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ALMOST INJECTIVE COLORINGS

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Abstract

We define an almost-injective coloring as a coloring of the vertices of a graph such that every closed neighborhood has exactly one duplicate. That is, every vertex has either exactly one neighbor with the same color as it, or exactly two neighbors of the same color. We present results with regards to the existence of such a coloring and also the maximum (minimum) number of colors for various graph classes such as complete k-partite graphs, trees, and Cartesian product graphs. In particular, we give a characterization of trees that have an almost-injective coloring. For such trees, we show that the minimum number of colors equals the maximum degree, and we also provide a polynomial-time algorithm for computing the maximum number of colors, even though these questions are NP-hard for general graphs.

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1. Introduction

Among the many variants of graph colorings that have been defined are injective and 2-distance colorings. *Injective colorings* [3] have the property that all vertices at distance two have different colors, while 2-*distance colorings* [4] have the property that all vertices at distance one or two have different colors. Equivalently, 2-distance colorings require that every closed neighborhood (meaning a vertex together with its neighbors) has all colors distinct. In this paper we consider a related variant.

We define an *almost-injective coloring* (*valid coloring* for short) as a coloring of the vertices such that every closed neighborhood has exactly one duplicate. That is, every vertex has either exactly one neighbor with the same color as it, or exactly two neighbors of the same color. We call this a *blemish*; that is, every vertex has exactly one blemish. For example, if the graph is a 5-cycle, then a valid coloring is given by coloring two nonadjacent vertices red and the remaining three vertices blue.

Almost-injective colorings grew out of RASH colorings [2] for regular graphs. These colorings include the more general idea of specifying the number of colors in each closed neighborhood.

It is immediate that not all graphs have an almost-injective coloring. For example, any graph containing an isolated vertex does not have such a coloring. If a graph has an almost-injective coloring, then we say the graph is *valid*. For a valid graph G, we define two parameters: $f^-(G)$ and $f^+(G)$ are respectively the minimum and maximum number of colors in a valid coloring. Clearly $f^-(G) \geq \Delta(G)$ where $\Delta(G)$ denotes the maximum degree of G.

We will proceed as follows. In Section 2 we present some basic observations and examples. In Section 3 we give a characterization of trees that have a valid coloring, and provide a polynomial-time algorithm for computing the maximum number of colors. Then in Section 4 and Section 5 we consider some other graph classes such as regular graphs, Cartesian product graphs, and random graphs. In Section 6 we show that the existence problem is NP-complete. Finally in Section 7 we conclude with some thoughts on future research.

2. Preliminary Results and Examples

We start by considering some standard graphs. The complete graph K_n has a valid coloring. All valid colorings use $n-1$ colors. Consider next the complete bipartite graph $K_{m,n}$. If $m \neq n$, there is a unique valid coloring up to symmetry: the colors in the partite sets are disjoint, and there is one duplicate in each partite set. (This requires $m, n > 1$.) If $m = n$, then the above coloring works, and there is a second coloring: take $m = n$ colors and use each color once in each partite set.

In particular, it follows from $K_{m,m}$ that the number of colors used in a valid coloring is not always a continuous spectrum between f^- and f^+.

We turn next to paths and cycles.

Lemma 1. (a) *For all $n \geq 3$, the cycle C_n is valid.*

(b) *For all $n \geq 3$, it holds that $f^-(C_n) = 2$.*

(c) *For all $n \geq 4$, it holds that $f^+(C_n) = \lfloor n/2 \rfloor$ (while $f^+(C_3) = 2$).*

Proof. (a,b) Any coloring with red and blue is valid provided there are not three consecutive vertices of the same color.

(c) The upper bound is immediate if every color is used at least twice. So assume some color, say red, is used once, say on vertex u. Then the two neighbors of u, say v_1 and v_2, have the same color, say blue. Further, the other neighbors of v_1 and v_2 are also blue. If $4 \leq n \leq 5$ the result is not a valid coloring. So assume $n \geq 6$. Let w_1 and w_2 be the vertices at distance two from v. Then if we remove vertices u, v_1 and v_2, and add an edge between w_1 and w_2, we obtain an $(n-3)$-cycle with a valid coloring. Further, the $(n-3)$-cycle has every color from the original cycle except possibly red; so the bound follows by induction.

