Discussiones Mathematicae
Graph Theory 39 (2019) 183–190
doi:10.7151/dmgt.2075

INDEPENDENCE NUMBER, CONNECTIVITY AND ALL FRACTIONAL (a, b, k)-CRITICAL GRAPHS

Yuan Yuan and Rong-Xia Hao[1]

Department of Mathematics
Beijing Jiaotong University
Beijing 100044, China

e-mail: kuailenanshi@126.com
rxhao@bjtu.edu.cn

Abstract

Let G be a graph and a, b and k be nonnegative integers with $1 \leq a \leq b$. A graph G is defined as *all fractional* (a, b, k)*-critical* if after deleting any k vertices of G, the remaining graph has all fractional $[a, b]$-factors. In this paper, we prove that if $\kappa(G) \geq \max\left\{\frac{(b+1)^2+2k}{2}, \frac{(b+1)^2\alpha(G)+4ak}{4a}\right\}$, then G is all fractional (a, b, k)-critical. If $k = 0$, we improve the result given in [Filomat 29 (2015) 757–761]. Moreover, we show that this result is best possible in some sense.

Keywords: independence number, connectivity, fractional $[a, b]$-factor, fractional (a, b, k)-critical graph, all fractional (a, b, k)-critical graph.

2010 Mathematics Subject Classification: 05C70, 05C72.

1. Introduction

All graphs considered here are finite, simple and undirected graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $x \in V(G)$, we use $d_G(x)$ and $N_G(x)$ to denote the degree and neighbourhood of x in G, respectively. For any $S \subseteq V(G)$, let $N_G(S)$ denote the union of $N_G(x)$ for each $x \in S$. We use $G[S]$ and $G - S$ to denote the subgraph of G induced by S and $V(G) - S$. A subset I of $V(G)$ is an *independent set* of G, if no two distinct vertices in I are adjacent. The cardinality of a maximum independent set in a graph G is called the *independence number* of G, denoted by $\alpha(G)$. A *vertex-cut* of a noncomplete

[1]Corresponding author.

graph G is a set of vertices of G such that $G-S$ is disconnected. A vertex-cut of minimum cardinality in G is called a *minimum vertex-cut* of G and this cardinality is called the *connectivity* of G and is denoted by $\kappa(G)$.

Let g, f be two integer-valued functions defined on $V(G)$ with $0 \leq g(x) \leq f(x)$ for all $x \in V(G)$. A *(g,f)-factor* of G is a spanning subgraph H of G satisfying $g(x) \leq d_H(x) \leq f(x)$ for all $x \in V(G)$. Let $a \leq b$ be two integers. A (g,f)-factor is called an *$[a,b]$-factor* if $g(x) \equiv a$ and $f(x) \equiv b$. Let $h : E(G) \to [0,1]$ be a function. If $g(x) \leq \sum_{x \in e} h(e) \leq f(x)$ holds for every $x \in V(G)$, then we call graph F with vertex set $V(G)$ and edge set E_h a *fractional (g,f)-factor* of G with indicator function h, where $E_h = \{e \in E(G) | h(e) > 0\}$. If $f(x) = g(x)$ for all $x \in V(G)$, then a fractional (g,f)-factor is called a *fractional f-factor.* If $g(x) \equiv a$ and $f(x) \equiv b$, then a fractional (g,f)-factor is called a *fractional $[a,b]$-factor.* Let p be an integer-valued function defined on $V(G)$ such that $g(x) \leq p(x) \leq f(x)$ for each $x \in V(G)$. We say that G has *all fractional (g,f)-factors* if G has a fractional p-factor for every p described above. If $g(x) \equiv a$ and $f(x) \equiv b$, then all fractional (g,f)-factors are said to be *all fractional $[a,b]$-factors.* A graph G is called an *all fractional (a,b,k)-critical* graph if after deleting any k vertices of G the remaining graph of G has all fractional $[a,b]$-factors.

Many authors have studied factors and fractional factors of graphs. For example, see [1, 3, 4, 5, 6, 7, 8, 9, 10, 13, 14]. Anstee [1] and Lu [6] gave necessary and sufficient conditions for a graph to have all fractional (g,f)-factors and all fractional $[a,b]$-factors, respectively. Liu *et al.* [5] proved the necessary and sufficient conditions for a graph to have a fractional (g,f)-factor. The following theorem, on the existence of fractional (g,f)-factors of graphs, is well known.

