_5) = 3$ and $\chi_o(T_5) = 5$.

Proposition 19 bounds χ_o for oriented graphs such that $\overrightarrow{\chi_i} \leqslant 3$. However, no such bound for χ_o can be obtained in this way for any case $k > 3$, as H_k, in these cases, is a digraph that is not an orientation. We further observe that the class of orientations of bipartite graphs has oriented incidence chromatic number four,

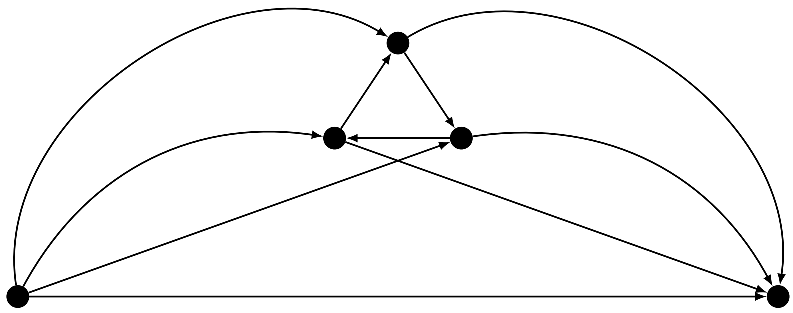

Figure 3. The tournament T_5.

but unbounded oriented chromatic number and conclude that no such bound can exist.

Using the homomorphism model of oriented incidence colouring, we can discuss the complexity of oriented incidence colouring. Let H be a fixed directed graph. We use Hom_H to denote the problem of deciding whether a given digraph D has a homomorphism to H. If G and H are homomorphically equivalent, then the complexity of Hom_G is the same as the complexity of Hom_H.

We will make use of the following theorems.

Theorem 20 [2]**.** *Let T be a semicomplete digraph. If T has at most one directed cycle, then Hom_T is Polynomial. If T has at least two directed cycles, then Hom_T is NP-complete.*

Theorem 21 [3]**.** *Let H be a directed graph in which each vertex has positive in-degree and positive out-degree. If every component of the core of H is a cycle, then Hom_H is Polynomial. Otherwise, Hom_H is NP-complete.*

Let S be a fixed directed graph with specified vertex s. The *subindicator construction* with respect to (S, s) transforms a given directed graph H into its subgraph H' induced by the set of all vertices x such that there is a homomorphism $f\colon S \to H$ with $f(s) = x$.

Theorem 22 [10], also see [2]**.** *Let H be a fixed core and H' be the result of applying the subindicator construction with respect to (S, s) to H. If $Hom_{H'}$ is NP-complete, then so is Hom_H.*

Let k be a fixed integer. We use OIC_k to denote the problem of deciding whether a given directed graph D has an oriented incidence k-colouring. Let H_k be the homomorphism model for oriented incidence k-colouring. Note that H_k has no loops.

Theorem 23. *The problems OIC_2 and OIC_3 are Polynomial. For all fixed integers $k \geqslant 4$, the problem OIC_k is NP-complete.*

Proof. The first statement follows from Theorem 20 as each of $\overrightarrow{P_2}$ and T_5 have at most one directed cycle.

Let $k \geqslant 4$. We show that Hom_{H_k} is NP-complete.

Let M be the subgraph of H_k induced by the pairs (A, B) belonging to $\{(\{1\},\{2\}),(\{2\},\{3\}),(\{3\},\{4\})\}$.

By definition of H_k, the digraph M consists of the directed 3-cycle $(\{1\},\{2\})$, $(\{2\},\{3\}),(\{3\},\{4\}),(\{1\},\{2\})$, and the arc $(\{1\},\{2\})(\{3\},\{4\})$.

Let G_k be the core of H_k. Then there is a homomorphism of M to G_k. Since G_k has no loops, it follows that G_k has a subgraph isomorphic to M.

