Discussiones Mathematicae
Graph Theory 39 (2019) 41–53
doi:10.7151/dmgt.2069

SOME PROGRESS ON THE DOUBLE ROMAN DOMINATION IN GRAPHS

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Abstract

For a graph $G = (V, E)$, a double Roman dominating function (or just DRDF) is a function $f : V \longrightarrow \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$ for a vertex v, then v has at least two neighbors assigned 2 under f or one neighbor assigned 3 under f, and if $f(v) = 1$, then vertex v must have at least one neighbor w with $f(w) \geq 2$. The weight of a DRDF f is the sum $f(V) = \sum_{v \in V} f(v)$, and the minimum weight of a DRDF on G is the double Roman domination number of G, denoted by $\gamma_{dR}(G)$. In this paper, we derive sharp upper and lower bounds on $\gamma_{dR}(G) + \gamma_{dR}(\overline{G})$ and also $\gamma_{dR}(G)\gamma_{dR}(\overline{G})$, where $\overline{G}$ is the complement of graph G. We also show that the decision problem for the double Roman domination number is NP-complete even when restricted to bipartite graphs and chordal graphs.

Keywords: Roman domination, double Roman domination.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

For notation and terminology not given here the reader is referred to [8]. Let $G = (V, E)$ be a graph of order $n = |V|$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u|uv \in E\}$, and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$.

The *degree* of a vertex v is $\deg(v) = |N(v)|$. The maximum and minimum degree among the vertices of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A vertex of degree one is referred as a *leaf* and its unique neighbor as a *support* vertex. We refer a vertex of degree $n-1$ as a *dominating vertex*, and a vertex of degree 0 as an *isolated vertex*. An *isolated edge* is an edge whose end-vertices are leaves. The open neighborhood of a set $S \subseteq V$ is $N(S) = \bigcup_{v \in S} N(v)$, and the closed neighborhood of S is $N[S] = N(S) \cup S = \bigcup_{v \in S} N[v]$. We denote by $G[S]$ the subgraph of G induced by S. A set $S \subseteq V$ in a graph G is called a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set in G, and a dominating set of G of cardinality $\gamma(G)$ is called a γ-*set* of G.

Let $f : V \longrightarrow \{0, 1, 2\}$ be a function having the property that for every vertex $v \in V$ with $f(v) = 0$, there exists a neighbor $u \in N(v)$ with $f(u) = 2$. Such a function is called a Roman dominating function. The weight of a Roman dominating function is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a Roman dominating function on G is called the Roman domination number of G and is denoted $\gamma_R(G)$. A Roman dominating function on G of weight $\gamma_R(G)$ is called a $\gamma_R(G)$-function (or a γ_R-function of G). The original study of Roman domination was motivated by the defense strategies used to defend the Roman Empire during the reign of Emperor Constantine the Great, 274–337 AD. He decreed that for all cities in the Roman Empire, at most two legions should be stationed. Further, if a location having no legions was attacked, then it must be within the vicinity of at least one city at which two legions were stationed, so that one of the two legions could be sent to defend the attacked city. This part of the history of the Roman Empire gave rise to the mathematical concept of Roman domination, as originally defined and discussed by Stewart [13] in 1999, and ReVelle and Rosing [12] in 2000, and subsequently developed by Cockayne *et al.* [7] in 2004. For references on Roman domination, see for example, [4, 5, 6, 9].

Beeler *et al.* [3] introduced the concept of double Roman domination in graphs. A function $f : V \longrightarrow \{0, 1, 2, 3\}$ is a *double Roman dominating function* (or just DRDF) on a graph G if the following conditions hold, where V_i denotes the set of vertices assigned i under f, for $i = 0, 1, 2, 3$: (1) If $f(v) = 0$, then v must have at least two neighbors in V_2 or one neighbor in V_3; (2) If $f(v) = 1$, then v must have at least one neighbor in $V_2 \cup V_3$. The weight of a DRDF f is the value $w(f) = f(V) = \sum_{v \in V} f(v)$. The *double Roman domination number*, $\gamma_{dR}(G)$, is the minimum weight of a DRDF on G, and a DRDF of G with weight $\gamma_{dR}(G)$ is called a γ_{dR}-function of G. Beeler *et al.* [3] observed that in a DRDF of minimum weight no vertex needs to be assigned the value 1. In fact for every DRDF $f : V \longrightarrow \{0, 1, 2, 3\}$, there is a DRDF $f' : V \longrightarrow \{0, 2, 3\}$ with $w(f') \leq w(f)$. Thus, since $\gamma_{dR}(G)$ is the minimum weight among all double Roman dominating functions on G, without loss of generality, we only consider double Roman dominating functions with no vertex assigned 1. We use the notation $f = (V_0, V_2, V_3)$ for a DRDF $f : V \longrightarrow \{0, 2, 3\}$.

For a graph parameter ρ, bounds on $\rho(G) + \rho(\overline{G})$ and $\rho(G)\rho(\overline{G})$ in terms of the number of vertices are called results of "Nordhaus-Gaddum" type, honoring the paper of Nordhaus and Gaddum [11]. Nordhaus-Gaddum type bounds for several domination parameters are investigated, see for example [2]. Chambers *et al.* [5] investigated Nordhaus-Gaddum type bounds for Roman domination.

In this paper we first present Nordhaus-Gaddum type bounds on the double Roman domination number. We then show that the decision problem for the double Roman domination number is NP-complete even when restricted to bipartite graphs and chordal graphs.

Let $\mathcal{H}$ be the family of connected graphs G of order n that can be built from $n/4$ copies of P_4 by adding a connected subgraph on the set of centers of $\frac{n}{4}P_4$. We make use of the following.