A suitable coloring is as follows. For n even, partition vertices into consecutive pairs and use different colors for each pair. For $n = 5$, color two nonadjacent vertices red and color the three remaining vertices blue. So assume n odd and $n \geq 7$. Start with two reds, one blue, two reds; then use new colors in consecutive pairs, as in the even case. (One can show that the optimal coloring is unique for $n \geq 7$.) ■

Lemma 2. (a) *The path P_n is valid for $n = 2$ and $n \geq 4$.*

(b) *For all $n \geq 4$, it holds that $f^-(P_n) = 2$.*

(c) *For all $n \geq 4$, it holds that $f^+(P_n) = \lfloor n/2 \rfloor$.*

Proof. (a,b) The path on three vertices does not have a valid coloring. For, the requirement of a valid coloring means that an end-vertex always has the same color as its neighbor. In P_3 this would mean that all three vertices have the same color, which is not valid. All other paths have such a coloring: For P_n with even n, alternate colors in pairs (two blues, two reds, etc); for P_n with odd n, start with two reds, one blue, two reds, after which alternate colors in pairs.

(c) One can use a similar argument as in Lemma 1(c) (or see Corollary 7). ■

Some of the above ideas can be generalized. For example, a valid coloring exists if the graph has a perfect matching M so that no edge of M is in a triangle. For, color each edge of M monochromatically. For instance, this shows that $f^+ \geq n/2$ for bipartite graphs of order n that have a perfect matching.

A valid coloring also exists if the graph has a perfect/exact double dominating set. (Recall that this is a set S such that every closed neighborhood contains exactly two vertices of S; see [1].) For, one can give all the vertices of S the same color, and then give every other vertex a unique color. For example, the 6-cycle has a perfect double dominating set on 4 vertices.

Finally, we note that it suffices to study connected graphs. For a disconnected graph, the existence of a valid coloring depends on the existence in all components; the parameter f^- is the maximum of the f^- over all components, while the parameter f^+ is the sum of the f^+ over all components.

3. Trees

We saw earlier that, except for P_3, all paths have a valid coloring. Here is a characterization of trees with a valid coloring:

Theorem 3. *A tree T is valid if and only if no two end-vertices have a common neighbor.*

Proof. In a valid coloring, an end-vertex must have the same color as its neighbor. It follows that two end-vertices having a common neighbor is forbidden.

To show that there is a valid coloring otherwise, we proceed by induction. We know from earlier that a path other than P_3 has a valid coloring. So assume the tree T is not a path.

Consider a longest path P in T. Say it ends in vertex u. The neighbor of u must have degree 2, since it cannot have more than one end-vertex neighbor. Move away from u until one reaches the first vertex of degree 3 or above, say x. Remove that part P' of the path P up to but not including x. Apply induction to the resultant tree $T - P'$. (The removal of P' does not create any new end-vertices.) If P' is not P_3, then we can apply induction to it as well, using a disjoint set of colors, and we are done. If P' is P_3, then give the neighbor of x a new color, and give u and its neighbor the same color as x. ∎

So we consider now the maximum and minimum number of colors in a valid coloring. The minimum is straightforward.

Theorem 4. *For any valid tree T, it holds that $f^-(T) = \Delta(T)$.*

Proof. This is immediate if $\Delta = 1$, and we proved it for paths above, so assume $\Delta \geq 3$. We use induction to find a valid coloring with this many colors.