Let G'_k be the result of applying the subindicator construction with respect to (P_3, s) to G_k, where s is the middle vertex of the directed 3-path P_3. Then every vertex of G'_k has positive in-degree and positive out-degree. Further, G'_k has a subdigraph isomorphic to M. By Theorem 21 we have that $\mathrm{Hom}_{G'_k}$ is NP-complete. Consequently, using Theorem 22, Hom_{G_k} and Hom_{H_k} are both NP-complete. ∎

Using our homomorphism models it is possible to determine the configurations which prevent a directed graph from having an oriented incidence 2-colouring, or an oriented incidence 3-colouring. Since the core of H_2 is a transitive tournament on 2 vertices, we know that a directed graph D has a homomorphism to H_2 if and only if there is no homomorphism of P_3, the directed path on 3 vertices, to D [17]. It follows that the minimal digraphs that do not have an oriented incidence 2-colouring are P_3 and C_2. The core of H_3 is a unicyclic tournament T_5 on 5 vertices with one vertex of in-degree 0 and one vertex of out-degree 0 (see Figure 3). It is shown in [15] that a directed graph D has a homomorphism to T_5 if and only if there is no digraph F in the family $\mathcal{F}$ described below to D. Let $\mathcal{U}_{\overline{3}}$ be the collection of oriented cycles such that the number of forwards arcs minus the number of backwards arcs is not a multiple of 3. To each oriented cycle $U \in \mathcal{U}_{\overline{3}}$ there corresponds a digraph $F \in \mathcal{F}$ which is obtained from U in two steps:

(i) for each vertex u of U with out-degree 0, add a new vertex u' and the arc uu'; and

(ii) for each vertex v of U with in-degree 0, add a new vertex v' and the arc $v'v$.

Since a homomorphic image of an element of $\mathcal{U}_{\overline{3}}$ contains another element of $\mathcal{U}_{\overline{3}}$ as an induced subgraph, it follows that the minimal digraphs which do not have an oriented incidence 3-colouring are the homomorphic images of elements $F \in \mathcal{F}$ in which all vertices of the oriented cycle $U \in \mathcal{U}_{\overline{3}}$ from which F is constructed have different images.

6. Planar Graphs and Outerplanar Graphs

Using results from the previous section we derive tight bounds for the classes of oriented outerplanar graphs and oriented planar graphs, respectively. By observing that every orientation of a planar graph admits a homomorphism to $\overrightarrow{K}_4$ and by applying Proposition 5, we have directly that the oriented incidence chromatic number of the class of oriented planar graphs is at most 5. Similarly, as every orientation of an outerplanar graph admits a homomorphism to $\overrightarrow{K}_3$, we have directly that the oriented incidence chromatic number of the class of oriented outerplanar planar graphs is at most 4. We show that both of these bounds are tight.

Consider the oriented graph L given in Figure 4. Since L contains a directed cycle of length 4, we have that $\overrightarrow{\chi_i}(L) \geqslant 4$. Moreover, Figure 4 gives an oriented incidence 4-colouring of L. We thus have.

Property 1. $\overrightarrow{\chi_i}(L) = 4$.

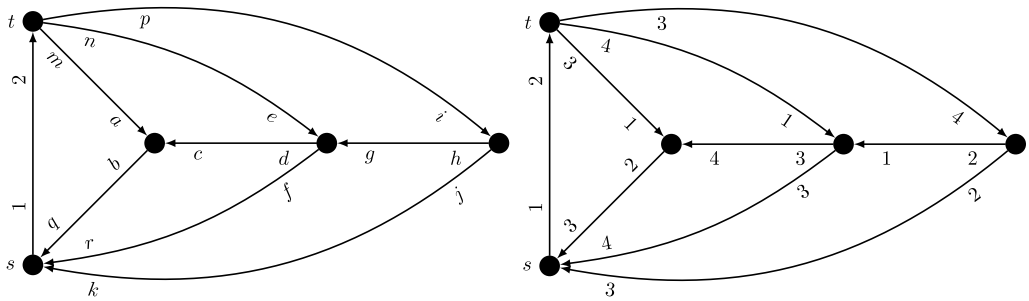

Figure 4. An oriented incidence 4-colouring of L.

