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ON EDGE COLORINGS OF 1-PLANAR GRAPHS WITHOUT 5-CYCLES WITH TWO CHORDS

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Abstract

A graph is 1-planar if it can be drawn on the plane so that each edge is crossed by at most one other edge. In this paper, it is proved that every 1-planar graph with maximum degree $\Delta \geq 8$ is edge-colorable with Δ colors if each of its 5-cycles contains at most one chord.

Keywords: 1-planar graphs, edge coloring, discharging method.

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1. Introduction

All graphs considered in this paper are simple, finite and undirected, and we follow [1] for the terminology and notation not defined here. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, $N_G(v)$ denotes the set of vertices adjacent to v, $d_G(v) = |N_G(v)|$ denotes the degree of v, $\delta_G(v) = \min\{d_G(u) : u \in N_G(v)\}$. The minimum degree of G is $\delta(G) = \min\{d_G(v) : v \in V(G)\}$ and the maximum degree of G is $\Delta(G) = \max\{d_G(v) : v \in V(G)\}$. A k-,

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k^+- and k^--vertex is a vertex of degree k, at least k and at most k, respectively. A vertex u is called a k-neighbor (respectively, k^--neighbor, k^+-neighbor) of a vertex v if $uv \in E(G)$ and $d_G(u) = k$ (respectively, $d_G(u) \leq k$, $d_G(u) \geq k$). Let $n_k(v)$ (respectively, $n_{\leq k}(v)$, $n_{\geq k}(v)$) be the number of k-neighbors (respectively, k^--neighbors, k^+-neighbors) of a vertex v in G. We use V_i to denote the set of all i-vertices in G. For $x, y \in V(G)$, $N_G(x, y) = N_G(x) \cup N_G(y)$ and in general, for any set $S \subseteq V(G)$, let $N_G(S) = \bigcup_{v \in S} N_G(v)$. A k-cycle is a cycle of length k. For a k-cycle C in G, an edge $xy \in E(G) \backslash E(C)$ is called a chord of C if $x, y \in V(C)$. If there is no confusion in the context, sometimes we write $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ simply as V, E, δ and Δ.

A *proper k-edge coloring* of a graph G is an assignment of k colors to the edges of G so that no two adjacent edges have the same color. The smallest number of colors needed in a proper edge coloring of G is the *chromatic index*, denoted by $\chi'(G)$. G is called k-edge colorable or edge colorable with k colors if G has a proper k-edge coloring. Edge colorings have some real-life applications in optimization and network design, such as file transfer in a computer network. In this model, we construct a graph H in which a vertex represents a computer, and an edge uv in H represents a file which one wishes to transfer between u and v. Each computer has only one available communication ports. Edges with the same color represent files that can be transferred in the network simultaneously. Thus a proper edge coloring of H using $\chi'(H)$ colors corresponds to a scheduling of file transfers with the minimum completion time.

The well-known Vizing's theorem tells us that $\chi'(G)$ equals $\Delta(G)$ or $\Delta(G)+1$. This theorem divides all graphs into two classes: *Class* 1 graphs have $\chi'(G) = \Delta(G)$; *Class* 2 graphs have $\chi'(G) = \Delta(G) + 1$. A graph G is Δ-*critical* if G is a graph with maximum degree Δ and G is of Class 2, but $G - e$ is of Class 1 for every edge $e \in E(G)$. It is clear that every critical graph is 2-connected. The exposition of critical graphs can be seen in [5]. Erdős and Wilson [3] proved that almost all graphs are of Class 1, that is, if p_n is the probability that a random graph of order n is of Class 1, then $p_n \to 1$ as $n \to \infty$.

So far there have been many results about classification of planar graphs in terms of proper edge colorings. For planar graphs, Vizing [9] presented examples of planar graphs of Class 2 with maximum degree Δ for each $\Delta \in \{2, 3, 4, 5\}$, and proved that every planar graph with maximum degree at least 8 is of Class 1. At the same time, he posed the following conjecture.

Conjecture 1 [9]**.** *Every planar graph with maximum degree* 6 *or* 7 *is of Class* 1.

Conjecture 1 is known as Vizing's Planar Graph Conjecture. It was confirmed for $\Delta = 7$ by Sanders and Zhao [8] and Zhang [13], independently. Conjecture 1 remains open for $\Delta = 6$. However, the case $\Delta = 6$ has been proved for some restricted families. Bu and Wang [2] proved that every planar graph with

maximum degree at least 6 is of Class 1 if it has no chordal 5-cycles and chordal 6-cycles. Ni [6] extended the above result by removing one of the conditions that every planar graph has no chordal 6-cycles. Later Xue and Wu obtained a further extension in which they proved that a planar graph with maximum degree at least 6 is of Class 1 if any 6-cycle contains at most one chord [11] or any 5-cycle contains at most one chord [12].

In this paper, we consider the proper edge coloring of 1-planar graphs. A graph is 1-*planar* if it can be drawn on the plane so that each edge is crossed by at most one other edge. This notion of 1-planar graphs was introduced by Ringel [7] while trying to simultaneously color the vertices and faces of a planar graph such that any pair of adjacent or incident elements receive different colors. The first result concerning proper edge colorings of 1-planar graphs is due to Zhang and Wu [15] who proved that every 1-planar graph with maximum degree at least 10 is of Class 1. Recently, Zhang [14] constructed 1-planar graphs of Class 2 with maximum degree 6 or 7. Later, in [16] Zhang and Liu proved that every 1-planar graph with maximum degree at least 8 is of Class 1 if G does not contain adjacent triangles and proposed the following conjecture.

