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GAPS IN THE SATURATION SPECTRUM OF TREES

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Abstract

A graph G is H-saturated if H is not a subgraph of G but the addition of any edge from the complement of G to G results in a copy of H. The minimum number of edges (the size) of an H-saturated graph on n vertices is denoted $\mathbf{sat}(n, H)$, while the maximum size is the well studied extremal number, $\mathbf{ex}(n, H)$. The saturation spectrum for a graph H is the set of sizes of H-saturated graphs between $\mathbf{sat}(n, H)$ and $\mathbf{ex}(n, H)$. In this paper we show that paths, trees with a vertex adjacent to many leaves, and brooms have a gap in the saturation spectrum.

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1. Introduction

We will consider only simple graphs. Let the vertex set and edge set of G be denoted by $V(G)$ and $E(G)$, respectively. Let $|G| = |V(G)|$, $e(G) = |E(G)|$ and $\overline{G}$ denote the complement of G. A graph G is called *H-saturated* if H is not a subgraph of G but for every $e \in E(\overline{G})$, H is a subgraph of $G + e$. Let $\mathbf{SAT}(n, H)$ denote the set of H-saturated graphs of order n. The saturation number of a graph H, denoted $\mathbf{sat}(n, H)$, is the minimum number of edges in an H-saturated graph on n vertices and $\underline{\mathbf{SAT}}(n, H)$ is the set of H-saturated graphs of order n with size $\mathbf{sat}(n, H)$. The extremal number of a graph H, denoted $\mathbf{ex}(n, H)$ (also called the Turán number) is the maximum number of edges in an H-saturated graph on n vertices and $\overline{\mathbf{SAT}}(n, H)$ is the set of H-saturated graphs of order n with size $\mathbf{ex}(n, H)$.

The *saturation spectrum* of a graph H, denoted $\mathbf{spec}(n, H)$, is the set of sizes of H-saturated graphs of order n, that is, $\mathbf{spec}(n, H) = \{e(G) : G \in \mathbf{SAT}(n, H)\}$.

In this paper we give a bound on the maximum average degree of a connected H-saturated graph that is not a complete graph when H is a member of a large family of trees. One of the consequences of this bound is that the saturation spectrum has a gap immediately preceding $\mathbf{ex}(n, H)$. The proof used to give this bound and hence the gap in the saturation spectrum is not dependent on the structure of the tree, hence the proof establishes a general technique that could be applied to a broad families of trees. We apply this technique to paths, trees with a vertex adjacent to many leaves, and brooms.

2. Background

Graph saturation is a well-studied area of extremal graph theory. Mantel [9] gave the first result when he determined $\mathbf{ex}(n, K_3)$. Following this, Turán [12] determined $\mathbf{ex}(n, K_{p+1})$ for all $p \geq 2$ and showed that there is a unique K_{p+1}-saturated graph of size $\mathbf{ex}(n, K_{p+1})$, namely the complete balanced p-partite graph of order n (often called the Turán graph, $T_{n,p}$). Erdős and Simonovits [5] observed that $T_{n,p}$ does not contain any graph H with $\chi(H) \geq p + 1$ and hence

$$\mathbf{ex}(n, H) \geq e(T_{n,p}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2).$$

Further, they showed this bound is sharp.

Theorem 2.1. *If H is a graph with $\chi(H) = p + 1$, then*

$$\mathbf{ex}(n, H) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2).$$

As a consequence of this theorem the magnitude of the extremal number for a graph G with $\chi(G) \geq 3$ is known. When $\chi(G) = 2$, less is known. Consider the n-vertex graph G that is the union of $\lfloor n/(k-1) \rfloor$ vertex disjoint copies of K_{k-1} and a K_r $(0 \leq r \leq k-2)$. It is not difficult to see that G does not contain any tree with k vertices. It follows that

$$\mathbf{ex}(n, T_k) \geq \frac{k-2}{2} n - \frac{1}{8} k^2.$$

Erdős and Sós (see [4]) conjectured that this bound is sharp.

Conjecture 2.2. *Let T_k be an arbitrarily fixed k-vertex tree. If a graph G is T_k-free, then*

$$e(G) \leq \frac{1}{2}(k-2)n.$$

It is not difficult to see that the Erdős-Sós conjecture holds for stars. It has also been shown to hold for some other families of trees including paths [6], trees of order ℓ with a vertex x adjacent to at least $\frac{\ell}{2}$ leaves [11], and trees of diameter at most 4 [10]. A solution for large k was announced by Ajtai, Komlós and Szemerédi in the early 1990's. Although an exposition of their solution has never appeared, a recent series of long papers ([14, 15, 16, 17]) of Hladký, Komlós, Piguet, Simonovits, Stein and Szemerédi proving a related conjecture of Loebl-Komlós-Sós has appeared on the arXiv. Its methods involve some of those used by Ajtai, Komlós and Szemerédi including a sparse decomposition theorem in the same vein as the regularity lemma.