Property 2. *If c is an oriented incidence 4-colouring of L, then it is not the case that $c(q) = c(r) = c(k) = c(m) = c(n) = c(p)$.*

Proof. Consider the partial colouring of L given in Figure 5. Colouring the remaining incidences is equivalent to finding a list-colouring for the graph in Figure 5. We show that no such list colouring exists.

If e is assigned 1, then the triangle b, c, d cannot be coloured. Therefore e is assigned 4. This implies f is assigned 2. Since e is assigned 4, the vertices c and d are assigned 1 and 2, in some order. Therefore b is assigned 4 and a is assigned 1. Since f is assigned 2, the vertices g and h are assigned 1 and 4, in some order. But g is adjacent to c and d, so g can not be assigned 1. Therefore g is assigned 4 and h is assigned 1. There is now no colour available for i. ■

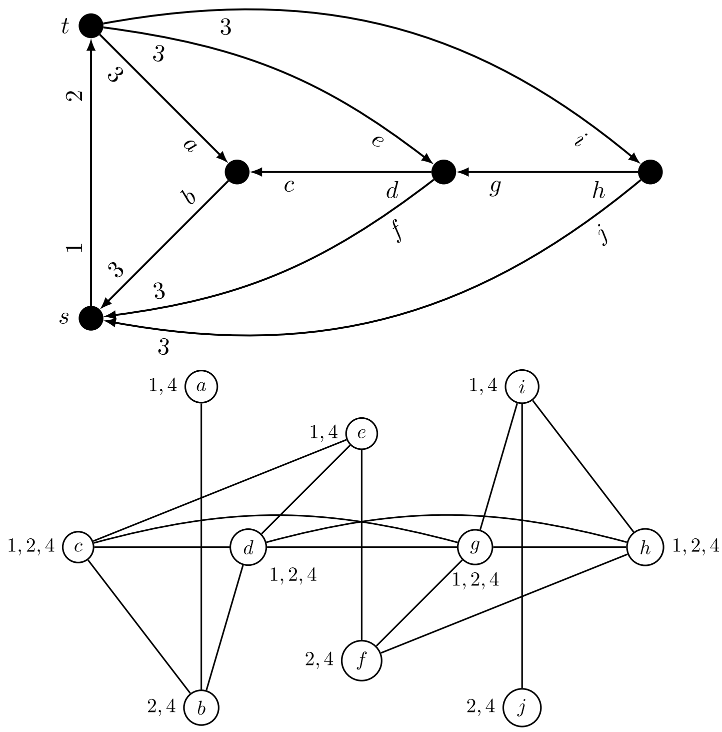

Figure 5. A partial oriented incidence colouring of L and a corresponding list colouring.

Lemma 24. *If G is a digraph with $\overrightarrow{\chi_i}(G) = 5$ that contains a 2-cycle, uv, then the digraph obtained by removing the arcs of the 2-cycle and identifying u with s and v with t in a copy of L has oriented incidence chromatic number at least 5.*

Proof. Let uv be a 2-cycle in G and let G_{uv} be the digraph obtained by removing the arcs of the 2-cycle and identifying u with s and v with t in a copy of L. Assume, for a contradiction, that $\overrightarrow{\chi_i}(G_{uv}) < 5$. Since G_{uv} contains L and $\overrightarrow{\chi_i}(L) = 4$, it must be that $\overrightarrow{\chi_i}(G_{uv}) = 4$. Let c be an oriented incidence 4-colouring of G_{uv}. Without loss of generality, assume that $c(s, st) = 1$, and $c(t, st) = 2$. By Property 2 we may assume that at least one of incidences q, r, or k receives colour 4 and at least one of incidences m, n, or p receives colour 3.