Conjecture 2 [16]**.** *Every* 1*-planar graph with maximum degree at least* 8 *is of Class* 1*.*

Recently, Zhang and Liu [17] proved that if G is a 1-planar graph with $\Delta \geq 9$ and has no chordal 5-cycles, then G is of Class 1. In this paper, we will improve the result by proving that if G is a 1-planar graph with $\Delta \geq 8$ and has no 5-cycles with two chords, then G is of Class 1.

2. Main Results and Their Proofs

First, we introduce some useful lemmas on Δ-critical graphs.

Lemma 3 [10]**.** *Let G be a Δ-critical graph and let v, w be adjacent vertices of G with $d_G(v) = k$.*

(1) *If $k < \Delta$, then w is adjacent to at least $(\Delta - k + 1)$ Δ-vertices.*

(2) *If $k = \Delta$, then w is adjacent to at least two Δ-vertices.*

The lemma is referred to as Vizing's Adjacency Lemma (VAL for short). By VAL, it is easy to get the following corollary.

Corollary 4. *Let G be a Δ-critical graph. Then*

(1) *every vertex is adjacent to at most one* 2*-vertex and at least two Δ-vertices,*

(2) *the sum of the degree of any two adjacent vertices is at least $\Delta + 2$, and*

(3) *if* $uv \in E(G)$ *with* $d(u) + d(v) = \Delta + 2$*, then all vertices in* $N_G(u, v) \setminus \{u, v\}$ *are* Δ*-vertices.*

Lemma 5 [8, 13]**.** *Let* G *be a* Δ*-critical graph and let* xy *be an edge in* G *with* $d_G(x) + d_G(y) = \Delta + 2$*. If* $\max\{d_G(x), d_G(y)\} < \Delta$*, then every vertex of* $N_G(N_G(x, y))\setminus\{x, y\}$ *is a* Δ*-vertex.*

Lemma 6 [4]**.** *Let* G *be a* Δ*-critical graph with* $\Delta \geq 6$ *and let* x *be a* 4*-vertex. Then the following holds.*

(1) *If* x *is adjacent to a* $(\Delta - 2)$*-vertex, say* y*, then every vertex of* $N_G(N_G(x))\setminus \{x, y\}$ *is a* Δ*-vertex.*

(2) *If* x *is not adjacent to any* $(\Delta - 2)$*-vertex and if one of the neighbors* y *of* x *is adjacent to* $d_G(y) - (\Delta - 3)$ $(\Delta - 2)^-$*-vertices, then each of the other three neighbors of* x *is adjacent to only one* $(\Delta - 2)^-$*-vertex, which is* x*.*

(3) *If* x *is adjacent to a* $(\Delta - 1)$*-vertex, then there are at least two* Δ*-vertices in* $N_G(x)$ *which are adjacent to at most two* $(\Delta - 2)^-$*-vertices. Moreover, if* x *is adjacent to two* $(\Delta - 1)$*-vertices, then each of the two* Δ*-neighbors is adjacent to exactly one* $(\Delta - 2)^-$*-vertex, which is* x*.*

In the following, we always assume that all planar graphs have been embedded on the plane such that edges meet only at points corresponding to their common ends, and all 1-planar graphs have been embedded on the plane such that every edge is crossed by at most one other edge and the number of crossings is as small as possible. Let G be a planar graph and $F(G)$ be the face set of G. A face $f \in F(G)$ is said to be *incident* with all edges and vertices in its boundary and f is usually denoted by $f = [u_1u_2 \cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the boundary vertices of f in a cyclic order. For convenience, $f = [u_1u_2 \cdots u_n]$ is called an $(a_1, a_2, \ldots, a_n)$*-face* if the degree of the vertex u_i is a_i for $i = 1, 2, \ldots, n$. The degree of f, denoted by $d_G(f)$, is the number of edges incident with f where each cut edge is counted twice. A k-, k^--, k^+*-face* is a face of degree k, at most k and at least k, respectively.

Let G be a 1-planar graph. The *associated plane graph* $G^\times$ of G is the graph obtained from G by turning all crossings of G into new 4-vertices. A vertex in $G^\times$ is called $false$ if it is not a vertex of G and $true$ otherwise. Note that every 3-face in $G^\times$ is incident with at most one false vertex, so we call a 3-face in $G^\times$ $false$ or $true$ according to whether it is incident with a false vertex or not. In [15], Zhang and Wu showed some basic properties between a 1-planar graph and its associated plane graph.

Lemma 7 [15]**.** *Let* G *be a* 1*-planar graph and let* $G^\times$ *be its associated plane graph. Then the following results hold.*

(1) *In* $G^\times$*, any two false vertices are not adjacent.*

(2) *If there is a* 3*-face* $[uvw]$ *in* $G^\times$ *such that* $d_G(v) = 2$*, then* u *and* w *are both true vertices.*

(3) *If a* 3*-vertex* u *in* G *is adjacent to a false vertex* v *in* $G^\times$*, then* uv *is not incident with two* 3*-faces.*

(4) *If a* 3*-vertex* v *in* G *is incident with two* 3*-faces and adjacent to two false vertices in* $G^\times$*, then* v *must also be incident with a* 5^+*-face.*

(5) *For any* 4*-vertex* u *in* G*,* u *is incident with at most three false* 3*-faces.*

Now we pay our attention to 1-planar graphs satisfying that each 5-cycle contains at most one chord and prove the following lemma.

