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THE DISTINGUISHING NUMBER AND DISTINGUISHING INDEX OF THE LEXICOGRAPHIC PRODUCT OF TWO GRAPHS

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Abstract

The distinguishing number (index) $D(G)$ ($D'(G)$) of a graph G is the least integer d such that G has a vertex labeling (edge labeling) with d labels that is preserved only by the trivial automorphism. The lexicographic product of two graphs G and H, $G[H]$ can be obtained from G by substituting a copy H_u of H for every vertex u of G and then joining all vertices of H_u with all vertices of H_v if $uv \in E(G)$. In this paper we obtain some sharp bounds for the distinguishing number and the distinguishing index of the lexicographic product of two graphs. As consequences, we prove that if G is a connected graph with $\text{Aut}(G[G]) = \text{Aut}(G)[\text{Aut}(G)]$, then for every natural number k, $D(G) \leq D(G^k) \leq D(G) + k - 1$ and all lexicographic powers of G, G^k ($k \geq 2$) can be distinguished by two edge labels, where $G^k = G[G[\dots]]$.

Keywords: distinguishing index, distinguishing number, lexicographic.

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1. Introduction

Let $G = (V, E)$ be a simple graph with n vertices. Throughout this paper we consider only simple graphs. The set of all automorphisms of G, with the operation of composition of permutations, is a permutation group on V and is denoted by $\text{Aut}(G)$. A labeling of G, $\phi : V \to \{1, 2, \dots, r\}$, is r-*distinguishing*, if no non-trivial automorphism of G preserves all of the vertex labels. In other words, ϕ is r-distinguishing if for every non-trivial $\sigma \in \text{Aut}(G)$, there exists x in V such that $\phi(x) \neq \phi(\sigma x)$. The distinguishing number of a graph G has been defined by Albertson and Collins [1] and is the minimum number r such that G has a labeling that is r-distinguishing. Similar to this definition, Kalinowski and Pilśniak [7] have defined the distinguishing index $D'(G)$ of G which is the least integer d such that G has an edge colouring with d colours that is preserved only by the trivial automorphism. These indices has developed and a number of papers published on this subject (see, for example [2, 4, 8, 9]). For every vertex $v \in V$, the open neighborhood of v is the set $N_G(v) = \{u \in V : uv \in E\}$ and the closed neighborhood is the set $N_G[v] = N_G(v) \cup \{v\}$.

For two graphs G and H, let $G[H]$ be the graph with vertex set $V(G) \times V(H)$, such that the vertex (a, x) is adjacent to vertex (b, y) if and only if a is adjacent to b (in G) or $a = b$ and x is adjacent to y (in H). The graph $G[H]$ is the lexicographic product of G and H. This product was introduced as the composition of graphs by Harary [6]. The lexicographic product is also known as *graph substitution*, a name that bears witness to the fact that $G[H]$ can be obtained from G by substituting a copy H_u of H for every vertex u of G and then joining all vertices of H_u with all vertices of H_v if $uv \in E(G)$. For example $K_2[K_3] = K_6$. It can be seen that the number of edges of $G[H]$ is $|V(G)||E(H)| + |E(G)||V(H)|^2$. Also the degree of an arbitrary vertex (g, h) of $G[H]$ is $deg_H h + |V(H)| deg_G g$. The distinguishing number and the distinguishing index of some operations of two graphs, such as Cartesian product and corona product have been studied in [2, 8]. Klavžar and Zhu in [8] have shown that the Cartesian powers of graphs can be distinguished by two labels.

In this paper we shall study the distinguishing number and the distinguishing index of the lexicographic product of two graphs. To do this, we consider a condition on the automorphism group of $G[H]$ in this section. In Section 2, we study the distinguishing number of $G[H]$. In Section 3, we study the distinguishing index of the lexicographic product of two graphs. As usual we use the notation $\binom{n}{m}$ for the number of subsets of size m of a set with cardinality n. Note that $\binom{n}{m} = 0$ when $n < m$. Here we state some properties of the automorphisms of $G[H]$.

Let β be an automorphism of H, and (g, h) a vertex of $G[H]$. The permutation of $V(G[H])$ that maps (g, h) into $(g, \beta h)$, clearly is in $\mathrm{Aut}(G[H])$. Also, if $\alpha \in \mathrm{Aut}(G)$, then the mapping $(g, h) \mapsto (\alpha g, h)$ is an automorphism of $G[H]$. The group generated by such elements is known as the wreath product $\mathrm{Aut}(G)[\mathrm{Aut}(H)]$. Evidently all its elements can be written in the form $(g, h) \mapsto (\alpha g, \beta_{\alpha g} h)$, where α is an automorphism of G and where the $\beta_{\alpha g}$ are automorphisms of H. As the example of $K_2[K_2]$ shows, $\mathrm{Aut}(G)[\mathrm{Aut}(H)]$ can be a proper subgroup of $\mathrm{Aut}(G[H])$. In fact the elements of $\mathrm{Aut}(G)[\mathrm{Aut}(H)]$ are the automorphisms that map the copies of H to each other. The next theorem describes when $\mathrm{Aut}(G)[\mathrm{Aut}(H)]$ is equal to $\mathrm{Aut}(G[H])$. For the statement of the theorem, we use the relations S and R that are defined as follows.