Consider an arc having its tail at s in G_{uv}. It cannot be that 4 is assigned to an incidence of such an arc. Similarly, it cannot be that 1 is assigned to an incidence of an arc having its head at s and 2 cannot be assigned to a head incidence of such an arc. Further, 3 cannot appear on an incidence of any arc having its tail at t and 2 cannot appear on an incidence of any arc having its head at t and 1 cannot be assigned to a tail incidence of such an arc. Using these facts we can construct an oriented incidence 4-colouring of G, as follows

$$c'(a) = \begin{cases} c(a), & a \in V(\mathcal{I}_G) \cap V(\mathcal{I}_{G_{uv}}), \\ 1, & a = (u, uv), \\ 2, & a = (v, uv), \\ 3, & a = (v, uv), \\ 4, & a = (u, uv). \end{cases}$$

This contradicts that $\overrightarrow{\chi_i}(G) = 5$. Therefore $\overrightarrow{\chi_i}(G_{uv}) \geqslant 5$. ■

Theorem 25. *The class of oriented planar graphs has oriented incidence chromatic number* 5.

Proof. Consider the oriented graph formed by replacing each 2-cycle of $\overrightarrow{K}_4$ with a copy of L, as in Lemma 24. The resulting oriented graph is planar. By Table 1 and Lemma 24, this oriented planar graph has oriented incidence chromatic number at least 5. Proposition 5 implies directly that the class of oriented planar graphs has oriented incidence chromatic number at most 5, as every orientation of a planar graph admits a homomorphism to $\overrightarrow{K}_4$ [1]. Therefore the class of oriented planar graphs has oriented incidence chromatic number 5. ■

Theorem 26. *The class of oriented outerplanar graphs has oriented incidence chromatic number* 4.

Proof. Every oriented cycle of length $\not\equiv 0 \pmod 3$ has oriented incidence chromatic number at least four. Every simple outerplanar graph has chromatic number at most 3, and so every oriented outerplanar graph has oriented incidence chromatic number at most 4 by Proposition 5. ■

7. Conclusions and Future Directions

The definition of oriented colouring enforces that if there is an arc with its tail coloured i and its head coloured j, then there is no arc with its tail coloured j and its head coloured i [7]. To enforce this constraint with respect to the colours of the incidences would not drastically change the analysis given above. This extra constraint would increase the oriented incidence chromatic number, but the methods used above may still be utilized. By considering a colour on an in-incidence to be a distinct colour from the identical colour on an out-incidence, our upper bounds are all increased by only a factor of two.

Some of the ideas in this paper can be extended for use in incidence colouring of simple graphs. The ideas contained in Theorem 6 can be used to give a set-based definition of incidence colouring of simple graphs. Essentially, we can consider the set of colours appearing at each vertex to be a set of distinct representatives of a collection of sets formed by considering the colours available at each incidence. Further using homomorphisms to construct incidence colourings is possible, provided that the homomorphisms are injective. If a simple graph G admits an injective homomorphism to a simple graph H, then any incidence colouring of H can be lifted back to be an incidence colouring of G.

As strong edge colouring was the genesis for incidence colouring, it would be reasonable to consider a definition of strong arc colouring of digraphs. By considering a strong arc colouring of a digraph to be an arc colouring in which a colour class does not induce a 2-dipath, we arrive at the same relationship between oriented incidence colouring and strong arc colouring as exists between incidence colouring and strong edge colouring. Though this type of colouring is unstudied, it seems as if oriented incidence colouring would be useful in its study.