In Section 3 we give a bound on the maximum average degree of a T_ℓ-saturated graph that is not the union of cliques, when T_ℓ belongs to certain family of trees that we call strongly ES-embeddable trees.

Definition. A tree T_ℓ of order ℓ is *strongly Erdős-Sós embeddable* (or *strongly ES-embeddable* for short) if every connected graph G of order at least ℓ with $\overline{d}(G) > \ell - 3$ and $\delta(G) \geq \left\lfloor \frac{\ell}{2} \right\rfloor$ contains a copy of T_ℓ.

We will observe below that every tree that is strongly ES-embeddable satisfies the Erdős-Sós conjecture, justifying the name. The converse is not true: stars, for instance, are not strongly ES-embeddable. It seems plausible to us that stars are the *only* trees which satisfy the Erdős-Sós conjecture but are not strongly ES-embeddable, however this is not the focus of this paper. While we do show that a few basic classes are strongly ES-embeddable, our main focus is instead to show how this property can be used to imply gaps in the saturation spectrum.

We now state our main theorem whose proof will be given in Section 3.

Theorem 2.3. *Let T_ℓ be a strongly ES-embeddable tree on $\ell \geq 6$ vertices. If G is T_ℓ-saturated with $|G| \equiv 0 \pmod{\ell - 1}$ and G is not a union of copies of $K_{\ell-1}$, then*

$$|E(G)| \leq \frac{|G|}{\ell - 1}\binom{\ell - 1}{2} - \left\lfloor \frac{\ell - 3}{2} \right\rfloor .$$

As a consequence of this upper bound we show that the saturation spectrum for strongly ES-embeddable trees does not contain a consecutive set of values near the extremal number.

The saturation spectrum for K_3 was determined in [2]. That result was extended in [1] by determining the saturation spectrum for K_p, $p \geq 4$.

The saturation spectrum for small paths is also known. Gould, Tang, Wei, and Zhang [8] observe that $\mathbf{sat}(n, P_3) = \mathbf{ex}(n, P_3)$, hence $\mathbf{spec}(n, P_3)$ is continuous. They also observe that is straightforward to evolve a graph in $\underline{\mathbf{SAT}}(n, P_4)$ into a graph in $\overline{\mathbf{SAT}}(n, P_4)$, where at each step the size of the graph increases by exactly 1. Hence, $\mathbf{spec}(n, P_4)$ is continuous. They also determine $\mathbf{spec}(n, P_5)$ and $\mathbf{spec}(n, P_6)$.

Theorem 2.4 (Gould *et al.* [8])**.** *Let $n \geq 5$ and $\mathbf{sat}(n, P_5) \leq m \leq \mathbf{ex}(n, P_5)$ be integers, $m \in \mathbf{spec}(n, P_5)$ if and only if $n = 1, 2 \pmod 4$, or*

$$m \notin \begin{cases} \left\{\frac{3n}{2} - 3, \frac{3n}{2} - 2, \frac{3n}{2} - 1\right\}, & \text{if } n \equiv 0 \pmod 4, \\ \left\{\frac{3n-5}{2}\right\}, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Theorem 2.5 (Gould *et al.* [8])**.** *Let $n \geq 10$ and $\mathbf{sat}(n, P_6) \leq m \leq \mathbf{ex}(n, P_6)$ be integers, $m \in \mathbf{spec}(n, P_6)$ if and only if $(n, m) \notin \{(10, 10), (11, 11), (12, 12), (13, 13), (14, 14), (11, 14)\}$ and*

$$m \notin \begin{cases} \{2n - 4, 2n - 3, 2n - 1\}, & \text{if } n \equiv 0 \pmod 5, \\ \{2n - 4\}, & \text{if } n \equiv 2 \pmod 5, \\ \{2n - 4\}, & \text{if } n \equiv 4 \pmod 5. \end{cases}$$

Let $B_{3,t}$ be the tree obtained by subdividing an edge of $K_{1,t+1}$. Faudree, Gould, Jacobson and Thomas determined the saturation spectrum of $B_{3,2}$ and $B_{3,3}$.