Definition [10]**.** Let G be a graph. The equivalence relations R and S are defined on $V(G)$ as follows:

$$g_1 R g_2 \Longleftrightarrow N_G(g_1) = N_G(g_2), \quad g_1 S g_2 \Longleftrightarrow N_G[g_1] = N_G[g_2].$$

An infinite graph whose vertices have finite degree, is called locally finite. If only finitely many vertices have infinite degree, we say that it is almost locally finite.

Theorem 1 [10]**.** *Let G be an almost locally finite graph, H a finite graph and let R and S be the relations on $V(G)$ in Definition 1. Then a necessary and sufficient condition that* $\mathrm{Aut}(G[H]) = \mathrm{Aut}(G)[\mathrm{Aut}(H)]$ *is that H is connected if $R \neq \Delta$, and that $\overline{H}$ (the complement of H) is connected if $S \neq \Delta$, where $\Delta = \{(g,g) : g \in V(G)\}$.*

We now reply to the question: What is $\mathrm{Aut}(G[H])$ when $\overline{H}$ is a disconnected graph and G has nontrivial automorphism? In [3], Bird *et al.* have replied to this question for partially ordered sets G and H, where $\mathrm{Aut}(G)$ consists of all permutations on G that preserve order (and have order preserving inverses). By considering $\overline{G[H]} = \overline{G}[\overline{H}]$, it can be replied to this question exactly the same as Bird *et al.* as follows.

Theorem 2. *Let G and H be two graphs, the S-equivalent pairs in G are denoted by $[g_{j1}, g_{j2}]$ for $j = 1, \ldots, \theta$ and the connected components of $\overline{H}$ by $\overline{H}_i$. Also consider the following elements:*

$$S(ij) = \begin{cases} (g,h) \mapsto (g,h) & h \in \overline{H}_i, \\ (g_{j1},h) \mapsto (g_{j2},h) & h \notin \overline{H}_i, \\ (g_{j_2},h) \mapsto (g_{j_1},h) & h \notin \overline{H}_i, \\ (g,h) \mapsto (g,h) & h \notin \overline{H}_i, g \neq g_{j1}, g_{j2}. \end{cases}$$

Then the automorphism group of the graph $G[H]$, $\mathrm{Aut}(G[H])$, is the group generated by adding the elements $S(ij)$ to the wreath product $\mathrm{Aut}(G)[\mathrm{Aut}(H)]$.

2. The Distinguishing Number of $G[H]$

In this section we study the distinguishing number of the lexicographic product of two graphs G and H. The following theorem gives sharp bounds for the distinguishing number of $G[H]$.

Theorem 3. *Let G and H be two connected graphs, then*

$$D(H) \leqslant D(G[H]) \leqslant |V(G)| \times D(H).$$

Proof. The statement holds when H only has the trivial automorphism. Next we assume that H has nontrivial automorphism. First we prove that $D(H) \leqslant D(G[H])$. By contradiction, we suppose that $D(H) > D(G[H])$. So in the distinguishing labeling of $G[H]$ with $D(G[H])$ labels, it can be seen that all copies of H have been labeled with less than $D(H)$ labels. Hence for each copy of H there exists a nontrivial automorphism β_g of H such that β_g does not preserve the labeling of that copy of H in the distinguishing labeling of $G[H]$. So there exists the following nontrivial automorphism f of $G[H]$

$$f : V(G[H]) \to V(G[H]) \text{ such that } f(g,h) = (g, \beta_g h),$$

such that f does not preserve the labeling of $G[H]$, which is a contradiction.

Now we want to show that $D(G[H]) \leqslant |V(G)| \times D(H)$. For this purpose, we label the vertices of ith copy of H with the labels $\{1 + (i-1)D(H), 2 + (i-1)D(H), \ldots, D(H) + (i-1)D(H)\}$ in a distinguishing way, where $1 \leqslant i \leqslant |V(G)|$. This labeling is a distinguishing labeling of $G[H]$, because if f is an automorphism of $G[H]$ preserving the labeling, then with respect to the labeling of copies of H, the map f maps each copy of H to itself, and since we labeled each copy of H in a distinguishing way, f is the identity automorphism. Since there are $|V(G)| \times D(H)$ labels used during the second part of the proof, the result follows. ■

The bounds of $D(G[H])$ in Theorem 3 are sharp. For the upper bound it is sufficient to consider the complete graphs K_n and K_m, as two graphs G and H respectively, because $K_n[K_m] = K_{nm}$. For the lower bound we consider $G = K_1$, then $G[H] = H$, and so $D(H) = D(G[H])$.