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Appendix

Colour Class	**Vertex List**
1	$(1,(7,1)),(3,(3,2)),(2,(4,2)),(3,(4,3)),(3,(3,5)),$ $(5,(4,5)),(5,(6,5)),(3,(3,7)),(2,(7,2)),(5,(7,5)),$ $(6,(7,6)),(1,(6,1)),(6,(6,4)),(6,(6,2))$
2	$(6,(1,6)),(7,(1,7)),(2,(3,2)),(4,(3,4)),(4,(4,1)),$ $(4,(4,2)),(2,(5,2)),(4,(4,5)),(6,(5,6)),(4,(4,7)),$ $(7,(3,7)),(7,(5,7)),(6,(6,7)),(6,(6,3))$
3	$(1,(1,2)),(1,(1,3)),(1,(1,4)),(1,(1,5)),(1,(1,6)),$ $(1,(1,7)),(3,(2,3)),(4,(2,4)),(5,(2,5)),(6,(2,6)),$ $(7,(2,7))$
4	$(1,(2,1)),(1,(3,1)),(1,(4,1)),(5,(3,5)),(5,(5,4)),$ $(5,(5,1)),(5,(5,2)),(7,(4,7)),(7,(7,2)),(7,(7,3)),$ $(6,(3,6)),(5,(5,6)),(6,(4,6)),(7,(7,6)),(7,(7,1))$
5	$(2,(2,1)),(1,(5,1)),(1,(6,1)),(1,(3,1)),(2,(2,4)),$ $(2,(2,5)),(3,(3,4)),(3,(5,3)),(4,(5,4)),(3,(3,6)),$ $(2,(6,2)),(7,(7,4)),(7,(7,5)),(4,(6,4)),(7,(6,7))$
6	$(2,(1,2)),(3,(1,3)),(4,(1,4)),(5,(1,5)),(2,(2,3)),$ $(4,(4,3)),(2,(2,6)),(2,(2,7)),(5,(5,3)),(3,(6,3)),$ $(3,(7,3)),(4,(7,4)),(5,(5,7)),(4,(4,6)),(5,(6,5))$

Table 2. An oriented incidence colouring $\overrightarrow{K}_7$ with six colours.

Colour Class	**Vertex List**
1	$(6,(6,8)),(1,(2,1)),(3,(3,1)),(3,(3,2)),(2,(2,4)),(3,(3,4)),(4,(5,4)),(6,(2,6)),(6,(6,1)),(6,(6,4)),(6,(6,5)),(4,(7,4)),(8,(5,8)),(3,(5,3)),(1,(7,1)),(6,(7,6)),(3,(7,3)),(1,(5,1))$
2	$(8,(8,5)),(8,(8,6)),(2,(2,1)),(2,(2,3)),(8,(3,8)),(1,(3,1)),(4,(4,1)),(4,(3,4)),(4,(4,5)),(4,(4,7)),(6,(3,6)),(2,(2,6)),(4,(4,6)),(6,(5,6)),(2,(5,2)),(7,(5,7)),(2,(2,7))$
3	$(1,(8,1)),(2,(8,2)),(3,(8,3)),(4,(8,4)),(5,(8,5)),(6,(8,6)),(7,(8,7)),(1,(1,2)),(1,(1,3)),(1,(1,4)),(1,(1,5)),(1,(1,6)),(7,(7,2)),(7,(7,4)),(7,(7,5)),(7,(7,6)),(7,(7,3))$
4	$(1,(1,8)),(1,(1,7)),(2,(2,8)),(3,(3,8)),(2,(2,4)),(3,(4,3)),(8,(4,8)),(1,(4,1)),(3,(3,6)),(1,(6,1)),(2,(6,2)),(5,(7,5)),(2,(2,5)),(3,(3,5)),(5,(4,5)),(5,(6,5)),(2,(7,2)),(3,(3,7)),(8,(6,8)),(8,(7,8))$
5	$(8,(1,8)),(4,(4,8)),(5,(5,8)),(7,(7,8)),(8,(2,8)),(3,(1,3)),(4,(1,4)),(3,(2,3)),(4,(4,2)),(4,(4,3)),(5,(2,5)),(5,(5,1)),(3,(6,3)),(4,(6,4)),(5,(5,3)),(7,(2,7)),(7,(7,1)),(5,(5,6)),(7,(6,7))$
6	$(8,(8,1)),(8,(8,2)),(8,(8,3)),(8,(8,4)),(8,(8,7)),(2,(1,2)),(2,(3,2)),(5,(1,5)),(6,(1,6)),(7,(1,7)),(2,(4,2)),(5,(5,2)),(6,(6,2)),(6,(6,3)),(6,(4,6)),(5,(3,5)),(7,(4,7)),(5,(5,4)),(5,(5,7)),(6,(6,7)),(7,(3,7))$