Let v be any vertex of maximum degree. If v has no end-vertex neighbor, then add one. Let the non-end-vertex neighbors of v be $w_1, \ldots, w_k$. For $i \in \{1, \ldots, k\}$, define the tree T_i as the component containing v when all edges vw_j are removed for $j \neq i$. By construction, these trees have maximum degree at most $\Delta(T)$, have fewer vertices than T, and are valid; so, by induction we have $f^-(T_i) \leq \Delta(T)$. One can then name the colors so that v has the same color in each T_i, and none of, or exactly two of, the w_i have the same color, depending on whether v had an end-vertex neighbor in T or not. ∎

3.1. Algorithms and bounds for f^+

We show that there is an algorithm to determine the maximum number of colors used in a tree.

Lemma 5. *Let T be a valid tree. Say there is a vertex v of degree 2 with a neighbor u of degree 1 and other neighbor w.*
(a) *If $T - \{u,v\}$ is valid, then $f^+(T) = 1 + f^+(T - \{u,v\})$.*
(b) *If $T - \{u,v\}$ is not valid, then $f^+(T) = 1 + f^+(T - \{u,v,w\})$.*

Proof. (a) Assume $T' = T - \{u,v\}$ is valid. Then any valid coloring of it can be extended to a valid coloring of T by giving the same new color to both u and v. Thus $f^+(T) \geq 1 + f^+(T')$.

Consider any valid coloring of T. The vertices u and v must have the same color. If this coloring restricted to T' is valid, then the total number of colors is at most $1 + f^+(T')$, as required. So assume the coloring restricted to T' is not valid. This means that vertex w has no blemish in T'; it follows that w has a neighbor, say x, with the same color as v. That is, the tree T' contains all the colors used on T. So it suffices to show that the number of colors used on T' is at most $1 + f^+(T')$. Equivalently, that the coloring on T' can be transformed to a valid coloring while losing at most one color.

If w has degree more than 2 in T, say with third neighbor y, then rename the colors in the component of $T' - xw$ containing x so that x has the same color as y. The result is a valid coloring of T' that loses at most one color.

So assume w has degree 2 in T. That is, w is an end-vertex in T'. Recolor w to have the same color as x. This might lose a color. If the result is a valid coloring, then we are done. So assume it is not. That means that x has two blemishes. There are two possibilities.

Assume x has another neighbor y with the same color as x. Since T' is valid, it follows that y does not have degree 1. Let z be another neighbor of y. In the component of $T' - xy$ containing x, rename the colors so that the color on x and w is the same as the color of z. This does not lose another color, and produces a valid coloring of T'.

Finally, assume that x has two neighbors y and z that have the same color. Then in the component of $T' - xy$ containing y, rename the colors so that y has a different color from z. This introduces a new color, and produces a valid coloring.

In all cases we have shown that the number of colors on T', which included all colors of T, was at most $1 + f^+(T')$, as required.

(b) Assume $T - \{u, v\}$ is not valid. Then by Theorem 3 it must be that w is an end-vertex in T'. Let x be w's other neighbor; it must be that in T that x has an end-vertex neighbor, say y.

Consider any valid coloring of $T'' = T - \{u, v, w\}$. This can be extended to a valid coloring of T by giving w a new color, and giving u and v the same color as x. So $f^+(T) \geq 1 + f^+(T'')$. Conversely, consider any valid coloring of T. Then x has the same color as y; so the coloring restricted to T'' is valid. Furthermore, u, v, and x must have the same color. That is, the total number of colors in T is at most $1 + f^+(T'')$. ■

The above lemma provides a polynomial-time algorithm for computing $f^+(T)$. Any diametrical path must have its penultimate vertex of degree 2, since a vertex has at most one end-vertex neighbor. Thus, there always exist u and v to apply the above lemma. Indeed, one can readily implement a postorder traversal that computes the value of f^+ in linear time.

The above lemma also enables one to determine the minimum and maximum values of f^+ for a tree of fixed order.