If $\mathrm{Aut}(G[H]) = \mathrm{Aut}(G)[\mathrm{Aut}(H)]$, then we can improve the upper bound of $D(G[H])$ in Theorem 3 as follows:

Theorem 4. *Let G and H be two connected graphs such that $\mathrm{Aut}(G[H]) = \mathrm{Aut}(G)[\mathrm{Aut}(H)]$. Then $D(H) \leqslant D(G[H]) \leqslant D(H) + M$, where $M = \min\big\{k : \sum_{m=0}^{k} y_m \geqslant D(G)\big\}$ and*

$$y_m = \begin{cases} 1 & m = 0, \\ D(H) & m = 1, \\ D(H) + \sum_{i=1}^{m-1} \binom{m-1}{i}\binom{D(H)}{i+1} & m \geqslant 2. \end{cases}$$

Proof. The proof of $D(H) \leqslant D(G[H])$ is exactly the same as Theorem 3. For obtaining the upper bound, we partition the vertices of G by a distinguishing labeling of G, i.e., we partition the vertices of G into $D(G)$ classes, say $[1], \ldots, [D(G)]$ such that ith class contains the vertices of G having the label i, in the distinguishing labeling of G, where $1 \leqslant i \leqslant D(G)$. By this partition we label the copies of H as follows: First we label the vertices of H with $D(H)$ labels in a distinguishing way, next we make the following changes on the labeling of H. Before starting the labeling of the copies of H, we introduce the notation $H^{[i]}$ for the set of copies of H corresponding to the elements of the ith class, where $1 \leqslant i \leqslant D(G)$. In fact we partition the copies of H into $D(G)$ classes such that $H^{[i]}$ is the symbol of the ith class. Now we present the labeling of $G[H]$ by the following steps.

Step 1. We label all the copies of H that are in $H^{[1]}$, exactly the same as the distinguishing labeling of H, i.e., we label the vertices of each copy of H in $H^{[1]}$ with $D(H)$ labels distinguishingly.

Step 2. For the labeling of the copies in $H^{[i]}$, where $2 \leqslant i \leqslant D(H)+1$, we use of the label set $\{1, 2, \ldots, i-2, D(H)+1, i, \ldots, D(H)\}$ for a distinguishing labeling of all elements in $H^{[i]}$, where $2 \leqslant i \leqslant D(H)+1$. Note that in this case, just we have replaced the label $i-1$ in the distinguishing labeling of H, by $D(H)+1$.

Step 3. For the labeling of the copies in $H^{[i]}$, where $D(H)+2 \leqslant i \leqslant 2D(H)+1$, we do the same work as Step 2, with the new label $D(H)+2$, instead of the label $D(H)+1$.

Step 4. By choosing two labels among the labels $\{1, \ldots, D(H)\}$, and replacing them by the two new labels $D(H)+1$ and $D(H)+2$, we can label the elements of $\binom{D(H)}{2}$ other classes of the classes $H^{[i]}$, i.e., copies in $H^{[i]}$ where $2D(H)+2 \leqslant i \leqslant 2D(H)+\binom{D(H)}{2}+1$.

Step 5. We do the same work as Step 2 with the new label $D(H)+3$ instead of labels $D(H)+1$, and so we can label $D(H)$ other classes $H^{[i]}$ where $2D(H)+\binom{D(H)}{2}+2 \leqslant i \leqslant 3D(H)+\binom{D(H)}{2}+1$. Next we label $2\binom{D(H)}{2}$ other classes $H^{[i]}$, with the two new labels $D(H)+1$ and $D(H)+3$, also with the labels $D(H)+2$ and $D(H)+3$, exactly the same as Step 4. In fact we label the elements of $H^{[i]}$ where $3D(H)+\binom{D(H)}{2}+2 \leqslant i \leqslant 3D(H)+2\binom{D(H)}{2}+1$, by the label sets which have been made by replacing $D(H)+1, D(H)+3$, and label the elements of $H^{[i]}$ where $3D(H)+2\binom{D(H)}{2}+2 \leqslant i \leqslant 3D(H)+3\binom{D(H)}{2}+1$, by the label sets which have been made by replacing $D(H)+2, D(H)+3$.

Step 6. Now we choose three labels among the labels $\{1, \ldots, D(H)\}$, and replace them by the three new labels $D(H)+1$, $D(H)+2$ and $D(H)+3$. For instance, suppose that we chose three labels p, q, r with $p < q < r$, among the labels $\{1, \ldots, D(H)\}$, and replaced them by the labels $D(H)+1$, $D(H)+2$ and $D(H)+3$. In this case the label set for distinguishing labeling of copies of H in $H^{[i]}$ is $\{1, \ldots, p-1, D(H)+1, p+1, \ldots, q-1, D(H)+2, q+1, \ldots, r-1, D(H)+3, r+1, \ldots, D(H)\}$. Therefore we can label $\binom{D(H)}{3}$other classes $H^{[i]}$.