Table 3. An oriented incidence colouring $\overrightarrow{K}_8$ with six colours.

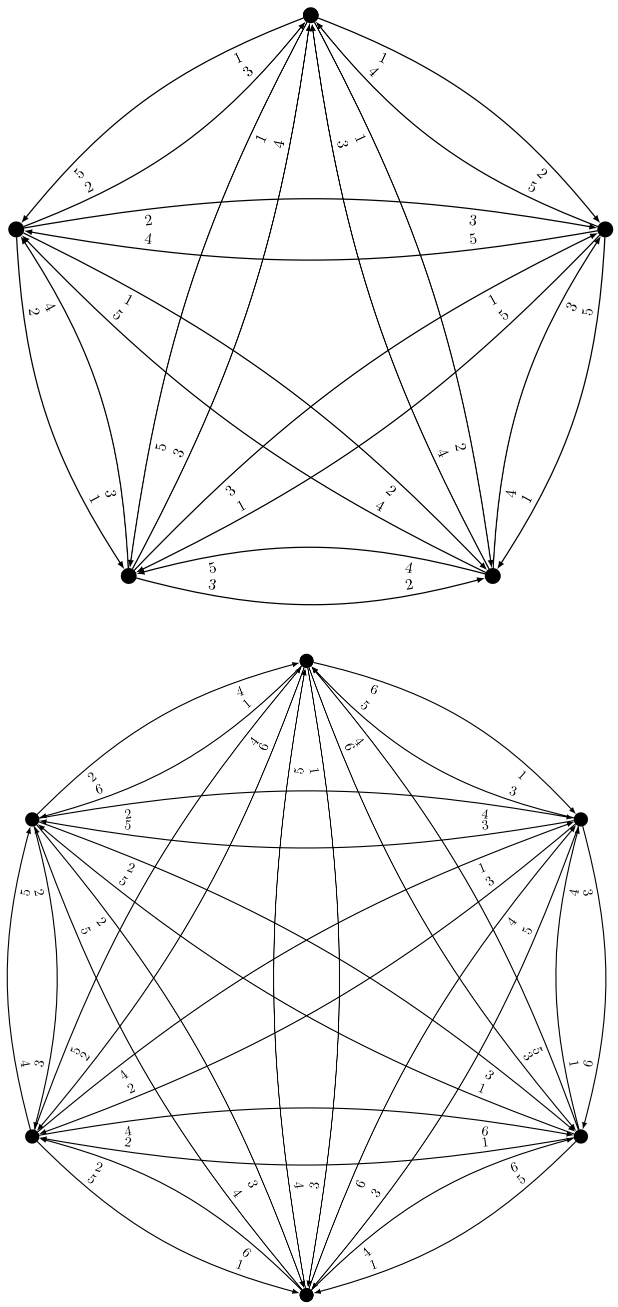

Figure 6. Oriented incidence colourings of $\overrightarrow{K}_4$,$\overrightarrow{K}_5$, $\overrightarrow{K}_6$ with the minimum number of colours. The colouring of $\overrightarrow{K}_4$ is obtained by deleting any vertex in the colouring of $\overrightarrow{K}_5$.