Theorem 1 [2]**.** *Let G be a graph, and let a, b and r be three nonnegative integers satisfying $1 \leq a \leq b-r$, and let g, f be two integer-valued functions defined on $V(G)$ with $a \leq g(x) \leq f(x) - r \leq b - r$ for every $x \in V(G)$. If*

$$\kappa(G) \geq \max\left\{\frac{(b+1)(b-r+1)}{2}, \frac{(b-r+1)^2\alpha(G)}{4(a+r)}\right\},$$

then G contains a fractional (g,f)-factor.

As far as we know, except a sufficient condition for graphs to be all fractional (a,b,k)-critical in terms of binding number $bind(G)$ in [11], there are few results for graphs to be all fractional (a,b,k)-critical. This is a motivation of this paper.

In this paper we use independent number and connectivity to obtain a new sufficient condition for a graph to be all fractional (a,b,k)-critical. The following theorem is the main result.

Theorem 2. *Let G be a graph and let a, b, k be nonnegative integers with $1 \leq a < b$. If $\kappa(G) \geq \max\left\{\frac{(b+1)^2+2k}{2}, \frac{(b+1)^2\alpha(G)+4ak}{4a}\right\}$, then G is all fractional (a,b,k)-critical.*

If $k = 0$ in Theorem 2, we can get the following corollary.

Corollary 3. *Let G be a graph and a, b nonnegative integers with $1 \leq a < b$. If $\kappa(G) \geq \max\left\{\frac{(b+1)^2}{2}, \frac{(b+1)^2\alpha(G)}{4a}\right\}$, then G has all fractional $[a, b]$-factors.*

2. The Proof of Theorem 2

Lemma 4 [12]**.** *Let a, b and k be nonnegative integers with $1 \leq a \leq b$, and let G be a graph of order n with $n \geq a+k+1$. Then G is all fractional (a, b, k)-critical if and only if for any $S \subseteq V(G)$ with $|S| \geq k$*

$$a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| \geq ak,$$

where $T = \{x : x \in V(G) \backslash S, d_{G-S}(x) < b\}$.

***Proof of Theorem* 2.** Let G be a graph satisfying the hypothesis of Theorem 2. We prove the theorem by contradiction. Suppose that G is not all fractional (a, b, k)-critical. Then by Lemma 4, there exists a subset S of $V(G)$ with $|S| \geq k$ such that

$$a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| < ak, \tag{1}$$

where $T = \{x : x \in V(G) \backslash S, d_{G-S}(x) < b\}$. Obviously, $T \neq \emptyset$. Otherwise,

$$a|S| + \sum_{x \in T} d_{G-S}(x) - b|T| = a|S| \geq ak,$$

contradicting to (1).

Now we consider the subgraph $G[T]$ of G induced by T. Set $T_1 = G[T]$. Choose $x_1 \in T_1$ with $d_{T_1}(x_1) = \delta(T_1)$ and $L_1 = N_{T_1}[x_1]$. Furthermore, for $i \geq 2$, choose $x_i \in T_i = T_1 - \bigcup_{1 \leq j < i} L_j$ with $d_{T_i}(x_i) = \delta(T_i)$ and $L_i = N_{T_i}[x_i]$. Set $|L_i| = d_i$. We continue these procedures until we reach the situation in which $T_i = \emptyset$ for some i, say for $i = r + 1$. Following the above definition we know that $\{x_1, x_2, \ldots, x_r\}$ is an independent set of G. Obviously, $r \geq 1$ and $|T| = \sum_{1 \leq i \leq r} d_i$. Let $U = V(G) \setminus (S \cup T)$ and $\kappa(G - S) = t$.

Now, we prove the following claims.

Claim 1. $r > 1$ or $U \neq \emptyset$.

Otherwise, we get $r = 1$ and $U = \emptyset$.