By continuing this method we obtain that the number of classes which can be labeled with the labels $1, \ldots, D(H)+m$, $m \geqslant 1$, such that the label $D(H)+m$

is used in the labeling of each element of classes, is y_m where

$$y_m = \begin{cases} 1 & m = 0, \\ D(H) & m = 1, \\ D(H) + \sum_{i=1}^{m-1} \binom{m-1}{i}\binom{D(H)}{i+1} & m \geqslant 2. \end{cases}$$

Therefore the number of labels that have been used for the labeling of all copies of H, is $D(H)+M$ where $M = \min\left\{k : \sum_{m=0}^{k} y_m \geqslant D(G)\right\}$. This labeling is a distinguishing vertex labeling of $G[H]$, because if f is an automorphism of $G[H]$ preserving the labeling, then since $\mathrm{Aut}(G[H]) = \mathrm{Aut}(G)[\mathrm{Aut}(H)]$, we have $f(g,h) = (\alpha g, \beta_{\alpha g} h)$, for some automorphism α of G and $\beta_{\alpha g}$ of H. With respect to the labeling of elements of each class $H^{[i]}$, it can be concluded that α is the identity automorphism on G. Since each copy of H has been labeled in a distinguishing way, $\beta_{\alpha g}$ is the identity automorphism on H, and so f is the identity automorphism on $G[H]$. ■

Here we shall show that the upper bound of $D(G[H])$ in Theorem 4 is sharp. To do this, suppose that G_n $(n \geq 3)$ is a spider graph which has been formed by subdividing all of the edges of a star $K_{1,n}$. We state and prove the following lemma.

Lemma 5. *For every $n \geqslant 3$, $D(G_n[K_2]) = \left\lceil \frac{1+\sqrt{1+8\sqrt{n}}}{2} \right\rceil$.*

Proof. In an r-distinguishing labeling of G_n, each of the pairs consisting of a noncentral-nonpendant vertex of a branch of G_n and its pendant neighbor must have different ordered pair of labels. There are r^2 possible ordered pairs of labels using r labels, hence $D(G_n) = \lceil \sqrt{n} \rceil$. Let $H = K_2$. It is easy to check that $\mathrm{Aut}(G_n[K_2]) = \mathrm{Aut}(G_n)[\mathrm{Aut}(K_2)]$, by Theorem 1. Let $L = \left\{(x_i, y_i, z_i, w_i) : 1 \leqslant i \leqslant n, x_i, y_i, z_i, w_i \in \mathbb{N}\right\}$ be a labeling of the vertices $G_n[K_2]$ except its central vertices (see Figure 1).

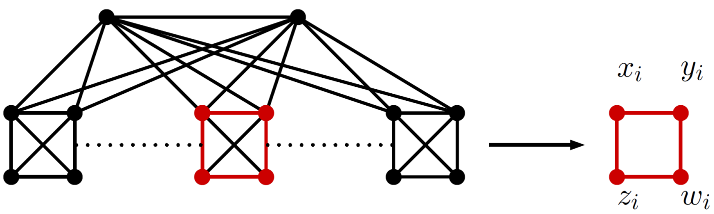

Figure 1. The place of labels x_i, y_i, z_i, w_i in $G_n[K_2]$.

If L is a distinguishing labeling then the label of two central vertices of $G_n[K_2]$ must be different, because there exists an automorphism f of $G_n[K_2]$ such that

f maps these two central vertices to each other and fixes the remaining vertices. In addition, the following conditions must be satisfied:

(i) $x_i \neq y_i$ and $w_i \neq z_i$, for all $i = 1, \dots, n$, because there exists an automorphism f_i (and also g_i) of $G_n[K_2]$ such that f_i (and also g_i) maps x_i and y_i (z_i and w_i) to each other and fixes the remaining vertices.

(ii) $\{x_i, y_i\} \neq \{x_j, y_j\}$ or $\{z_i, w_i\} \neq \{z_j, w_j\}$, for all $i, j \in \{1, \dots, n\}$ where $i \neq j$, because there exists an automorphism f_{ij} of $G_n[K_2]$ such that f_{ij} maps $\{x_i, y_i\}$ to $\{x_j, y_j\}$ and $\{z_i, w_i\}$ to $\{z_j, w_j\}$ and fixes the remaining vertices.