Lemma 8. *Let* G *be a* 1*-planar graph in which each* 5*-cycle has at most one chord and let* $G^\times$ *be its associated plane graph. Then every* 7^+*-vertex* $v \in V(G)$ *is incident with at most* $\lfloor \frac{6}{7} d_G(v) \rfloor$ 3*-faces in* $G^\times$*.*

Proof. It suffices to prove that there are no seven consecutive 3-faces incident with v in $G^\times$. Suppose to the contrary that there are consecutive seven 3-faces $f_i = [vv_iv_{i+1}]$ in $G^\times$, $i = 1, 2, \ldots, 7$ and $v_1 = v_8$ if $d_G(v) = 7$. If f_3 is a true 3-face, then f_2 is a true 3-face or v, v_1, v_3 form a 3-cycle of G. At the same time, f_4 is a true 3-face or v, v_4, v_6 form a 3-cycle. Now a 5-cycle with two chords appears. So f_3 is a false 3-face. Without loss of generality, assume that v_3 is false and v_4 is true. Then v, v_2, v_4 form at least one 3-cycle in G, that is, f_4 is a true 3-face or v, v_4, v_6 form a 3-cycle of G. If f_4 is a true 3-face, then f_5, f_6 can construct at least one 3-cycle in G, which implies that there is a 5-cycle with two chords, a contradiction. If v, v_4, v_6 form a 3-cycle of G, then f_6, f_7 form at least one 3-cycle in G, which implies that there is a 5-cycle with two chords, a contradiction. ■

Theorem 9. *Let* G *be a* 1*-planar graph. If* $\Delta(G) \geq 8$ *and every* 5*-cycle in* G *contains at most one chord, then* $\chi'(G) = \Delta(G)$*.*

Proof. Since it is proved in [15] that every 1-planar graph with maximum degree at least 10 has chromatic index Δ, we assume that $8 \leq \Delta(G) = \Delta \leq 9$ in the following proof. Suppose that G is a counterexample to the theorem with the smallest number of edges. Then G is a Δ-critical 1-planar graph. By VAL, $\delta(G) \geq 2$. Let $G^\times$ be the associated plane graph of G. By Euler's formula $|V(G^\times)| - |E(G^\times)| + |F(G^\times)| = 2$ and $\sum_{v \in V(G^\times)} d_{G^\times}(v) = \sum_{f \in F(G^\times)} d_{G^\times}(f) = 2|E(G^\times)|$, we can easily deduce that

$$\sum_{v \in V(G^\times)} (d_{G^\times}(v) - 4) + \sum_{f \in F(G^\times)} (d_{G^\times}(f) - 4) = -8 < 0. \tag{1}$$

Now we assign an initial charge c on $V(G^\times) \cup F(G^\times)$ by letting $c(v) = d_G(v) - 4$ for every vertex $v \in V(G^\times)$ and $c(f) = d_{G^\times}(f) - 4$ for every face

$f \in F(G^\times)$. So $\sum_{x\in V(G^\times)\cup F(G^\times)} c(x) < 0$. In the following, we will devise a set of discharging rules for redistributing charges among the elements of $V(G^\times)\cup F(G^\times)$ so that the final charge on each vertex and each face becomes nonnegative while the total charge is preserved. Finally, there is a contradiction to (1), completing our proof.

We use $\tau(x_1 \to x_2)$ to denote the charge move from x_1 to x_2. For any vertex $v \in V(G)$, we denote by $f_k(v)$ the number of k-faces in $G^\times$ incident with v. In the following, we should state that $n_k(v)$ (respectively, $n_{\le k}(v), n_{\ge k}(v)$)denotes the number of k-neighbors (respectively, k^--neighbors, k^+-neighbors) of a vertex v in G rather than in $G^\times$. The discharging rules are given as follows.

R1. Let f be a 3-face in $G^\times$. If f is false, then f receives $\frac{1}{2}$ from each of its incident true vertices. Otherwise, let $f = [x, y, z]$ such that $d_G(x) \le d_G(y) \le d_G(z)$. If $d_G(x) \ge 5$, then $\tau(x \to f) = \tau(y \to f) = \tau(z \to f) = \frac{1}{3}$. Otherwise, $\tau(y \to f) = \tau(z \to f) = \frac{1}{2}$.

R2. Every 2-vertex in G receives 1 from each of its neighbors in G.

R3. Every 3-vertex in G receives $\frac{1}{2}$ from each of neighbors in G.

R4. Every 5^+-face in $G^\times$ sends $\frac{1}{2}$ to each of its incident 3-vertices.

R5. Let v be a 4-vertex in G and $uv \in E(G)$. Then

R5.1. Suppose $n_{\Delta-2}(v) = 1$. Then $\tau(u \to v) = \frac{1}{2}$ if $d_G(u) = \Delta$.