Theorem 2.6 (Faudree *et al.* [7])**.** *Let $n \geq 5$ and $\mathbf{sat}(n, B_{3,2}) \leq m \leq \mathbf{ex}(n, B_{3,2})$. Then $m \in \mathbf{spec}(n, B_{3,2})$ if and only if $n \equiv 1, 2 \pmod 4$ or*

$$m \notin \begin{cases} \left\{\frac{3n}{2} - 3, \frac{3n}{2} - 2, \frac{3n}{2} - 1\right\}, & \text{if } n \equiv 0 \pmod 4, \\ \left\{\frac{3n-3}{2}\right\}, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Theorem 2.7 (Faudree *et al.* [7]). *Let m and n be integers such that $n \geq 17$. If $\mathbf{sat}(n, B_{3,3}) \leq m \leq \mathbf{ex}(n, B_{3,3})$, then $m \in \mathbf{spec}(n, B_{3,3})$ if and only if $n \equiv 1, 2, 3 \pmod 5$ or*

$$m \notin \begin{cases} \{2n-1, 2n-2, 2n-3, 2n-4\}, & \text{if } n \equiv 0 \pmod 5, \\ \{2n-3, 2n-4\}, & \text{if } n \equiv 4 \pmod 5. \end{cases}$$

In the cases of P_5, P_6, $B_{3,2}$, and $B_{3,3}$ there is a gap in the saturation spectrum when $n \equiv 0 \pmod{\ell - 1}$ where ℓ is the order of the tree. Corollary 3.9 shows that in general, for $\ell \geq 6$, there is a gap in the saturation spectrum of P_ℓ and $B_{\ell-t,t}$ in addition to another large family of trees where a single vertex is adjacent to at least $\lfloor \ell/2 \rfloor$ leaves.

3. Main

In this section, we give general results showing that for given graphs there are particular values, m, between $\mathbf{sat}(n, G)$ and $\mathbf{ex}(n, G)$ for which there cannot exist G-saturated graphs having m edges.

It is not difficult to show that if G is a vertex minimum counterexample to the Erdős–Sós conjecture, then G has order at least ℓ with $\overline{d}(G) > \ell - 2$ and $\delta(G) \geq \left\lfloor \frac{\ell}{2} \right\rfloor$. Thus if a tree is strongly ES-embeddable, then the Erdős–Sós conjecture holds for that tree.

The following result was obtained by Erdős and Gallai and independently by Dirac (cf. [3].)

Theorem 3.1. *If G is a connected graph with minimum degree δ and order at least $2\delta + 1$, then G contains a path on at least $2\delta + 1$ vertices.*

Since $2 \left\lfloor \frac{\ell}{2} \right\rfloor + 1 \geq \ell$, we obtain the following corollary.

Corollary 3.2. *Paths are strongly ES-embeddable.*

A tree T of order ℓ, $T \neq K_{1,\ell-1}$, having a vertex that is adjacent to at least $\left\lfloor \frac{\ell}{2} \right\rfloor$ leaves will be referred to as a *scrub-grass* tree. The set of scrub-grass trees of order ℓ will be denoted SG_ℓ. We will show that scrub-grass trees are strongly ES-embeddable (a slight strengthening of the result of Sidorenko from [11]). First we give the following lemma, which is a slight strengthening of a well-known embedding result.

Lemma 3.3. *Let F_ℓ be a forest of order ℓ and G be a graph with $\delta(G) \geq \ell - 1$. If $X \subseteq V(F_\ell)$ contains exactly one vertex from each component of F_ℓ and Y is a set of $|X|$ vertices in G, then there is an embedding of F_ℓ in G so that the vertices in X are mapped to Y.*

Proof. Notice that there are 2 forests on 2 vertices, namely P_2 and $2K_1$. In both cases it follows that the forest can be embedded into any graph G with $\delta(G) \geq 1$. Further this can be done having any vertex of G correspond to either vertex of P_2 and any 2 vertices of G correspond to the vertices of $2K_1$. Hence the result holds when $\ell = 2$.

Assume the result holds for every forest of order at most $\ell - 1$. Let F_ℓ be a forest of order ℓ and X be a set of vertices of F_ℓ containing exactly one vertex from each component of F_ℓ. Additionally choose $v \in X$. Let G be a graph with $\delta(G) \geq \ell - 1$ with a vertex $u \in V(G)$. $F_\ell - v$ is a forest, $F_{\ell-1}$, of order $\ell - 1$ and $G - u$ has $\delta(G - u) \geq \ell - 2$. Set $Y = X - \{v\}$. By the induction hypothesis $F_{\ell-1}$ can be embedded into $G - u$ so that the neighbors of v in $F_{\ell-1}$ are mapped to neighbors of u in $G - u$ and the vertices in Y are mapped to other vertices in $G - u$. This embedding of $F_{\ell-1}$ into $G - u$ together with u form an embedding of F_ℓ in G. Since the choice of v and u were arbitrary, it follows that this embedding satisfies the condition of mapping any set of vertices, X, containing exactly one vertex from each component of F_ℓ into any set of $|X|$ vertices of G. ■

We are now ready to prove that scrub-grass trees are strongly ES-embeddable.