So there are $\binom{r}{2}\binom{r}{2}$ possible 4-arrays (x_i, y_i, z_i, w_i) of labels using r labels such that they satisfy (i) and (ii), hence $D(G_n[K_2]) = \min\left\{r : \binom{r}{2}^2 \geqslant\right\} = \left\lceil \frac{1+\sqrt{1+8\sqrt{n}}}{2} \right\rceil$ (see Figure 2 for a 3-distinguishing labeling of $G_9[K_2]$ (note that we do not sketch some edges for blinding clarity)). ■

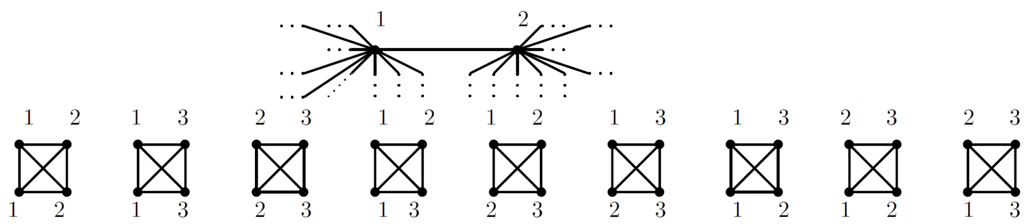

Figure 2. 3-distinguishing labeling of $G_9[K_2]$.

Now by Lemma 5 we see that for $n = 9$, we have $D(G_9) = 3$, $D(G_9[K_2]) = 3$ and $M = 1$. Hence there exists n such that the graph $G_n[K_2]$ obtains the upper bound of Theorem 4.

As a corollary of Theorem 4 we would like to present bounds for the distinguishing number of $G^k = G[G[\dots]]$.

Corollary 6. *Let G be a connected graph such that* $\mathrm{Aut}(G[G]) = \mathrm{Aut}(G)[\mathrm{Aut}G]$, *then*

(i) *If $D(G) > 1$, then $D(G) \leqslant D(G^k) \leqslant D(G) + k - 1$.*

(ii) *If $D(G) = 1$, then $D(G^k) = 1$.*

Proof. By Theorem 1 it can be seen that if $\mathrm{Aut}(G[G]) = \mathrm{Aut}(G)[\mathrm{Aut}(G)]$, then $\mathrm{Aut}(G^k) = \mathrm{Aut}(G)\big[\mathrm{Aut}\big(G^{k-1}\big)\big]$ for $k \geq 2$.

(i) Proof is by induction on k. For $k = 2$, we observe that by Theorem 4 the value of M is one, and so the result follows.

(ii) If $D(G) = 1$, we observe that by Theorem 4 the value of M is zero, and so the proof is complete. ■

3. The Distinguishing Index of $G[H]$

In this section we shall study the distinguishing index of the lexicographic product of two graphs. We begin with the following theorem:

Theorem 7. *Let G and H be two connected graphs such that $H \neq K_2$ and $\mathrm{Aut}(G[H]) = \mathrm{Aut}(G)[\mathrm{Aut}(H)]$. Then $D'(G[H]) \leqslant \max\{D'(G), D'(H)\}$.*

Proof. First we partition the edge set of G into $D'(G)$ classes, say $[1], \ldots, [D'(G)]$, by a distinguishing edge labeling of G. In fact, the ith class contains the edges of G with the label i in the distinguishing edge labeling of G, where $1 \leqslant i \leqslant D'(G)$. For a labeling of $G[H]$, we label the edge set of each copy of H in a distinguishing way with $D'(H)$ labels. By the definition of $G[H]$ we know that each edge of G, such as e is replaced by the edges which join the corresponding two copies of H, in $G[H]$. We denote the set of these replacement edges by $\mathbf{E}$. Now we assign all edges in $\mathbf{E}$, the same label of the edge e in the distinguishing edge labeling of G. This labeling is a distinguishing edge labeling of $G[H]$, because if f is an automorphism of $G[H]$ preserving the labeling, then since $\mathrm{Aut}(G[H]) = \mathrm{Aut}(G)[\mathrm{Aut}(H)]$, we have $f(g, h) = (\alpha g, \beta_{\alpha g} h)$, for some automorphism α of G and $\beta_{\alpha g}$ of H. With respect to the labeling of edges in the set $\mathbf{E}$, where $e \in E(G)$, it can be concluded that α is the identity automorphism on G. Since the edges of each copy of H have been labeled in a distinguishing way, $\beta_{\alpha g}$ is the identity automorphism on H, and so f is the identity automorphism on $G[H]$. ■

By Theorem 7 the lexicographic product $P_m[P_n]$ of two path of orders m and $n > 2$, respectively, has distinguishing index equal to 2, unless $m = 2$ (since $\mathrm{Aut}(P_2[P_n]) \neq \mathrm{Aut}(P_2)[\mathrm{Aut}(P_n)]$). For the lexicographic product of a cycle C_n with a path P_m we also have $D'(P_m[C_n]) = 2$ where $m > 2$ and $n > 5$. The lexicographic product of two cycles C_n and C_m also has distinguishing index equal to two, where $n, m > 5$. It is worth noting that these results do not depend on the relation between n and m.