R5.2. Suppose $n_{\Delta-1}(v) = 2$. Then $\tau(u \to v) = \begin{cases} \frac{2}{3}, & d_G(u) = \Delta; \\ \frac{1}{12}, & d_G(u) = \Delta - 1. \end{cases}$

R5.3. Suppose that $n_{\Delta-2}(v) = 0$ and $n_{\Delta-1}(v) = 1$. If some Δ-neighbor y of v is adjacent to three $(\Delta-2)^-$-vertices, then

$\tau(u \to v) = \begin{cases} \frac{1}{3}, & u = y; \\ \frac{2}{3}, & u \in V_\Delta \setminus \{y\}. \end{cases}$ Otherwise, $\tau(u \to v) = \begin{cases} \frac{5}{12}, & d_G(u) = \Delta; \\ \frac{1}{4}, & d_G(u) = \Delta - 1. \end{cases}$

R5.4. Suppose $n_\Delta(v) = 4$. If some Δ-neighbor y of v is adjacent to three $(\Delta-2)^-$-vertices, then $\tau(u \to v) = \frac{2}{3}$ if $u \in V_\Delta \setminus \{y\}$. Otherwise, $\tau(u \to v) = \frac{5}{12}$.

R6. Suppose $d_G(v) = 5$ and $uv \in E(G)$. Then

R6.1. $\tau(u \to v) = \frac{2}{9}$ if $f_3(v) \le 4$, $d_G(u) = 7$;

R6.2. $\tau(u \to v) = \frac{1}{3}$ if $f_3(v) = 5$, $d_G(u) = 7$ and $f_3(u) \le 5$;

R6.3. $\tau(u \to v) = \frac{3}{20}$ if $f_3(v) = 4$, $d_G(u) = 6$;

R6.4. $\tau(u \to v) = \frac{1}{3}$ if $f_3(v) = 5$, $d_G(u) = 6$ and $f_3(u) \le 4$;

R6.5. If $8 \le d_G(u) \le \Delta$, then

R6.5.1. $\tau(u \to v) = \frac{1}{3}$ if $f_3(v) = 5$, $\delta_G(v) = \Delta - 3$;

R6.5.2. $\tau(u \to v) = \frac{7}{24}$ if $f_3(v) = 5$, $\delta_G(v) \ge 8$ and the edge uv is incident with a $(5, 8^+, 8^+)$-face;

R6.5.3. $\tau(u \to v) = \frac{1}{4}$ otherwise.

R7. Suppose $d_G(v) = 6$ and $uv \in E(G)$. Then

R7.1. $\tau(u \to v) = \frac{1}{5}$ if $8 \le d_G(u) \le \Delta$;

R7.2. $\tau(u \to v) = \frac{1}{10}$ if $f_3(v) \le 5$ and $d_G(u) = 7$;

R7.3. $\tau(u \to v) = \frac{1}{6}$ if $f_3(v) = 6$, $d_G(u) = 7$ and $f_3(u) \le 5$;

R7.4. $\tau(u \to v) = \frac{1}{6}$ if $f_3(v) = 6$, $d_G(u) = 6$ and $f_3(u) \le 4$;

R7.5. $\tau(u \to v) = \frac{1}{8}$ if $f_3(v) = 6$, $d_G(u) = 5$ and $f_3(u) \le 3$.

R8. Suppose $d_G(v) = 7$ and $uv \in E(G)$. Then $\tau(u \to v) = \frac{1}{6}$ if $8 \le d_G(u) \le \Delta$.

Let $f \in F(G^\times)$. Suppose that f is a 5^+-face. Then the number of 3-vertices incident with f is at most $\left\lfloor \frac{d_{G^\times}(f)}{2} \right\rfloor$ by VAL, and it follows that $c'(f) \ge (d_{G^\times}(f) - 4) - \left\lfloor \frac{d_{G^\times}(f)}{2} \right\rfloor \times \frac{1}{2} \ge 0$ by R4. Suppose that f is a 4-face. Then $c'(f) = c(f) = 0$. Suppose that f is a 3-face in $G^\times$. If f is a false 3-face, then f is incident with two true vertices by (1) of Lemma 7, and it follows that $c'(f) \ge (3-4) + 2 \times \frac{1}{2} = 0$ by R1. Otherwise, $c'(v) \ge (3-4) + \min\left\{2 \times \frac{1}{2}, 3 \times \frac{1}{3}\right\} = 0$ by R1. Hence $c'(f) \ge 0$ for every face $f \in F(G^\times)$.

Let $v \in V(G^\times)$. Note that if $d_{G^\times}(v) \le 3$ or $d_{G^\times}(v) \ge 5$, then the vertex v is true and it is easy to check that $d_{G^\times}(v) = d_G(v)$. Hence, in the following, except for 4-vertices in $G^\times$, we will not distinguish between true and false vertices. Suppose that $d_{G^\times}(v) = 2$. Then v is incident with no false 3-faces in $G^\times$ by (2) of Lemma 7, and it follows that $c'(v) = (2-4) + 2 \times 1 = 0$ by R2. Suppose that $d_{G^\times}(v) = 3$. If v is incident with two false 3-faces in $G^\times$, then v is incident with a 5^+-face by (4) of Lemma 7, and we have $c'(v) \ge (3-4) - 2 \times \frac{1}{2} + \frac{1}{2} + 3 \times \frac{1}{2} = 0$ by R1, R3 and R4. Otherwise, $c'(v) \ge (3-4) - \frac{1}{2} + 3 \times \frac{1}{2} = 0$. Suppose that v is a 4-vertex in $G^\times$. If v is a false vertex, then $c'(v) = 4 - 4 = 0$. Otherwise, v is incident with at most three false 3-faces by (5) of Lemma 7, and by R5 v receives at least $\frac{3}{2}$ from its neighbors in G, so $c'(v) \ge (4-4) - 3 \times \frac{1}{2} + 3 \times \frac{1}{2} = 0$.