Theorem 6. *For any valid tree T on n vertices, $f^+(T) \leq n/2$. Furthermore, equality holds if and only if T has a perfect matching.*

Proof. In order to maximize f^+, one must use condition (a) in Lemma 5 above as many times as possible. That means that one removes vertices in adjacent pairs; thus the tree has a perfect matching. Conversely, if T has a perfect matching, then we saw earlier that $f^+(T) \geq n/2$. ■

Corollary 7. *For the path P_n for $n \geq 4$, it holds that $f^+(P_n) = \lfloor n/2 \rfloor$.*

Theorem 8. *For any valid tree T on n vertices, $f^+(T) \geq (n - 1)/3$, and this is sharp.*

Proof. In order to minimize f^+, one must use condition (b) in Lemma 5 above as many times as possible. That means that one removes vertices in triples.

We get equality for trees constructed as follows. Start with K_2. Repeatedly introduce a path of length 3 and identify one end of it with a vertex that has an end-vertex neighbor. One example of equality is illustrated in Figure 1: the dark vertices form a perfect double dominating set and all receive the same color, each light vertex receives a unique color. ■

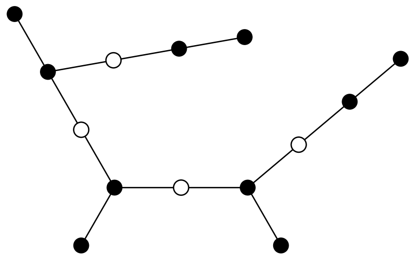

Figure 1. A tree with smallest possible f^+.

4. Some Other Graph Families

4.1. Complete multipartite graphs

We discussed complete bipartite graphs earlier. For complete multipartite graphs in general, we have the following result:

Observation 9. *A complete k-partite graph for $k \geq 3$ has a valid coloring if and only if at least two of the partite sets are singletons.*

Proof. Given a complete k-partite graph where at least two of the partite sets are singletons, a valid coloring is achieved by giving two singleton vertices the same color and every other vertex a unique color.

Suppose that a complete k-partite graph for $k \geq 3$ has a valid coloring. If some partite set is singleton, then its vertex has closed neighborhood the whole graph. So there is exactly one blemish in the graph, say vertices b_1 and b_2 have the same color. Then every closed neighborhood must contain both b_1 and b_2. This means that b_1 and b_2 are adjacent, and every other vertex is adjacent to both of them. That is, both b_1 and b_2 lie in singleton sets. Hence there are at least two singleton partite sets.

So assume none of the partite sets $A, B, C, \ldots$ is singleton. Suppose that vertex $a \in A$ and $b \in B$ have the same color. Then all vertices outside A have distinct colors and all vertices outside B have distinct colors. So for another vertex $a' \in A$, it must have the same color as a vertex of B, but that is a contradiction for vertices of C. So assume repeated colors occur only within a partite set. Then since every pair of partite sets is seen by some vertex, there can be only one partite set that contains such a blemish. But then vertices in that partite set do not see a blemish, a contradiction. ■

Recall that a vertex is *dominating* if it is adjacent to all other vertices. The above result shows that if a complete multipartite graph is valid, then all colorings use $n-1$ colors. More generally, a graph G of order n has a valid coloring with

$n-1$ colors if and only if it has at least two dominating vertices, and if so, $f^-(G) = f^+(G) = n-1$.

4.2. Regular graphs

We conjectured in [2] that a valid coloring always exists for 3-regular graphs. This remains open.

Conjecture 10 [2]**.** *If G is a cubic graph, then G has a valid coloring.*

We noted there that the 4-regular octahedron $K_{2,2,2}$ does not have a valid coloring (see also Observation 9), but found no other 4-regular graph that fails. This raises the question:

Question 11. *Does a valid coloring exist for "most" regular graphs?*

A rarer situation is r-regular graphs where $f^- = r$. Note that in this case, every vertex has every color in its closed neighborhood. Recall that the *domatic number* $d(G)$ of a graph G is the maximum number of disjoint dominating sets.