Theorem 3.4. *Scrub-grass trees are strongly ES-embeddable.*

Proof. Let $T_\ell \in SG_\ell$ and suppose that $v \in T_\ell$ is adjacent to at least $\lfloor \frac{\ell}{2} \rfloor$ leaves. Let G be a graph with $\delta(G) \geq \lfloor \frac{\ell}{2} \rfloor$ and $\overline{d}(G) > \ell - 3$. Suppose that there is vertex u in G such that $d(u) \geq \ell - 1$. Then $G - u$ is a graph with $\delta(G - u) \geq \lfloor \frac{\ell}{2} \rfloor - 1$. Notice the forest obtained by removing v and all the leaf neighbors of v from T_ℓ has order at most $\ell - \lfloor \frac{\ell}{2} \rfloor - 1 \leq \lfloor \frac{\ell}{2} \rfloor$. Thus by Lemma 3.3 the forest may be embedded into $G - u$ so that the non leaf neighbors of v correspond to neighbors of u. Since $d(u) \geq \ell - 1$, there are at least $\lfloor \frac{\ell}{2} \rfloor$ neighbors of u unused in this embedding. It follows that G contains T_ℓ.

Now suppose that $\Delta(G) = \ell - 2$ and let $u \in V(G)$ such that $d(u) = \ell - 2$. Since T_ℓ is not a star, there is some vertex w in T_ℓ so that $d(v, w) = 2$. There are $\ell - 1$ vertices in $N[u]$, which implies there is a vertex $x \in G - N[u]$. More specifically, there is a vertex $x \in V(G)$ such that $d_G(u, x) = 2$. Choose a vertex x' such that $x' \in N_G(u) \cap N_G(x)$ and let w' be the common neighbor of v and w in T_ℓ. Now removing v along with all the leaf neighbors of v and w' from T_ℓ creates a forest T'_ℓ with at most $\ell - \lfloor \frac{\ell}{2} \rfloor - 2 \leq \lfloor \frac{\ell}{2} \rfloor - 1$ vertices and $G - \{u, x'\}$ is a graph with minimum degree at least $\lfloor \frac{\ell}{2} \rfloor - 2$. Lemma 3.3 implies that T'_ℓ can be embedded into $G - \{u, x'\}$ so that the neighbors of v in T_ℓ, other than w', are neighbors of u in G and neighbors of w' in T_ℓ, other than v, are neighbors of x' in G. Thus the embedding of T'_ℓ in $G - \{u, x'\}$ can be extended to an embedding of T_ℓ in G by mapping w' to x' and v to u. ■

The *broom* $B_{\ell-s,s}$, for $2 \le s \le \ell - 2$ is obtained by taking a $K_{1,s+1}$ and subdividing one edge $\ell - s - 2$ times. The following theorem will be useful in the proof of Theorem 3.6.

Theorem 3.5 (Erdős and Gallai [6]). *If G is a 2-connected graph with at least $2k$ vertices and every vertex with the exception of one vertex v has degree at least k, then G contains a path with at least $2k$ vertices which starts at v.*

Note that in the following theorem we explicitly exclude the star $K_{1,\ell-1}$ which would correspond to $B_{2,\ell-2}$ from the family.

Theorem 3.6. *Brooms $B_{\ell-s,s}$ are strongly ES-embeddable, for $2 \le s \le \ell - 3$.*

Proof. Note that if $s \ge \left\lfloor \frac{\ell}{2} \right\rfloor$, then $B_{\ell-s,s}$ is a scrub-grass tree and already covered, so without loss of generality $s < \left\lfloor \frac{\ell}{2} \right\rfloor$. Let us fix a $B_{\ell-s,s}$, and we shall show that it embeds into a graph satisfying the conditions.

Suppose G is a graph satisfying $\delta(G) \ge \left\lfloor \frac{\ell}{2} \right\rfloor$ and $\overline{d}(G) > \ell - 3$, with at least ℓ vertices.

First consider the case where G is 2-connected. By the average degree condition G contains a vertex v of degree at least $\ell - 2$. By Theorem 3.5, G contains a path with at least $2\left\lfloor \frac{\ell}{2} \right\rfloor \ge \ell - 1$ vertices starting from any vertex, and in particular from v. Note that if $\deg(v) \ge \ell - 1$, or v is not adjacent to some vertex of the path, then it is easy to find a copy of the broom in this structure (finding the tail as part of the long path, and taking v's remaining neighbors as its s leaf neighbors.) In the remaining scenario $\deg(v) = \ell - 2$, and it is incident to all vertices on a path $P = \{v_1, \ldots, v_{\ell-2}\}$ on $\ell - 2$ vertices. At least one vertex of the graph is not in $P \cup \{v\}$, and by two connectedness there are at least two vertices x and y on the path which have neighbors outside of $P \cup \{v\}$. If one can find a path P' on the graph induced in P on $\ell - s - 2$ vertices ending in either x or y, then we have our desired broom as follows. Assume P' starts at x and let z be a vertex in $V(G) \setminus (P \cup \{v\})$ adjacent to x. Now it is easy to see that $zxP'v$ is a path on $\ell - s$ vertices ending at v. Note that v has s neighbors that are not on this path, which give the desired broom.