Now assume that $d_{G^\times}(v) = k \ge 5$. Let $v_1, v_2, \ldots, v_k$ be its neighbors in $G^\times$ in a clockwise order. We denote by f_i the face incident with vv_i and vv_{i+1} in $G^\times$, $i = 1, \ldots, k$, where the addition on subscripts are taken modulo k.

Suppose that $d_{G^\times}(v) = 5$. Let v be a 5-vertex in G with $f_3(v) \le 3$. By VAL, v is adjacent to at least two 8-vertices in G. If v is adjacent to no 6-vertex incident with six 3-faces in $G^\times$, by R1 and R6.5.3, $c'(v) \ge (5-4) - 3 \times \frac{1}{2} + 2 \times \frac{1}{4} = 0$. Otherwise, by VAL, v is adjacent to at most two 6-vertices and at least three Δ-vertices in G, since $\Delta \ge 8$. So by R1, R6.5.3 and R7.5, $c'(v) \ge (5-4) - 3 \times \frac{1}{2} + 3 \times \frac{1}{4} - 2 \times \frac{1}{8} = 0$. Let v be a 5-vertex in G with $f_3(v) = 4$. If $\delta_G(v) = 5$, by VAL, R1 and R6.5.3, $c'(v) \ge (5-4) - 4 \times \frac{1}{2} + 4 \times \frac{1}{4} = 0$. If $\delta_G(v) \ge 6$, then by VAL, R1, R6.1, R6.3 and R6.5.3, $c'(v) \ge (5-4) - 4 \times \frac{1}{2} + \min\left\{3 \times \frac{1}{4} + 2 \times \frac{3}{20}, 2 \times \frac{1}{4} + 3 \times \frac{2}{9}, 5 \times \frac{1}{4}\right\} = \frac{1}{20} > 0$. Let v be a 5-vertex in G

with $f_3(v) = 5$. Since $f_3(v) = 5$, v is incident with at least one true 3-faces, and by VAL, the degree of every vertex incident with such true 3-faces is at least 5. So v sends at most $\frac{7}{3}$ to its incident faces by R1. Suppose that v_1, v_3, v_5 are true vertices, v_2, v_4 are false vertices. Let v_i' $(i = 2, 4)$ be a vertex such that vv_i' is an edge in G that goes through the false vertex v_i in $G^\times$. We will show that $f_3(u) \leq d_G(u) - 2$ for each neighbor u of v in G. Firstly, $v_1v_2' \notin E(G)$, for otherwise there is a 5-cycle $vv_2'v_1v_5v_3$ with three chords vv_1, vv_5 and v_1v_3 in G. Similarly, $v_3v_2' \notin E(G)$, $v_3v_4' \notin E(G)$ and $v_5v_4' \notin E(G)$. Hence, we get that $f_3(v_2') \leq d_G(v_2') - 2$, $f_3(v_3) \leq d_G(v_3) - 2$, $f_3(v_4') \leq d_G(v_4') - 2$. Next, let x, y be two neighbors of v_1 in $G^\times$ such that x, y, v_5, v, v_2 are consecutive neighbors in an anticlockwise order of v_1 in $G^\times$. Since v_5 is a true vertex, the case that $xy \in E(G^\times)$ and $yv_5 \in E(G^\times)$ will result in at least one 3-cycle containing the edge vv_5, which will lead to a 5-cycle with at least two chords in G, a contradiction to the assumption of the theorem. So we can get that $f_3(v_1) \leq d_G(v_1) - 2$. Similarly, then we can have $f_3(v_5) \leq d_G(v_5) - 2$. If $\delta_G(v) = 5$, then by VAL, R1 and R6.5.1, $c'(v) \geq (5-4) - \frac{7}{3} + 4 \times \frac{1}{3} = 0$. If $\delta_G(v) = 6$, then v is adjacent to at least three Δ-vertices in G by VAL. Hence, by VAL, R1, R6.2, R6.4 and R6.5.3, $c'(v) \geq (5-4) - \frac{7}{3} + 3 \times \frac{1}{4} + \min\left\{2 \times \frac{1}{3}, 2 \times \frac{1}{3}, \frac{1}{3} + \frac{1}{4}\right\} = 0$. If $\delta_G(v) = 7$, then v is adjacent to at least two Δ-vertices in G by VAL. Hence, by VAL, R1, R6.2 and R6.5, $c'(v) \geq (5-4) - \frac{7}{3} + 4 \times \frac{1}{4} + \frac{1}{3} = 0$. If $\delta_G(v) \geq 8$, then v is incident with at least one true 3-face in $G^\times$, and there are at least two 8^+-neighbors incident with such 3-faces. So by R1, R6.5.2 and R6.5.3, $c'(v) \geq (5-4) - \frac{7}{3} + 3 \times \frac{1}{4} + 2 \times \frac{7}{24} = 0$.