First, we prove an inequality $\frac{(a+b+1)^2}{4a} \leq \frac{(b+1)^2}{2}$, which is used later. In fact, this inequality is equivalent to $2(a + b + 1)^2 - 4a(b + 1)^2 \leq 0$. Now, let

$f(a) = 2(a+b+1)^2 - 4a(b+1)^2$, and so

$$\begin{aligned} f(a) &= 12(a^2+b^2+2a+2b+2ab+1) - 4a(b^2+2b+1) \\ &= 2a^2+2b^2+4a+4b+4ab+2-4ab^2-8ab-4a \\ &= 2a^2+2b^2+4b-4ab+2-4ab^2. \end{aligned}$$

By differential, we get $f'(a) = 4a-4b-4b^2 < 0$. So $f(a)$ is decreasing in $2 \le a \le b$ and we obtain

$$\begin{aligned} f(a) \le f(2) &= 2(3+b)^2 - 8(b+1)^2 = 2(9+b^2+6b) - 8(b^2+1+2b) \\ &= 18+2b^2+12b-8b^2-8-16b = 10-6b^2-4b \\ &= -2(3b^2+2b-5) = -2(b-1)(3b+5) < 0, \end{aligned}$$

which gives a proof of $\frac{(a+b+1)^2}{4a} \le \frac{(b+1)^2}{2}$.

By (1), we have

$$ak > a|S| + \sum_{x\in T} d_{G-S}(x) - b|T| = a|S| + d_1(d_1-1) - bd_1,$$

so $|S| < \frac{-d_1^2+d_1+bd_1+ak}{a}$. Then,

$$\begin{aligned} |V(G)| = |S| + d_1 &< \frac{-d_1^2+d_1+bd_1+ak}{a} + d_1 = \frac{-d_1^2+d_1+bd_1+ad_1}{a} + k \\ &= \frac{-d_1^2+(a+b+1)d_1}{a} + k \le \frac{(a+b+1)^2}{4a} + k \le \frac{(b+1)^2}{2} + k, \end{aligned}$$

which contradicts the assumption that $|V(G)| > \kappa \ge \frac{(b+1)^2+2k}{2}$. This completes the proof of Claim 1.

Claim 2. $\sum_{x\in T} d_{G-S}(x) \ge \sum_{1\le i\le r}(d_i^2 - d_i) + \frac{rt}{2}$.

In fact, by the choice of x_i, we know that every vertex in L_i has degree at least $d_i - 1$ in T_i, which implies that $\sum_{1\le i\le r}(\sum_{x\in L_i} d_{T_i}(x)) \ge \sum_{1\le i\le r} d_i(d_i-1)$.

Because an edge joining $x \in L_i$ and $y \in L_j$ $(i<j)$ is counted only once, we obtain that

$$\sum_{x\in T} d_{G-S}(x) \ge \sum_{1\le i\le r}(d_i^2-d_i) + \sum_{1\le i<j\le r} e_G(L_i,L_j) + e_G(T,U). \tag{2}$$

For each $L_i (1 \le i \le r)$, by $\kappa(G-S) = t$, we have

$$e_G(L_i, \bigcup_{j\ne i} L_j) + e_G(L_i, U) \ge t. \tag{3}$$

Summing up these inequalities for all i $(1 \leq i \leq r)$, we get

$$\sum_{1\leq i\leq r}\Big(e_G(L_i, \bigcup_{j\neq i} L_j) + e_G(L_i, U)\Big) = 2\sum_{1\leq i<j\leq r} e_G(L_i, L_j) + e_G(T, U) \geq rt. \tag{4}$$

According to (4), it is obvious that

$$\sum_{1\leq i<j\leq r} e_G(L_i, L_j) + e_G(T, U) \geq \frac{rt}{2}. \tag{5}$$

In terms of (2) and (5), we have

$$\sum_{x\in T} d_{G-S}(x) \geq \sum_{1\leq i\leq r}(d_i^2 - d_i) + \frac{rt}{2}. \tag{6}$$

This completes the proof of Claim 2.

Now we continue to prove the main theorem. Combining (1) and (6), obtain

$$\begin{aligned} ak &> a|S| + \sum_{x\in T} d_{G-S}(x) - b|T| \geq a|S| + \sum_{1\leq i\leq r}(d_i^2 - d_i) + \frac{rt}{2} - b\sum_{1\leq i\leq r} d_i \\ &= a|S| + \sum_{1\leq i\leq r}(d_i^2 - (b+1)d_i) + \frac{rt}{2} \geq a|S| - \frac{(b+1)^2 r}{4} + \frac{rt}{2}, \end{aligned}$$