Lemma 12. *For an r-regular graph G, it holds that $f^-(G) = r$ if and only if the domatic number $d(G) \geq r$.*

Proof. If there is a valid coloring with only r colors, then every color is a dominating set. Conversely, if there are r disjoint dominating sets, then color each vertex by the set it is in, and color any remaining vertices arbitrarily. Since each vertex sees exactly r colors and its closed neighborhood has $r+1$ elements, this is a valid coloring. ∎

4.3. Random graphs

We consider the Erdős–Renyi random graph $G(n,p)$. Almost surely $G(n,p)$ for p fixed does not have a valid coloring. For, it almost surely does not have two dominating vertices, and thus we need more than two blemishes. If we use the same color three times, almost surely there is a vertex adjacent to all three vertices, and if we use two colors each twice, almost surely there is a vertex adjacent to all four vertices. Either case precludes a valid coloring.

5. Cartesian Product Graphs

We consider next the Cartesian product $G \square H$ of graphs G and H. A simple observation is that if G has a valid coloring, then so does the product: one uses the same coloring on each G-fiber but with a different palette. It follows also that:

Lemma 13. *If G has a valid coloring, then $f^+(G \square H) \geq f^+(G) \times |H|$.*

But note that it is possible for $G \square H$ to have a valid coloring even if neither G nor H does. For example, $P_3 \square P_3$ is discussed below.

5.1. Grid graphs

The minimum number of colors for the infinite grid is $f^- = 4$. This can be achieved by repeating the following pattern.

$$\begin{array}{cccccccc} 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 \\ 3 & 3 & 4 & 4 & 3 & 3 & 4 & 4 \\ 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 \\ 4 & 4 & 3 & 3 & 4 & 4 & 3 & 3 \end{array}$$

We consider next the finite grid.

Lemma 14. *If $m, n \geq 2$, then $f^-(P_m \square P_n) = \Delta(P_m \square P_n)$.*

Proof. Since the maximum degree is a lower bound, we need only describe suitable colorings. We think of the grid as having m rows and n columns.

If $m = 2$, then an optimal coloring can be obtained by using a 2-distance coloring in the first row and duplicating the coloring in the second row, illustrated below.

$$\begin{array}{cccccccc} 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 \\ 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 \end{array}$$

If $m = n = 3$, then an optimal coloring is illustrated below.

$$\begin{array}{ccc} 2 & 2 & 1 \\ 1 & 4 & 1 \\ 1 & 3 & 3 \end{array}$$

So assume $m \geq 3$ and $n \geq 4$. Then color the first row as a valid 2-coloring of the path, using colors 1 and 3. For each subsequent row, give each vertex the color 1 more than the color of the vertex above it (modulo 4). A valid coloring of $P_4 \square P_5$ is illustrated below.

$$\begin{array}{ccccc} 1 & 1 & 3 & 1 & 1 \\ 2 & 2 & 0 & 2 & 2 \\ 3 & 3 & 1 & 3 & 3 \\ 0 & 0 & 2 & 0 & 0 \end{array}$$

■

We turn next to the maximum number of colors in a valid coloring. In the infinite grid, a valid coloring is obtained by taking a perfect dominating set D and adding every vertex to the right of a vertex in D to form set X; then giving all the vertices in X the same color and every other vertex a unique color.

In the finite grid, the vertices on the outer rows and columns cause problems. Nevertheless one can adapt the approach. Illustrated below is a valid coloring for the case of $m = n = 10$. A dot means that the vertex has a unique color.

$$\begin{array}{cccccccccc}
X & X & . & 1 & 1 & X & X & . & 2 & 2 \\
0 & . & X & X & . & . & . & X & X & 3 \\
0 & . & 0 & . & X & X & . & . & . & X \\
9 & X & X & . & . & . & X & X & . & 3 \\
9 & . & . & X & X & . & . & . & X & X \\
X & X & . & . & . & X & X & . & . & 4 \\
8 & . & X & X & . & . & . & X & X & 4 \\
X & . & . & . & X & X & . & 5 & . & 5 \\
8 & X & X & . & . & . & X & X & . & 5 \\
7 & 7 & . & X & X & 6 & 6 & . & X & X
\end{array}$$

A similar idea works in general when m and n are multiples of 5. It follows that $f^+(P_m \square P_n) \geq 3mn/5 - O(m+n)$. But we do not know what the correct value is, even asymptotically.