Suppose that $d_{G^\times}(v) = 6$. By VAL, $\delta_G(v) \geq 4$ and v is adjacent to at least $9 - \delta_G(v)$ Δ-vertices in G, since $\Delta \geq 8$. Suppose that $f_3(v) \leq 4$. By VAL, R1, R6.3, R6.4, R7.1, R7.2 and R7.4, $c'(v) \geq (6-4) - 4 \times \frac{1}{2} + \min\left\{5 \times \frac{1}{5}, 4 \times \frac{1}{5} - 2 \times \frac{1}{3}, 3 \times \frac{1}{5} - 3 \times \frac{1}{6}, 4 \times \frac{1}{10} + 2 \times \frac{1}{5}, 6 \times \frac{1}{5}\right\} = \frac{1}{10} > 0$. If $f_3(v) = 5$, then by VAL, R1, R6.3, R7.1 and R7.2, $c'(v) \geq (6-4) - 5 \times \frac{1}{2} + \min\left\{5 \times \frac{1}{5}, 4 \times \frac{1}{5} - 2 \times \frac{3}{20}, 3 \times \frac{1}{5}, 2 \times \frac{1}{5} + 4 \times \frac{1}{10}\right\} = 0$. Let v be a 6-vertex in G with $f_3(v) = 6$. Suppose that v_1, v_3, v_5 are false vertices and v_2, v_4, v_6 are true vertices. Let v_i' $(i = 1, 3, 5)$ be a vertex such that vv_i' is an edge in G that goes through the false vertex v_i in $G^\times$. If $\delta_G(v) = 4$, then by VAL, R1 and R7.1, $c'(v) \geq (6-4) - 6 \times \frac{1}{2} + 5 \times \frac{1}{5} = 0$. If $\delta_G(v) \geq 5$, then we shall show that $f_3(v_i') \leq d_G(v_i') - 2, i = 1, 3, 5$ and $f_3(v_j) \leq d_G(v_j) - 2, j = 2, 4, 6$. We claim that $v_1'v_2 \notin E(G)$, for otherwise there is a 5-cycle $vv_1'v_2v_4v_6$ with three chords vv_2, vv_4 and v_2v_6 in G. Similarly, we can get that $v_1'v_6 \notin E(G)$. So $f_3(v_1') \leq d_G(v_1') - 2$. By the same argument, the above results hold. Hence, as for every 5-neighbor r, 6-neighbor s and 7-neighbor t of v in G, we have $f_3(r) \leq 3$, $f_3(s) \leq 4$ and $f_3(t) \leq 5$. Therefore, by VAL, R1 and R7, $c'(v) \geq (6-4) - 6 \times \frac{1}{2} + \min\left\{2 \times \frac{1}{8}, \frac{1}{8} + \frac{1}{6}, \frac{1}{8} + \frac{1}{6}\right\} + 4 \times \frac{1}{5} = \frac{1}{20} > 0$ for $\delta_G(v) = 5$; $c'(v) \geq (6-4) - 6 \times \frac{1}{2} + \min\left\{3 \times \frac{1}{5} + 3 \times \frac{1}{6}, 2 \times \frac{1}{5} + 4 \times \frac{1}{6}, 6 \times \frac{1}{5}\right\} = \frac{1}{15} > 0$ for $\delta_G(v) \geq 6$.

Suppose that $d_{G^\times}(v) = 7$. By Lemma 8, v is incident with at most six 3-faces

in $G^{\times}$. If $f_3(v) \leq 5$, then by VAL, R1, R3, R5, R6.2, R7.3 and R8, we have $c'(v) \geq (7-4) - 5 \times \frac{1}{2} + \min\left\{-\frac{1}{2} + 6 \times \frac{1}{6}, 2 \times (-\frac{1}{4}) + 5 \times \frac{1}{6}, -\frac{1}{4} - \frac{1}{3} + 5 \times \frac{1}{6}, 3 \times (-\frac{1}{3}) + 4 \times \frac{1}{6}, 4 \times (-\frac{1}{6}) + 3 \times \frac{1}{6}, 2 \times \frac{1}{6}, 7 \times \frac{1}{6}\right\} = \frac{1}{6}$. If $f_3(v) = 6$, then by VAL, R1, R3, R5, R6.1, R7.2 and R8, $c'(v) \geq (7-4) - 6 \times \frac{1}{2} + \min\left\{6 \times \frac{1}{6} - \frac{1}{2}, 5 \times \frac{1}{6} - 2 \times \frac{1}{4}, 4 \times \frac{1}{6} - 3 \times \frac{2}{9}, 3 \times \frac{1}{6} - 4 \times \frac{1}{10}, 2 \times \frac{1}{6}\right\} = 0$.

Now we consider two cases.