which implies that

$$ak > a|S| - \frac{(b+1)^2 r}{4} + \frac{rt}{2}. \tag{7}$$

Since $|S| \geq k$, from (7) we get that $-\frac{(b+1)^2 r}{4} + \frac{rt}{2} < 0$, which implies that

$$-\frac{(b+1)^2}{4} + \frac{t}{2} < 0. \tag{8}$$

By (7), (8), $\alpha(G) \geq \alpha(G[T]) \geq r$ and the assumption

$$\kappa(G) \geq \max\left\{\frac{(b+1)^2+2k}{2}, \frac{(b+1)^2\alpha(G)+4ak}{4a}\right\},$$

we get

$$\begin{aligned} ak &> a|S| - \frac{(b+1)^2 r}{4} + \frac{rt}{2} \geq a\left(\kappa(G) - t\right) - \frac{(b+1)^2}{4}\alpha(G) + \frac{t}{2}\alpha(G) \\ &\geq a\left(\kappa(G) - t\right) - \frac{(b+1)^2}{4}\frac{4a\kappa(G) - 4ak}{(b+1)^2} + \frac{t}{2}\frac{4a\kappa(G) - 4ak}{(b+1)^2} \\ &= at\Big(\frac{2\kappa(G) - 2k}{(b+1)^2} - 1\Big) + ak \geq at\Big(\frac{(b+1)^2 + 2k - 2k}{(b+1)^2} - 1\Big) + ak = ak, \end{aligned}$$

which is a contradiction. Therefore, G is all fractional (a, b, k)-critical. ■

3. Remarks

Remark 1. Let us know that the condition $\kappa(G) \geq \frac{(b+1)^2+2k}{2}$ cannot be replaced by $\frac{(b+1)^2+2k}{2} - 1$. In fact, let $1 \leq a < b$ and $k \geq 0$ be three integers, and let $G = K_{\frac{(b+1)^2+2k}{2}-1} \vee \frac{a\left((b+1)^2-2\right)+2}{2b} K_1$. Let $S = K_{\frac{(b+1)^2+2k}{2}-1}$ and $T = \frac{a\left((b+1)^2-2\right)+2}{2b} K_1$. Obviously, $\kappa(G) = \frac{(b+1)^2+2k}{2} - 1 > k$, $|S| = \frac{(b+1)^2+2k}{2} - 1$, $|T| = \frac{a\left((b+1)^2-2\right)+2}{2b}$. So,

$$\begin{aligned} a|S| + d_{G-S}(T) - b|T| &= a\left(\frac{(b+1)^2+2k}{2} - 1\right) - b\frac{a\left((b+1)^2-2\right)+2}{2b} \\ &= a\frac{(b+1)^2}{2} + ak - a - b\frac{a\left((b+1)^2-2\right)+2}{2b} \\ &= ak - 1 < ak, \end{aligned}$$

a contradiction to Lemma 4, which implies that G is not all fractional (a, b, k)-critical.

Remark 2. The condition $\kappa(G) \geq \frac{(b+1)^2\alpha(G)+4ak}{4a}$ is equivalent to $a\kappa(G) \geq \frac{(b+1)^2\alpha(G)}{4} + ak$. Now we show that the condition $a\kappa(G) \geq \frac{(b+1)^2\alpha(G)}{4} + ak$ is best possible in the following sense. We cannot replace $a\kappa(G) \geq \frac{(b+1)^2\alpha(G)}{4} + ak$ by $a\kappa(G) \geq \frac{(b+1)^2\alpha(G)}{4} + ak - 1$, which is showed by the following example.

Let $b > a \geq 1$, $r \geq 1$ and $k \geq 0$ be four integers such that b is odd and $(\frac{b+1}{2})^2 r + ak - 1 \equiv 0 (\mathrm{mod}\ a)$. Let $G = K_p \vee rK_q$, where $p = \frac{(\frac{b+1}{2})^2 r+ak-1}{a}$ and $q = \frac{b+1}{2}$. It is obvious that $\alpha(G) = r$ and $\kappa(G) = p = \frac{(\frac{b+1}{2})^2 r+ak-1}{a}$. Let $S = V(K_p) \subseteq V(G)$ and $T = V(rK_q) \subseteq V(G)$, then $|S| = p = \frac{(\frac{b+1}{2})^2 r+ak-1}{a} \geq k$ and $|T| = r\frac{b+1}{2}$. So, we have