It remains to show that there is always a path with at least $\ell - s - 2$ vertices in the graph induced on P ending at a vertex that has a neighbor outside of $P \cup \{v\}$. This happens, for instance, if P induces a Hamiltonian cycle, or even a cycle on $\ell - 3$ vertices. To this end, consider the endpoints v_1 and $v_{\ell-2}$ of P. They must have degree at least $\lfloor \ell/2 \rfloor - 1$ to vertices other than v, and cannot have any neighbors outside of the path, otherwise we have the desired broom. No neighbor of v_1 can be the successor of the neighbor of $v_{\ell-2}$. Unfortunately, this need not force a Hamiltonian cycle in the case where ℓ is odd, but it does force the neighbors of v_1 to be $v_2, v_3, \ldots, v_{(\ell-1)/2}$ and likewise the neighbors of $v_{\ell-2}$ to be $v_{\ell-3}, v_{\ell-4}, \ldots, v_{\ell-2-\lfloor \ell/2 \rfloor+1} = v_{(\ell-1)/2}$. These adjacencies, however, easily force a path of length $\ell - s - 2$ in P, for any $2 \leq s \leq \ell - 3$, ending at any vertex other than $v_{(\ell-1)/2}$. Since there are two vertices in P with neighbors outside of $P \cup \{v\}$ this is enough to complete this part of the proof.

Finally, we treat the case where G is not 2-connected. In this case we consider the block decomposition of the graph. Consider two leaf-blocks L_1, and L_2 in the block decomposition. Note that L_1 and L_2 may share a common cut-vertex. Each contains a vertex of degree at least $\left\lfloor \frac{\ell}{2} \right\rfloor$, and hence contains (including the cut-vertex) at least $\left\lfloor \frac{\ell}{2} \right\rfloor + 1$ vertices. By Theorem 3.5 there is a path in each of L_1 and L_2 starting at the cut-vertex on at least $\left\lfloor \frac{\ell}{2} \right\rfloor + 1$ vertices. Label the vertices on the path in L_1 in order as $x_0, x_1, \ldots, x_{\lfloor \ell/2 \rfloor}$, where x_0 is the cut-vertex. Then note that for $1 \leq i \leq \lfloor \ell/2 \rfloor - 2$, there are at least $\lfloor \ell/2 \rfloor - i$ neighbors of x_i which are not $x_0, \ldots, x_{i-1}$. Since we are only interested in $B_{\ell-s,s}$ for $2 \leq s \leq \lfloor \ell/2 \rfloor - 1$ this allows us to find our desired broom: Start with $x_{\lfloor \ell/2 \rfloor - s}$, and take the s neighbors of this which are not in $x_0, \ldots, x_{\lfloor \ell/2 \rfloor - s - 1}$ to be the bristles of the broom. Then the remainder of the broom is the path starting with $x_{\lfloor \ell/2 \rfloor - s}$ to x_0, and then following the path through the block components to L_2, and a sufficient amount of the path in L_2. This path starting at $x_{\lfloor \ell/2 \rfloor - s}$ will have at least $\lfloor \ell/2 \rfloor - s + 1 + \lfloor \ell/2 \rfloor \geq \ell - s$ vertices, which completes the broom. ∎

The remainder of the results in this section will expand on the properties of the class of strongly ES-embeddable trees and use these results to show there is gap in the saturation spectrum of all strongly ES-embeddable trees by using Theorem 2.3.

Our next result gives a bound on the number of edges in a T_ℓ free graph which satisfies a 'local sparsity' condition.

Lemma 3.7. *Let T_ℓ be a strongly ES-embeddable tree on $\ell \geq 6$ vertices. If G is a T_ℓ-free graph and there is no set of $\ell - 1$ vertices that induce more than $\binom{\ell-1}{2} - \left\lfloor \frac{\ell-3}{2} \right\rfloor$ edges, then*

$$\sum_{v \in V(G)} d(v) \leq (\ell - 2)|G| - 2 \left\lfloor \frac{\ell - 3}{2} \right\rfloor.$$

Proof. First note that every graph of order at most $\ell - 2$ is T_ℓ-free and does not contain a set of $\ell - 1$ vertices. Let $|G| \leq \ell - 2$ with $\ell \geq 6$. The degree sum of the vertices of G is at most

$$\begin{aligned} \sum_{v \in V(G)} d(v) &\leq |G|(|G| - 1) \leq (\ell - 2)(|G| - 1) \\ &= (\ell - 2)|G| - (\ell - 2) \leq (\ell - 2)|G| - 2 \left\lfloor \frac{\ell - 3}{2} \right\rfloor. \end{aligned}$$

Hence, $\sum_{v\in V(G)} d(v) \leq (\ell-2)|G| - 2\left\lfloor\frac{\ell-3}{2}\right\rfloor$ for every graph G such that $|G| \leq \ell-2$. If $|G| = \ell-1$ then the result follows from the condition that no set of $\ell-1$ vertices induce more than $\binom{\ell-1}{2} - \left\lfloor\frac{\ell-3}{2}\right\rfloor$ edges.