Case 1. $\Delta = 8$. Suppose that v is an 8-vertex. By Lemma 8, v is incident with at most six 3-faces in $G^{\times}$. Assume that $\delta_G(v) \geq 5$. If v is adjacent to a 5-vertex w with $n_5(w) = 1$, then by Lemma 5, v has at least six 8-neighbors. So by R1 and R6.5.1, $c'(v) \geq (8-4) - 6 \times \frac{1}{2} - 2 \times \frac{1}{3} = \frac{1}{3} > 0$. If v is adjacent to at least one 5-vertex which has the special case in R6.5.2, then $n_5(v) \leq 4$ by VAL, and it follows that $c'(v) \geq (8-4) - 5 \times \frac{1}{2} - \frac{1}{3} - 4 \times \frac{7}{24} = 0$ by R1 and R6.5.2. If $\delta_G(v) \notin \{3, 4\}$ and the above cases are excluded, then $c'(v) \geq (8-4) - 6 \times \frac{1}{2} - \max\left\{1, 4 \times \frac{1}{4}, 5 \times \frac{1}{5}, 6 \times \frac{1}{6}\right\} = 0$ by VAL, R1, R2, R6.5.3, R7.1 and R8.

Let $\delta_G(v) = 3$. By VAL, v is adjacent to at least six 8-vertices in G. If $n_3(v) = 2$, then by VAL, R1 and R3, $c'(v) \geq (8-4) - 6 \times \frac{1}{2} - 2 \times \frac{1}{2} = 0$. If $n_3(v) = n_4(v) = 1$ and let u be the 4-neighbor of v in G, by Lemma 6, only when every 8-neighbor of u is adjacent to at most two $(\Delta-2)^-$-vertices in G, v sends out at most $\frac{5}{12}$ to u by R5.3 and R5.4. So $c'(v) \geq (8-4) - 6 \times \frac{1}{2} - \frac{1}{2} - \frac{5}{12} = \frac{1}{12} > 0$. If $n_3(v) = 1$ and $n_{5^+}(v) = 7$, then by VAL, R1, R6.5, R7.1 and R8, $c'(v) \geq (8-4) - 6 \times \frac{1}{2} - \frac{1}{2} - \max\left\{\frac{7}{24}, \frac{1}{5}, \frac{1}{6}\right\} = \frac{5}{24} > 0$. Let $\delta_G(v) = 4$. By VAL, v is adjacent to at least five 8-vertices in G. Assume that u is a 4-neighbor of v in G. If v sends some charge to u by R5.1, then by (1) of Lemma 6, VAL, R1 and R7, $c'(v) \geq (8-4) - 6 \times \frac{1}{2} - \frac{1}{2} - \frac{1}{5} = \frac{3}{10} > 0$. If v sends some charge to u by R5.2, then by (3) of Lemma 6, R1 and R8, $c'(v) \geq (8-4) - 6 \times \frac{1}{2} - \frac{2}{3} - 2 \times \frac{1}{6} = 0$. If v sends some charge to u by R5.3, then we will consider three cases. The first case is that v is adjacent to three $(\Delta-2)^-$-vertices in G. Then v will send at most $3 \times \frac{1}{3}$ to its $(\Delta-2)^-$-neighbors in G. So we have $c'(v) \geq (8-4) - 6 \times \frac{1}{2} - 3 \times \frac{1}{3} = 0$ by R1 and R5-R7. The second case is that there is another 8-neighbor of u, not v, which is adjacent to three $(\Delta-2)^-$-vertices in G. Then by (2) of Lemma 6, R1 and R8, we shall have $c'(v) \geq (8-4) - 6 \times \frac{1}{2} - \frac{2}{3} - 2 \times \frac{1}{6} = 0$. The third case is that every 8-neighbor of u is adjacent to at most two $(\Delta-2)^-$-vertices in G, then v will send $\frac{5}{12}$ to u. Suppose that v is also adjacent to another 4-vertex w in G. Then by R5.3 and R5.4, v can send at most $\frac{5}{12}$ to w only when every 8-neighbor of w is adjacent to at most two $(\Delta-2)^-$-vertices in G. This implies that $c'(v) \geq (8-4) - 6 \times \frac{1}{2} - \max\left\{2 \times \frac{5}{12} + \frac{1}{6}, \frac{5}{12} + \frac{7}{24} + \frac{1}{6}\right\} = 0$ by Lemma 5, VAL, R1, R5.3, R5.4, R6.5 and R8. If v sends some charge to u by R5.4, then we divide this problem into cases similar to the above. If some 8-neighbor of u is adjacent to three $(\Delta-2)^-$-vertices in G, the argument of which is the same to those of the first and second cases above, so we have $c'(v) \geq (8-4) - 6 \times \frac{1}{2} - \max\left\{2 \times \frac{7}{24}, \frac{2}{3} + 2 \times \frac{1}{6}\right\} = 0$ by VAL, Lemma 5, Lemma 6, R1,

R5.4, R6.5 and R8. Otherwise, $c'(v) \geq (8-4) - 6 \times \frac{1}{2} - 2 \times \frac{5}{12} - \frac{1}{6} = 0$ by VAL, R1, R5.3, R5.4, R6.5 and R8.