In [5] Gorzkowska *et al.* have obtained the distinguishing index of the Cartesian product $K_{1,n} \square P_m$ and $K_{1,n} \square C_m$. We use their method to obtain an upper bound for the distinguishing index of the lexicographic product $K_{1,n}[H]$ where H is a graph of order $m \geqslant 2$. For $n = 1$, since the graph $K_2[H]$ has a nontrivial automorphism, $D'(K_2[H]) \geqslant 2$. We present a distinguishing edge labeling of $K_2[H]$ with two labels. First we label all edges of the first copy of

H with label 1, and all edges of the second copy of H with the label 2. Let $V(K_2) = \{x_1, x_2\}$ and $V(H) = \{y_1, \ldots, y_n\}$ where $n \geqslant 3$. We label the edges $(x_1, y_j)(x_2, y_1), \ldots, (x_1, y_j)(x_2, y_n)$ with $j-1$ labels 2 and $n-(j-1)$ labels 1, where $1 \leqslant j \leqslant n$. By Theorem 2 this labeling is distinguishing, and so $D'(K_2[H]) = 2$. The following proposition gives an upper bound for $D'(K_{1,n}[H])$, when $n \geq 2$.

Proposition 8. *If H is a connected graph of order $m \geqslant 2$ and $K_{1,n}$ is the star graph with $n \geqslant 2$, then $2 \leqslant D'(K_{1,n}[H]) \leqslant \max\left\{D'(H), \lceil \sqrt[m^2]{n} \rceil\right\}$, unless $m = 2$ and $n = r^4$ for some integer r. In the latter case, $2 \leqslant D'(K_{1,n}[K_2]) \leqslant \sqrt[4]{n} + 1$.*

Proof. Since the graph $K_{1,n}[H]$ has a nontrivial automorphism, so $D'(K_{1,n}[H]) \geqslant 2$. Now we present a distinguishing edge labeling of $K_{1,n}[H]$. First we label the edges of each copy of H with $D'(H)$ labels in a distinguishing way. Let d be a positive integer such that $(d-1)^{m^2} < n \leqslant d^{m^2}$. Denote by x_0 the central vertex of the star $K_{1,n}$, by $x_1, \ldots, x_n$ its pendant vertices, and by $y_1, \ldots, y_m$ vertices of H where $m \geqslant 2$. First suppose that $m \geqslant 3$. By Theorem 1 every automorphism of $K_{1,n}[H]$ is of the form $f(x, y) = (\alpha x, \beta_{\alpha x} y)$ where α is an automorphism of $K_{1,n}$ and $\beta_{\alpha x}$ is an automorphism of H. Since we labeled the edges of each copy of H in a distinguishing way, $\beta_{\alpha x}$ is the identity automorphism of H if f is the automorphism of $K_{1,n}[H]$ preserving the labeling.

We want to show that the remaining edges of $K_{1,n}[H]$ can be labeled such that the copies of H also cannot be interchanged, since then the identity automorphism is the only automorphism of $K_{1,n}[H]$ preserving the labeling. A labeling of all edges yet unlabeled can be fully described by defining a matrix L with m^2 rows and n columns such that in the jth column the initial m elements are labels of the edges $(x_0, y_1)(x_j, y_1), \ldots, (x_0, y_1)(x_j, y_m)$, and the next m elements are labels of the edges $(x_0, y_2)(x_j, y_1), \ldots, (x_0, y_2)(x_j, y_m)$, and finally, the last m elements are labels of the edges $(x_0, y_m)(x_j, y_1), \ldots, (x_0, y_m)(x_j, y_m)$. If matrix L contains at least two identical columns, then there exists a permutation of copies of H preserving the labeling, and vice versa. There are exactly d^{m^2} sequences of length m^2 with elements from the set $\{1, \ldots, d\}$, hence there exists a labeling with d colours such that the columns of L are all distinct. Therefore, $D'(K_{1,n}[H]) \leqslant \max\{D'(H), d\} = \max\left\{D'(H), \lceil \sqrt[m^2]{n} \rceil\right\}$.

For $m = 2$, we label the edges of $K_{1,n}[K_2]$ in the same way. The only difference is that each copy of K_2 has only one edge, hence the edges of the copies of K_2 are fixed. This is the case when $n = d^4$, because then each element of $\{1, \ldots, d\}^4$ is a column in L, and there exists a permutation of columns of L which together with the transposition of rows of L defines a non-trivial automorphism of $K_{1,n}[K_2]$ preserving the colouring. Thus we need an additional label for one edge in a copy of K_2. When $n < d^4$, we put the sequence $(1, 1, 1, 2)$ as the first column of L, and we do not use the sequence $(1, 1, 2, 1)$ any more, thus this labeling breaks the transposition of the rows of L, and so all automorphisms of $K_{1,n}[K_2]$. ■

The following proposition implies that the lexicographic product of P_n ($n \geq 3$) with any connected graph, can be distinguished by two edge labels.