5.2. Rooks graphs

We next consider almost-injective colorings for the Rooks graph $K_m \square K_n$. We think of this as having m rows and n columns. We start with the maximum number of colors.

Theorem 15. *For $2 \leq m \leq n$, $f^+(K_m \square K_n) = m(n-1)$.*

Proof. Since $f^+(K_n) = n-1$, by Lemma 13 it follows that $f^+(K_m \square K_n) \geq m(n-1)$. A coloring of $K_3 \square K_4$ is illustrated below.

$$\begin{array}{cccc}
1 & 2 & 3 & 3 \\
4 & 5 & 6 & 6 \\
7 & 8 & 9 & 9
\end{array}$$

Next we show that $f^+(K_m \square K_n) \leq m(n-1)$. Assume to the contrary that there is a valid coloring using more than $m(n-1)$ colors. Then some row must use n colors, say $1, 2, \ldots, n$. Since every closed neighborhood contains exactly $m+n-2$ colors, it follows that every column contains exactly $m-2$ colors not in $\{1, 2, \ldots, n\}$. Thus the total number of colors used is at most $(m-2)n + n = n(m-1) \leq m(n-1)$, a contradiction. ∎

Theorem 16. *For $2 \leq m \leq n$,*

$$f^-(K_m \square K_n) = \begin{cases} \frac{mn}{2}, & \textit{if } mn \textit{ is even}, \\ \frac{m(n+1)}{2} - 1, & \textit{if } mn \textit{ is odd}. \end{cases}$$

Proof. Construction: If n is even, give every vertex in odd columns a distinct color; then give the set of colors in the $(2k-1)^{\text{th}}$ column to the $(2k)^{\text{th}}$ column, but shift the colors up by one. If instead m is even, apply a similar strategy for rows. Clearly every closed neighborhood has exactly one repeat, and the total number of colors used is $mn/2$. The coloring of $K_3 \square K_4$ is illustrated below.

$$\begin{array}{cccc} 1 & 2 & 4 & 5 \\ 2 & 3 & 5 & 6 \\ 3 & 1 & 6 & 4 \end{array}$$

If mn is odd, the pattern is similar except that there is exactly one repeat in the last column. The coloring of $K_5 \square K_5$ with 14 colors is illustrated below.

$$\begin{array}{ccccc} 1 & 2 & 6 & 7 & 11 \\ 2 & 3 & 7 & 8 & 12 \\ 3 & 4 & 8 & 9 & 13 \\ 4 & 5 & 9 & 10 & 14 \\ 5 & 1 & 10 & 6 & 14 \end{array}$$

Next we show the above colorings are optimal. For any coloring of $K_m \square K_n$, let k_i denote the number of color classes of size i. Let k denote the total number of colors used. By counting the number of colors and vertices, we have

$$k = \sum_{i \geq 1} k_i \qquad \text{and} \qquad mn = \sum_{i \geq 1} i k_i. \tag{1}$$

Note that the total number of closed neighborhoods is mn, and each closed neighborhood is "double covered" by exactly one color class. A color class of size 1 does not double cover any closed neighborhood. A color class of size 2 double covers exactly 2, m, or n closed neighborhoods; say there are a, b, and c such colors respectively. A color class of size $i \geq 3$ cannot have two vertices located in the same row or column, otherwise it would not be a valid coloring. So the vertices of that class are in different rows and different columns, and that color class covers exactly $2\binom{i}{2}$ closed neighborhoods. Therefore, we have

$$mn = 2a + mb + nc + \sum_{i \geq 3} 2\binom{i}{2} k_i \geq 2k_2 + \sum_{i \geq 3} 2\binom{i}{2} k_i = \sum_{i \geq 1} i(i-1)k_i. \tag{2}$$