Case 2. $\Delta = 9$. Suppose that v is a 8-vertex. By Lemma 8, v is incident with at most six 3-faces in $G^{\times}$. If v is incident with at least one 5-vertex which has the special case in R6.5.2, then $n_5(v) \leq 3$ by VAL, and it follows that $c'(v) \geq (8-4) - 6 \times \frac{1}{2} - 3 \times \frac{7}{24} = \frac{1}{8}$ by R1 and R6.5.2. If the above case is excluded, then $c'(v) \geq (8-4) - 6 \times \frac{1}{2} - \max\left\{\frac{1}{2}, 2 \times \frac{1}{4}, 3 \times \frac{1}{4}, 4 \times \frac{1}{5}, 5 \times \frac{1}{6}\right\} = \frac{1}{6}$ by VAL, R1, R2, R6.5.3, R7.1 and R8.

Suppose that v is a 9-vertex. By Lemma 8, v is incident with at most seven 3-faces in $G^{\times}$. Assume that $\delta_G(v) \geq 5$. If v is adjacent to a 5-vertex w with $n_5(w) = 1$ in G, then by Lemma 5, v has at least seven 9-neighbors in G. So by R1 and R6.5.1, $c'(v) \geq (9-4) - 7 \times \frac{1}{2} - 2 \times \frac{1}{3} = \frac{5}{6} > 0$. If v is incident with at least one 5-vertex which has the special case in R6.5.2, then $n_5(v) \leq 4$ by VAL, and it follows that $c'(v) \geq (9-4) - 6 \times \frac{1}{2} - \frac{1}{3} - 4 \times \frac{7}{24} = \frac{1}{2}$ by R1 and R6.5.2. If $\delta_G(v) \notin \{3, 4\}$ and the above cases are excluded, then $c'(v) \geq (9-4) - 7 \times \frac{1}{2} - \max\left\{1, 4 \times \frac{1}{4}, 5 \times \frac{1}{5}, 6 \times \frac{1}{6}\right\} = \frac{1}{2}$ by VAL, R1, R2, R6.5.3, R7.1 and R8.

Let $\delta_G(v) = 3$. If $n_3(v) = 2$, then by VAL, R1 and R3, $c'(v) \geq (9-4) - 7 \times \frac{1}{2} - 2 \times \frac{1}{2} = \frac{1}{2}$. If v has a 4-neighbor u in G, by Lemma 6, only when every 9-neighbor of u is adjacent to at most two $(\Delta-2)^-$-vertices in G, v sends out at most $\frac{5}{12}$ to u by R5.3 and R5.4. So by R1, R3, R5.3, R5.4, R6.5, R7.1 and R8, if $n_3(v) = 1$, then $c'(v) \geq (9-4) - 7 \times \frac{1}{2} - \frac{1}{2} - \frac{5}{12} = \frac{7}{12} > 0$. Let $\delta_G(v) = 4$. By VAL, v is adjacent to at least six 9-vertices in G. Assume that u is a 4-neighbor of v in G. If v sends some charge to u by R5.1, then by (1) of Lemma 6, VAL, R1 and R8, $c'(v) \geq (9-4) - 7 \times \frac{1}{2} - \frac{1}{2} - \frac{1}{6} = \frac{5}{6} > 0$. If v sends some charge to u by R5.2, then by (3) of Lemma 6 and R1, $c'(v) \geq (9-4) - 7 \times \frac{1}{2} - \frac{2}{3} = \frac{5}{6}$. If v sends some charge to u by R5.3, then we will consider three cases. The first case is that v is adjacent to three $(\Delta-2)^-$-vertices in G. Then v will send at most $3 \times \frac{1}{3}$ to its $(\Delta-2)^-$-neighbors in G. So we have $c'(v) \geq (9-4) - 7 \times \frac{1}{2} - 3 \times \frac{1}{3} = \frac{1}{2}$ by R1 and R5-R7. The second case is that there is another 9-neighbor of u, not v, which is adjacent to three $(\Delta-2)^-$-vertices in G. Then by (2) of Lemma 6, R1 and R8, we shall have $c'(v) \geq (9-4) - 7 \times \frac{1}{2} - \frac{2}{3} = \frac{5}{6}$. The third case is that every 9-neighbor of u is adjacent to at most two $(\Delta-2)^-$-vertices in G, then v will send $\frac{5}{12}$ to u. Suppose that v is also adjacent to another 4-vertex w in G. Then by R5.3 and R5.4, v can send at most $\frac{5}{12}$ to w only when every 9-neighbor of w is adjacent to at most two $(\Delta-2)^-$-vertices in G. This implies that $c'(v) \geq (9-4) - 7 \times \frac{1}{2} - \max\left\{2 \times \frac{5}{12}, \frac{5}{12} + \frac{7}{24}\right\} = \frac{2}{3}$ by VAL, R1, R5.3, R5.4 and R6.5. If v sends some charge to u by R5.4, then we divide this problem into cases similarly. If some 9-neighbor of u is adjacent to three $(\Delta-2)^-$-vertices in G, then $c'(v) \geq (9-4) - 7 \times \frac{1}{2} - \max\left\{2 \times \frac{7}{24}, \frac{2}{3}\right\} = \frac{5}{6}$ by VAL, R1, R5.4 and R6.5. Otherwise, $c'(v) \geq (9-4) - 7 \times \frac{1}{2} - 2 \times \frac{5}{12} = \frac{2}{3}$ by VAL, R1, R5.3, R5.4

and R6.5.

Now, we have checked that $c'(x) \geq 0$ for all $x \in V(G^{\times}) \cup F(G^{\times})$. Therefore, the proof of this theorem is completed. ■

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