Proposition 9. *Let P_n be the path of order $n \geqslant 3$ and H be a connected graph of order $m \geqslant 1$. Then $D'(P_n[H]) = 2$.*

Proof. Since the graph $P_n[H]$ has a nontrivial automorphism, so $D'(P_n[H]) \geqslant 2$. If $m = 1$, then $P_n[H] = P_n$, and so $D'(P_n[H]) = 2$. Let $m \geqslant 2$, we present a 2-distinguishing labeling for $P_n[H]$ as follows: We label all edges of each copy of H with the label 1. If we denote the consecutive vertices of P_n by $x_1, \ldots, x_n$ and vertices of H by $h_1, \ldots h_m$, then for every $1 \leqslant i \leqslant n-2$ we label the edges $(x_i, h_j)(x_{i+1}, h_1), \ldots, (x_i, h_j)(x_{i+1}, h_m)$ with $j-1$ labels 2 and $m-(j-1)$ labels 1 for $1 \leqslant j \leqslant m$. We label the edges $(x_{n-1}, h_j)(x_n, h_1), \ldots, (x_{n-1}, h_j)(x_n, h_m)$ with $j-1$ labels 1 and $m-(j-1)$ labels 2 for $1 \leqslant j \leqslant m$. By Theorem 1, $\mathrm{Aut}(P_n[H]) = \mathrm{Aut}(P_n)[\mathrm{Aut}(H)]$, and so the labeling is distinguishing, because if f is an automorphism of $P_n[H]$ preserving the labeling, then $f(x, h) = (\alpha x, \beta_{\alpha x} h)$, for some automorphism α of P_n and $\beta_{\alpha x}$ of H. With respect to the labeling of edges between copies of H, it is concluded that $\beta_{\alpha x}$ is the identity automorphism on H. Regarding to the labeling of the edges between the first and the second copies of H and the labeling of the edges between the $(n-1)$-th and the last copies of H, it follows that α is the identity automorphism of P_n. Therefore f is the identity automorphism of $P_n[H]$. ∎

The following theorem gives an upper bound for the distinguishing index of $G[K_2]$.

Theorem 10. *Let G be a connected graph with $\mathrm{Aut}(G[K_2]) = \mathrm{Aut}(G)[\mathrm{Aut}(K_2)]$. Then*

$$D'(G[K_2]) \leqslant \min \left\{ k : \sum_{m=2}^{k} \left(2\binom{m-1}{1} + m\binom{m-1}{2} + \binom{m-1}{3} \right) \geqslant D'(G) \right\}.$$

Proof. First we partition the edge set of G by a distinguishing labeling into $D'(G)$ classes, say $[1], \ldots, [D'(G)]$ such that ith class contains the edges of G having label i in the distinguishing labeling of G. Let $[i] = \{e_{i1}, \ldots, e_{is_i}\}$ such that s_i is the size of ith class, where $1 \leqslant i \leqslant D'(G)$. So each of $e_{i1}, \ldots, e_{is_i}$ is replaced by four edges in $G[K_2]$. We denote the set of four edges corresponding to the edge e_{ij} of G by the symbol $\mathbf{E}_{ij}$. For labeling the edges of $G[K_2]$ we first label all copies of K_2 with the label 1. We continue the labeling by the following steps.

Step 1. For every $1 \leqslant j \leqslant s_1$, we label the edges in $\mathbf{E}_{1j}$ with three labels 1 and one label 2.

Step 2. For every $1 \leqslant j \leqslant s_2$, we label the edges in $\mathbf{E}_{2j}$ with three labels 2 and one label 1.

So we labeled the corresponding edges to the edges in the first and second classes of G with labels 1 and 2.

Step 3. For every $1 \leqslant j \leqslant s_3$, we do the same work as Step 1 for labeling the edges in $\mathbf{E}_{3j}$ with the labels 1 and 3. Also for every $1 \leqslant j \leqslant s_4$, we do the same work as Step 2 for labeling the edges in $\mathbf{E}_{4j}$ with the labels 1 and 3.

Step 4. For every $1 \leqslant j \leqslant s_5$ and $1 \leqslant j \leqslant s_6$, we do the same work as Steps 1 and 2, respectively, with the labels 2 and 3.

Step 5. For every $1 \leqslant j \leqslant s_7$, we label four edges in $\mathbf{E}_{7j}$ with the labels $1, 2, 3, 1$. For every $1 \leqslant j \leqslant s_8$, we label four edges in $\mathbf{E}_{8j}$ with the labels $1, 2, 3, 2$. For every $1 \leqslant j \leqslant s_9$, we label four edges in $\mathbf{E}_{9j}$ with the labels $1, 2, 3, 3$.

So we labeled the corresponding edges to the classes $[3], \dots, [9]$ of G with the new label 3.

Step 6. For every $1 \leqslant j \leqslant s_k$, $10 \leqslant k \leqslant 15$ we label four edges in $\mathbf{E}_{10j}$ and $\mathbf{E}_{11j}$ with the labels $1, 4$, the edges in $\mathbf{E}_{12j}$ and $\mathbf{E}_{13j}$ with the labels $2, 4$, and the edges in $\mathbf{E}_{14j}$ and $\mathbf{E}_{15j}$ with the labels $3, 4$ as Step 1 and 2, respectively.