Thus, any counterexample to the theorem must have order at least ℓ. Let G be a vertex minimal counterexample to the statement. That is, suppose G is the graph of smallest order that is T_ℓ-free and has no set of $\ell-1$ vertices which induce more than $\binom{\ell-1}{2} - \left\lfloor\frac{\ell-3}{2}\right\rfloor$ edges and

$$\sum_{v\in V(G)} d(v) > (\ell-2)|G| - 2\left\lfloor\frac{\ell-3}{2}\right\rfloor.$$

If G is disconnected then there is some component X of G such that $\overline{d}(G) \leq \overline{d}(G[X])$, hence $G[X]$ is a smaller counterexample. Thus we may assume that G is connected.

Suppose there is a vertex $u \in V(G)$ such that $d(u) < \left\lfloor\frac{\ell}{2}\right\rfloor$. Notice that since $G-u$ is a subgraph of G, it is T_ℓ-free and does not contain a set of $\ell-1$ vertices that induce more than $\binom{\ell-1}{2} - \left\lfloor\frac{\ell-3}{2}\right\rfloor$ edges. Now we calculate the degree sum of $G-u$ as follows:

$$\begin{aligned}\sum_{v\in V(G-u)} d(v) &> (\ell-2)|G| - 2\left\lfloor\frac{\ell-3}{2}\right\rfloor - 2\left(\left\lfloor\frac{\ell}{2}\right\rfloor - 1\right)\\ &\geq (\ell-2)|G| - 2\left\lfloor\frac{\ell-3}{2}\right\rfloor - (\ell-2)\\ &= (\ell-2)|G-u| - 2\left\lfloor\frac{\ell-3}{2}\right\rfloor.\end{aligned}$$

Thus $G-u$ is a smaller order counterexample than G, hence we may assume that $\delta(G) \geq \left\lfloor\frac{\ell}{2}\right\rfloor$. Finally note that $\overline{d}(G) > (\ell-2) - \frac{2}{|G|}\left\lfloor\frac{\ell-3}{2}\right\rfloor > (\ell-3)$. Therefore, G contains every strongly ES-embeddable tree on ℓ vertices, a contradiction. ∎

Our next result allows us to exploit the local sparsity condition in the previous theorem. We show that a graph close to complete graph on $\ell-1$ vertices, along with a pendant vertex contains all small trees.

Lemma 3.8. *Let T be a tree on ℓ vertices which is not $K_{1,\ell-1}$, and G be a graph on ℓ vertices consisting of a $K_{\ell-1}$, with at most $\left\lfloor\frac{\ell-3}{2}\right\rfloor$ edges removed, having a single pendant edge adjacent to one of the vertices. Then there is an embedding of T into G.*

Proof. Note that in any tree that is a non-star, there are always at least two vertices, which are not leaves, all of whose neighbors except one are leaves, and at least one of these vertices must be adjacent to at most $\lfloor\frac{\ell-2}{2}\rfloor$ leaves. Let $z^* \in V(T)$ be a vertex, that is not a leaf, such that all but one of the neighbors of z^* are leaves, and which has at most $\lfloor\frac{\ell-2}{2}\rfloor$ leaf neighbors. We prove the slightly stronger statement: If $x \in V(G)$ is the neighbor of the pendant vertex of G, then there is an embedding of T, so that z^* maps to x.

Inspection quickly verifies that this holds for $4 \le \ell \le 5$. (Note that P_4 cannot be embedded into $K_{1,3}$, so the statement is false if $\lfloor\frac{\ell-3}{2}\rfloor$ is replaced by $\lceil\frac{\ell-3}{2}\rceil$. Thus our statement is best possible.) We now assume $\ell \ge 6$ and proceed by induction.

Let H be the graph formed by removing $s \le \lfloor\frac{\ell-3}{2}\rfloor$ edges from $K_{\ell-1}$ so that $G = H + e$, where e is the pendant edge. Also, let x denote the vertex of the H incident to the pendant edge in G. Note that since $2 \cdot s \le \ell - 3$, there are at least two vertices of H not incident to any of the s removed edges (that is H contains two dominating vertices), at least one of these dominating vertices is not x, denote it by y.