$$\begin{aligned} a|S| + d_{G-S}(T) - b|T| &= a\frac{\left(\frac{b+1}{2}\right)^2 r + ak - 1}{a} + r\left(\frac{b+1}{2}\right)\left(\frac{b+1}{2} - 1\right) \\ &- br\left(\frac{b+1}{2}\right) \\ &= \left(\frac{b+1}{2}\right)^2 r + ak - 1 + r\left(\frac{b+1}{2}\right)^2 - r\left(\frac{b+1}{2}\right) \\ &- br\left(\frac{b+1}{2}\right) \\ &= \left(\frac{b+1}{2}\right)^2 r + ak - 1 + r\left(\frac{b+1}{2}\right)^2 - r\left(\frac{b+1}{2}\right)(1+b) \end{aligned}$$

$$= \left(\frac{b+1}{2}\right)^2 r + ak - 1 + r\left(\frac{b+1}{2}\right)^2 - 2r\left(\frac{b+1}{2}\right)^2$$
$$= \left(\frac{b+1}{2}\right)^2 r + ak - 1 - r\left(\frac{b+1}{2}\right)^2 = ak - 1 < ak.$$

In terms of Lemma 4, G is not all fractional (a, b, k)-critical.

Acknowledgments

The authors would like to thank the anonymous referees for their comments on this paper. This work was supported by the National Natural Science Foundation of China (Nos.11731002), the Fundamental Research Funds for the Central Universities (Nos.2016JBZ012) and the 111 Project of China (B16002).

References

[1] R.P. Anstee, *Simplified existence theorems for (g, f)-factors*, Discrete Appl. Math. **27** (1990) 29–38.
doi:10.1016/0166-218X(90)90126-W

[2] Q. Bian and S. Zhou, *Independence number, connectivity and fractional (g, f)-factors in graphs*, Filomat **29** (2015) 757–761.
doi:10.2298/FIL1504757B

[3] J.R. Correa and M. Matamala, *Some remarks about factors of graphs*, J. Graph Theory **57** (2008) 265–274.
doi:10.1002/jgt.20284

[4] S. Liu and J. Cai, *Toughness and existence of fractional (g, f)-factors in graphs*, Ars Combin. **93** (2009) 305–311.

[5] G. Liu and L. Zhang, *Fractional (g, f)-factors of graphs*, Acta Math. Sci. Ser. B Engl. Ed. **21** (2001) 541–545.
doi:10.1016/S0252-9602(17)30443-5

[6] H. Lu, *Simplified existence theorems on all fractional $[a, b]$-factors*, Discrete Appl. Math. **161** (2013) 2075–2078.
doi:10.1016/j.dam.2013.02.006

[7] T. Niessen, *A characterization of graphs having all (g, f)-factors*, J. Combin. Theory Ser. B **72** (1998) 152–156.
doi:10.1006/jctb.1997.1797

[8] T. Nishimura, *Independence number, connectivity, and r-factors*, J. Graph Theory **13** (1989) 63–69.
doi:10.1002/jgt.3190130109

[9] S. Zhou, *Independence number, connectivity and (a, b, k)-critical graphs*, Discrete Math. **309** (2009) 4144–4148.
doi:10.1016/j.disc.2008.12.013

[10] S. Zhou, *Some results about component factors in graphs*, RAIRO Oper. Res., accepted.
doi:10.1051/ro/2017045

[11] S. Zhou, Q. Bian and Z. Sun, *Binding numbers for all fractional* (a, b, k)*-critical graphs*, Filomat **28** (2014) 709–713.
doi:10.2298/FIL1404709Z

[12] S. Zhou and Z. Sun, *On all fractional* (a, b, k)*-critical graphs*, Acta Math. Sin. Engl. Ser. **30** (2014) 696–702.
doi:10.1007/s10114-014-2629-2

[13] S. Zhou, Z. Sun and Z. Xu, *A result on* r*-orthogonal factorizations in digraphs*, European J. Combin. **65** (2017) 15–23.
doi:10.1016/j.ejc.2017.05.001

[14] S. Zhou, J. Wu and T. Zhang, *The existence of* $P_{\geq 3}$*-factor covered graphs*, Discuss. Math. Graph Theory **37** (2017) 1055–1065.
doi:10.7151/dmgt.1974

Received 5 April 2017
Revised 5 July 2017
Accepted 5 July 2017