Combining the second equation of (1) and the inequality of (2) gives that $\sum_{i \geq 1} i k_i \geq \sum_{i \geq 1} i(i-1)k_i$. That is, $\sum_{i \geq 1} i(2-i)k_i \geq 0$. It follows from (1) that

$$2k - mn = k_1 - \sum_{i \geq 3}(i-2)k_i \geq k_1 - \sum_{i \geq 3} i(i-2)k_i = \sum_{i \geq 1} i(2-i)k_i \geq 0.$$

If mn is odd, then by the first equality of (2), it must be that $b+c$ is odd, and in particular $b+c>0$. Since $m \le n$, we have

$$mn \geq 2(k_2-1)+m+\sum_{i\geq 3} 2\binom{i}{2}k_i = \sum_{i\geq 1} i(i-1)k_i + m - 2. \tag{3}$$

By the same reasoning as above, it follows that $mn \le 2k-(m-2)$, whence the bound. ■

5.3. Hypercube

Let Q_k be the k-dimensional cube. Since the graph has a perfect matching but no triangle, a valid coloring exists that uses $2^k/2$ colors, as noted earlier. However, this coloring is neither the minimum nor the maximum.

We observed earlier (Lemma 12) that an r-regular graph G has $f^-(G)=r$ if and only if domatic number $d(G) \ge r$. It is well known that, for dimension k where $k=2^m-1$, the cube Q_k has a partition into $k+1$ dominating sets (also known as a fall coloring). Also Zelinka [7] observed that the domatic number for $k=2^m$ is k. Thus we have:

Lemma 17. *If $k=2^m-1$ or $k=2^m$, then $f^-(Q_k)=k$.*

However, Laborde [5] observed that $d(Q_5)=4$, since Q_5 has domination number 7. So the above lemma does not generalize for all k. Indeed, it can be checked by computer that $f^-(Q_5)=6$. It is unclear what the behavior is in general.

Next we consider the maximum number of colors. Consider for example Q_3. The maximum number of colors is 5. One can achieve this by taking a strong matching of size 2 (that is, two disjoint edges such that the graph induced by their ends is $2K_2$) and giving all such vertices the same color, and giving all other vertices unique colors. This generalizes:

Lemma 18. *If $k=2^m-1$, then $f^+(Q_k) \ge 2^k\left(1-\frac{2}{k+1}\right)+1$.*

Proof. We know that such a cube has a perfect dominating set D (meaning every closed neighborhood contains exactly one vertex of D; see [6]). So there is a strong matching of size $|D|$ (consider a perfect matching M that separates Q_k into two copies of Q_{k-1} and for each vertex of D take its partner in M). (Effectively the vertices of the strong matching form a perfect double dominating set.) Then a valid coloring is achieved by giving all vertices in the strong matching the same color, and all other vertices unique colors. Since $|D|=2^k/(k+1)$, the number of colors in such a coloring is $2^k\left(1-\frac{2}{k+1}\right)+1$. ■

For Q_4, a computer search says the maximum number of colors is 10, which is twice that of Q_3. It is unclear what the behavior of f^+ is in general.

6. Complexity

It is easy to see that f^- is hard to compute. For example, if one constructs the corona G' (take G and add a new end-vertex adjacent to each vertex), then $f^-(G') = \chi(G)$.

Theorem 19. *Deciding whether a graph has a near-injective coloring is NP-hard.*

Proof. We reduce from *Not-all-equal* $3SAT$. This is the NP-complete decision problem of: given a collection of clauses where each clause contains three literals, is there a truth assignment for the variables such that each clause contains both a false and a true literal.

First, pick some variable u, and for every other literal ℓ, add the triple $\{u, \overline{u}, \ell\}$ to the list of clauses. Note that this preserves satisfiability, since such a clause automatically has a false and a true literal.

Then we proceed as usual in such a reduction. Build the variable gadget as follows. For each variable u, start with a triangle and label two vertices with u and $\overline{u}$. For these two literal-vertices, add a common neighbor y and a leaf neighbor z of y. Call this pair $\{y, z\}$ a regulator.