Let z^* be a specified internal vertex of T incident to only one non-leaf and with at most $\lfloor\frac{\ell-2}{2}\rfloor$ leaves. Let z be an internal vertex of T in $V(T) \setminus \{z^*\}$ incident to only one non-leaf and incident to as many leaves as possible. Suppose z is incident to t leaves. Consider $G' = G-(\{y\} \cup L)$ where L consists of t vertices of $H - \{x\}$ which are incident to as many removed edges as possible. Of such choices, select as many vertices whose incident removed edges are also incident to x as possible. In particular, note that L is incident to at least $\min\{t, s\}$ removed edges. So G' consists of H', which is obtained by removing at most $s - \min\{t, s\}$ edges from $K_{\ell-t-2}$, and the pendant edge.

Let T' be the subtree of T with z and its t leaf neighbors removed. We proceed by showing that G' contains a copy of T', with $z^* \in T'$ as x, so that we can add back z and the corresponding leaves by mapping z to y and the corresponding leaves to the remaining vertices of H.

Case 1. T' is a star. If T' is a star $K_{1,\ell-t-2}$, note that T is either a double star or a two stars connected by a path of length two and hence there are at most three internal vertices and two with leaves. Since z^* is incident to at most $\lfloor\frac{\ell-2}{2}\rfloor$ leaves, z is incident to at least $\ell - 3 - \lfloor\frac{\ell-2}{2}\rfloor \ge \lfloor\frac{\ell-3}{2}\rfloor$ leaves. Hence $t \ge \lfloor\frac{\ell-3}{2}\rfloor$ and it follows that $t \ge s$. This implies that H' is a complete graph. Thus G' is composed of $K_{\ell-t-2}$ with a pendant vertex, and it is easy to find an embedding of $K_{1,\ell-t-2}$. Note that in the case where T' consists of two stars connected by a path of length 2 then we require that the 'midpoint' be mapped to one of the internal vertices of the $K_{\ell-t-2}$, but this is obviously possible.

Case 2. T' is not a star. Suppose that z^* is incident to $0 < k \le \lfloor\frac{\ell-2}{2}\rfloor$ leaves in T'. Since T' is not a star, removing z and its leaf neighbors did not increase the number of leaves incident to z^* in T'.

We may assume that $H' - \{x\}$ is not complete. If it were, we would be done easily. Indeed, if x is incident to at most $s - t \le \lfloor \frac{\ell-3}{2} \rfloor - t$ missing vertices, and is incident (including the pendant edge) to

$$\ell - t - 2 - (s-t) \ge \ell - 2 - \left\lfloor \frac{\ell-3}{2} \right\rfloor \ge \left\lfloor \frac{\ell-2}{2} \right\rfloor + 1,$$

where the last inequality holds as it is equivalent to $\ell - 3 \ge \left\lfloor \frac{\ell-3}{2} \right\rfloor + \left\lfloor \frac{\ell-2}{2} \right\rfloor$, which holds as at least one of the floors must involve rounding down. This means that z^* can be embedded to x, all of its neighbors can be embedded to vertices incident to x, and its internal neighbor can be embedded within H'. Embedding the remainder of T' is now easy as is it is to be embedded in the (complete) $H' - \{x\}$.

Let k' denote the smallest non-negative integer so that

$$1 \le k - k' \le \left\lfloor \tfrac{\ell-(t+1)-k'-2}{2} \right\rfloor.$$

Note that it may be the case that $k' = 0$, however, regardless if $k' = t+1$, then

$$k - (t+1) \le \left\lfloor \frac{\ell-2}{2} \right\rfloor - (t+1) \le \left\lfloor \frac{\ell-2t-4}{2} \right\rfloor \le \left\lfloor \frac{\ell-(t+1)-(t+1)-2}{2} \right\rfloor,$$

so it is the case that $k' \le t+1$.

Since $H' - \{x\}$ is not complete, there is a missing edge f in $H' - \{x\}$. One of two things occurs: If the edge between either end of f and x is missing, then by our choice vertices to remove from H maximizing the incident missing edges indicates that there were at least $t+1$ missing edges incident to the removed vertices. In this case, H' has at most

$$\left\lfloor \frac{\ell-3}{2} \right\rfloor - (t+1) = \left\lfloor \frac{\ell-3-2(t+1)}{2} \right\rfloor$$

missing edges. A simple edge count shows there are least $t+2$ dominating vertices in H'.

Otherwise both ends of f are connected to x in H'. A simple edge count here shows there are at least $t+1$ dominating vertices in H'.