Step 7. For every $1 \leqslant j \leqslant s_k$, $16 \leqslant k \leqslant 27$ we label all four edges in $\mathbf{E}_{16j}, \dots, \mathbf{E}_{19j}$ with the labels $(1, 2, 4, 1), (1, 2, 4, 2), (1, 2, 4, 3), (1, 2, 4, 4)$, the all four edges in $\mathbf{E}_{20j}, \dots, \mathbf{E}_{23j}$ with the labels $(1, 3, 4, 1)$, $(1, 3, 4, 2)$, $(1, 3, 4, 3)$, $(1, 3, 4, 4)$, and all four edges in $\mathbf{E}_{24j}, \dots, \mathbf{E}_{27j}$ with the labels $(2, 3, 4, 1)$, $(2, 3, 4, 2)$, $(2, 3, 4, 3)$, $(2, 3, 4, 4)$, respectively.

Step 8. For every $1 \leqslant j \leqslant s_{28}$ we label the four edges in $\mathbf{E}_{28j}$ with the labels $1, 2, 3, 4$, respectively.

So we labeled nineteen corresponding classes of G with the new label 4. Continuing this method we obtain that the number of corresponding classes of G that can be labeled with the new label m, $m \geqslant 2$ is $2\binom{m-1}{1} + m\binom{m-1}{2} + \binom{m-1}{3}$.

This labeling is distinguishing, because if f is an automorphism of $G[K_2]$ preserving the labeling then there exist the automorphism α of G and $\beta_{\alpha g}$ of K_2 such that $f(g, x) = (\alpha g, \beta_{\alpha g} x)$, where $g \in V(G)$ and $x \in V(K_2)$. With respect to the method of labeling it is concluded that α is the identity automorphism of G, because we labeled the set of four edges corresponding to the edge e_{ij} of G, for every $1 \leqslant j \leqslant s_i$ the same and different from the corresponding edges to the edge e_{kj} of G where $i \neq k$. On the other hand $\beta_{\alpha g}$ is the identity automorphism on K_2, because for each four edges corresponding to an edge of G, none of two distinct labels can not be repeated (at most, one of labels can be repeated). Therefore f is the identity automorphism of $G[K_2]$. Since we used $\min\big\{k : \sum_{m=2}^{k} \big(2\binom{m-1}{1} + m\binom{m-1}{2} + \binom{m-1}{3}\big) \geqslant D'(G)\big\}$ labels, the result follows. ■

Theorem 11. *Let G and H be two connected graphs such that* $\mathrm{Aut}(G[H]) = \mathrm{Aut}(G)[\mathrm{Aut}(H)]$. *If* $|V(G)| \leqslant |E(H)| + 1$, *then* $D'(G[H]) \leqslant 2$.

Proof. Since $|V(G)| \leqslant |E(H)| + 1$, we can label the edges of ith copy of H with $i-1$ labels 1 and $|E(H)| - (i-1)$ labels 2, for every $1 \leqslant i \leqslant |V(G)|$. On the other hand each edge of G is corresponds to $|V(H)|^2$ edges in $G[H]$. Let $V(G) = \{g_1, \ldots, g_{|V(G)|}\}$ and $V(H) = \{h_1, \ldots, h_{|V(H)|}\}$. If $e = g_i g_j$ is an edge of G then e is replaced by the edges $(g_i, h_k)(g_j, h_{k'})$, where $k, k' \in \{1, \ldots, |V(H)|\}$. We label the edges $(g_i, h_p)(g_j, h_1), \ldots, (g_i, h_p)\big(g_j, h_{|V(H)|}\big)$ with $p-1$ labels 2 and $|V(H)| - (p-1)$ labels 1, where $1 \leqslant p \leqslant |V(H)|$. We do the similar labeling for the remaining edges of G.

As every copy of H has a different number of edges with label 2, they can not be interchanged. The same is true for the edges of each copy of H. Therefore the labeling is 2-distinguishing labeling. ■

Corollary 12. *If G is a connected graph such that* $\mathrm{Aut}(G[G]) = \mathrm{Aut}(G)[\mathrm{Aut}(G)]$, *then* $D'(G[G]) \leqslant 2$.

Proof. Since G satisfies the conditions of Theorem 11, the result follows. ■

Corollary 13. *If G is a connected graph such that* $\mathrm{Aut}(G[G]) = \mathrm{Aut}(G)[\mathrm{Aut}(G)]$, *then* $D'(G^k) \leqslant 2$, *for every* $k \geqslant 2$.

Proof. The proof is by induction on k. Let $k = 2$, then the result is obtained from Corollary 12. For the induction step, we apply Theorem 7 by taking $H = G^{k-1}$, because $|V(G)| \leqslant \big|E\big(G^{k-1}\big)\big| + 1$. ■

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