Build the clause gadget as follows. For each clause c, start with a single vertex. For each literal ℓ in c, add the edge between the clause-vertex for c and the literal-vertex for ℓ, as well as a regulator: a common neighbor y and a leaf neighbor z of y. Finally, add edges to make all the y vertices of all the regulators into a clique. See Figure 2 for an illustration.

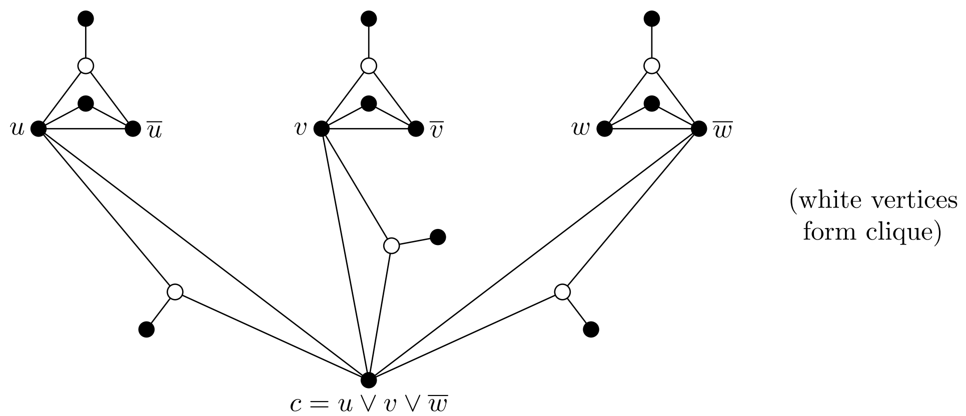

Figure 2. Example of the reduction.

We need to show that the NAE-3SAT formula ϕ has a suitable truth assignment if and only if the resultant graph G_ϕ has a valid coloring.

Suppose the original boolean formula has a suitable truth assignment. Then so does the extended formula. For each literal-vertex, color it T or F depending on whether it is true or false. For each regulator, color its two vertices the same color but use different colors for each regulator. Color the third vertex in each variable gadget with color F. And give all clause-vertices unique colors. (This uses $4c + v + 2$ colors, where c is the number of clauses and v the number of variables.)

The regulators have a blemish and do not contribute to blemishes in the other vertices. The variable triangle has the desired coloring—the three vertices see one blemish. Each clause-vertex has two literal neighbors of one color and one of the other. Thus, this is a valid coloring of G_ϕ.

Conversely, suppose the graph has a valid coloring. In each regulator, the two vertices y and z have the same color. So, the vertex y cannot have two neighbors of the same color. This means that the y-vertices have distinct colors, and furthermore, there cannot be a literal- or clause-vertex with the same color as any regulator.

In every variable triangle there is a blemish for the third vertex. So each literal-vertex has a color distinct from each of its clause-vertex neighbors. It follows that, for the blemish for clause-vertex c, it must be that (exactly) two of its literal neighbors have the same color. Because we added the clauses $\{u, \overline{u}, \ell\}$ for every literal ℓ, this means that ℓ has the same color as one of u or $\overline{u}$. In particular, exactly two colors are used on the literals. Call one of these colors false and one true. This is a suitable truth assignment for ϕ. ∎

Note that the coloring of G_ϕ created above clearly uses the maximum number of colors for a valid coloring. It follows that the same proof shows that the parameter f^+ is NP-complete as well.

7. Possible Directions

We conclude with some thoughts on possible directions for future research.

In general, for graphs of given order n, it is clear that the minimum value of f^- is 2, achieved by the path, and the maximum value of f^- is $n-1$, achieved by the complete graph inter alia. We noted that the maximum value of f^+ is $n-1$, achieved by the complete graph inter alia. However, it is unclear what the minimum value of f^+ is. Maybe it is true that for every valid graph the value f^+ is linear.

Another direction that looks interesting is to consider colorings of other specific graph classes. These might include outerplanar graphs and their generalizations planar graphs and chordal graphs.

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