Create T'' by removing k' leaves incident to z^* from T'. Create H'' by either removing k' of the dominating vertices (in the case where e is not connected to x) or one vertex incident to e, and $k'-1$ of the dominating vertices. This ensures that H'' is missing at most $s-(t+1)$ edges, as opposed to merely $s-t$ edges. We claim that T'', z^* and H'' plus a pendant edge attached to x satisfy the inductive hypothesis.

- The degree of z^* in T'' is at most $\left\lfloor \frac{\ell-(t+1)-k'-2}{2} \right\rfloor = \left\lfloor \frac{(|H''|+1)-2}{2} \right\rfloor$ by choice of k'.

- H'' is missing at most $\left\lfloor \frac{\ell-2(t+1)-3}{2} \right\rfloor \le \left\lfloor \frac{|H''|+1-3}{2} \right\rfloor$ edges.
- T'' is a $|H''|+1$ vertex tree.
- T'' is not a star, as T' was not a star.

Thus an embedding of T'' in H'' mapping z^* to x exists by induction. We complete the embedding of T' into H' by mapping the remaining leaves incident to z^* to the removed dominating vertices (and one vertex incident to f if applicable). ∎

We now restate and prove our main theorem.

Theorem 2.3. *Let T_ℓ be a strongly ES-embeddable tree on $\ell \ge 6$ vertices. If G is T_ℓ-saturated with $|G| \equiv 0 \pmod{\ell-1}$ and G is not a union of copies of $K_{\ell-1}$, then*

$$|E(G)| \le \frac{|G|}{\ell-1}\binom{\ell-1}{2} - \left\lfloor \frac{\ell-3}{2} \right\rfloor.$$

Proof. There is a component S of G which is not a copy of $K_{\ell-1}$. Note that this implies $|S| \ne \ell-1$. It will be shown that $G[S]$ satisfies the hypothesis of Lemma 3.7. Since G is T_ℓ-saturated it follows that $G[S]$ is T_ℓ-free. If $|S| \le \ell-2$ then there is not a set of $\ell-1$ vertices in $G[S]$. Suppose that $|S| \ge \ell$ and there is a set of $\ell-1$ vertices X in $V(G[S])$ that induce more than $\binom{\ell-1}{2} - \left\lfloor \frac{\ell-3}{2} \right\rfloor$ edges. Since $G[S]$ is connected there is an additional vertex in S that is connected to $G[X]$. Thus, Lemma 3.8 implies that $G[X \cup \{v\}]$ contains every non-star tree of order ℓ. Therefore $G[S]$ satisfies the hypothesis of Lemma 3.7. Hence $\sum_{v \in V(S)} d(v) \le (\ell-2)|S| - 2\left\lfloor \frac{\ell-3}{2} \right\rfloor$ and any component S that is not a copy of $K_{\ell-1}$ has degree sum at most $(\ell-2)|S| - 2\left\lfloor \frac{\ell-3}{2} \right\rfloor$. Let X be the vertices of G that are in a $K_{\ell-1}$ component and $Y = V(G) \setminus X$. It follows that the degree sum of G can be bounded as follows:

$$\begin{aligned}\sum_{v \in V(G)} d(v) = \sum_{v \in X} d(v) + \sum_{v \in Y} d(v) &\le (\ell-2)|X| + (\ell-2)|Y| - 2\left\lfloor \frac{\ell-3}{2} \right\rfloor \\ &\le (\ell-2)|G| - 2\left\lfloor \frac{\ell-3}{2} \right\rfloor.\end{aligned}$$

∎

From the proof of Theorem 2.3 it is easy to see that when $|G| \equiv 0 \pmod{\ell-1}$ the extremal graph is a union of copies of $K_{\ell-1}$. Hence, as an immediate corollary to Theorem 2.3 we obtain the following result.

Corollary 3.9. *Let T_ℓ be a path, scrub grass tree or broom on $\ell \ge 6$ vertices and $n = |G| \equiv 0 \pmod{\ell-1}$ and m be an integer such that $1 \le m \le \left\lfloor \frac{\ell-3}{2} \right\rfloor - 1$. There is no graph with $\frac{|G|}{\ell-1}\binom{\ell-1}{2} - m$ edges in $\mathbf{spec}(n, T_\ell)$. Hence there is a gap in $\mathbf{spec}(n, T_\ell)$.*

In [13] a similar result, which applies only to paths, to Corollary 3.9 was obtained. While Corollary 3.9 implies the existence of a gap in the saturation spectrum, it is possible that the size of the gap is larger than what the corollary guarantees. However, note that if T_ℓ is a tree of order ℓ, where T_ℓ is not $K_{1,\ell-1}$, then $2K_{\ell-2} \cup K_2$ is T_ℓ-saturated. This example shows that, for any non-star, any gap in the saturation spectrum is at most $O(\ell)$, and hence our result gives a gap of the correct order.

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