Theorem 1 (Beeler, Haynes, Hedetniemi [3])**.** *If G is a connected graph of order $n \geq 3$, then $\gamma_{dR}(G) \leq \frac{5n}{4}$, with equality if and only if $G \in \mathcal{H}$.*

2. Nordhaus-Gaddum Inequalities

A *good vertex* in a graph G is a vertex that belongs to a minimum dominating set of G. Let *good*(G) denote the set of all good vertices of G, and $G - good(G)$ denotes the subgraph of G induced by $V(G) - good(G)$. Given a graph H, we define an H-*partition* as follows. An H-partition is a partition of $V(H)$ into $k+1$ nonempty sets $A_0, A_1, \ldots, A_k$ for some integer $k < n$ such that the following hold:

(1) If $k \geq 2$, then for $i \geq 1$ the subgraph of H induced by $V(H) - A_i$ has domination number at least two, or a $\gamma(H[V(H) - A_i])$-set is contained in A_0.

(2) If $1 \leq \gamma(H) \leq 2$, then the following hold:

- (2–1) If $\gamma(H) = 1$, then $good(H) \subseteq A_0$; and every $\gamma(H - good(H))$-set has at most one common vertex with $\bigcup_{i=1}^{i=k} A_i$ whenever $\gamma(H - good(H)) = 2$.
- (2–2) If $\gamma(H) = 2$, then $\bigcup_{i=1}^{i=k} A_i$ contains at most one vertex of a $\gamma(H)$-set, for $i = 1, 2, \ldots, k$; otherwise a $\gamma(H)$-set is contained in A_i for $i \in \{1, \ldots, k\}$ and no $\gamma(H)$-set is contained in $\bigcap_{u \in A_0} N(u)$.

Note that for any graph H, $A_0 = V(H)$ is an H-partition, and thus we have the following.

Observation 2. *Every graph H has an H-partition.*

Now we introduce a family of graphs as follows. Let $\mathcal{G}$ be the family of graphs G that can be obtained from an arbitrary graph H as follows. Let $A_0, A_1, \ldots, A_k$

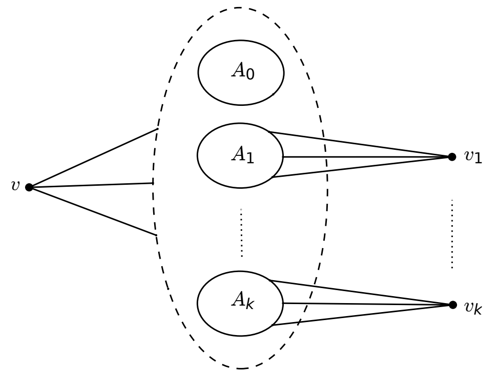

Figure 1. Structure of graphs in the family $\mathcal{G}$.

be an H-partition of H. Then G is obtained from H by adding $k+1$ new vertices $v, v_1, \ldots, v_k$, joining v to all of the vertices of H, and joining v_i to all of the vertices of A_i for $i = 1, 2, \ldots, k$. Figure 1 demonstrates the structure of graphs in the family $\mathcal{G}$.

Theorem 3. *If G is graph of order n, then $\gamma_{dR}(G) \leq 2n - 2\Delta(G) + 1$, with equality if and only if $G \in \mathcal{G}$.*

Proof. Let G be a graph of order n and v be a vertex of maximum degree. Then the function $h = (N(v), V(G) - N[v], \{v\})$ is a DRDF for G of weight $2n - 2\Delta(G) + 1$, and thus $\gamma_{dR}(G) \leq 2n - 2\Delta(G) + 1$.

We next prove the equality part. Assume that $\gamma_{dR}(G) = 2n - 2\Delta + 1$. Let $v \in V(G)$ be a vertex of maximum degree. We form a partition P of $N(v)$ as follows: If $V(G) - N[v] = \emptyset$, then $P : A_0$, where $A_0 = N(v)$, is the desired partition. Thus assume that $V(G) - N[v] \neq \emptyset$. If $N(v) \subseteq N(V(G) - N[v])$, then the function $g = (N(v), V(G) - N(v), \emptyset)$ is a DRDF for G of weight less than $2n - 2\Delta(G)+1$, a contradiction. Thus $N(v) - N(V(G) - N[v]) \neq \emptyset$. Let $V(G) - N[v] = \{v_1, v_2, \ldots, v_k\}$. Let $A_0 = N(v) - N(V(G) - N[v])$ and $A_i = N(v_i)$, for $i = 1, ..k$. Clearly, $\bigcup_{i=0}^{i=k} A_i = N(v)$. To show that $P : A_0, A_1, \ldots, A_k$ is a partition of $N(v)$, we need to show that any pair A_i, A_j $(i \neq j)$ are disjoint. We prove a stronger result by showing that $V(G) - N[v]$ is a 2-packing in G. Assume that there exist two vertices $v_i, v_j \in V(G) - N[v]$ such that $N[v_i] \cap N[v_j] \neq \emptyset$ and $u \in N[v_i] \cap N[v_j]$. Then the function $g = ((N(v) \cup \{v_i, v_j\}) - \{u\}, V(G) - (N[v] \cup \{v_i, v_j\}), \{v, u\})$ is a DRDF for G of weight less than $2n - 2\Delta(G) + 1$, a contradiction. We deduce that $V(G) - N[v]$ is a 2-packing. Thus $P : A_0, A_1, \ldots, A_k$ is the desired partition of $N(v)$.

Let H be the subgraph induced by $N(v)$. We show that P is an H-partition. To check the first condition for being an H-partition, assume that $k \geq 2$ and suppose there exists $i \in \{1, 2, \ldots, k\}$, such that the subgraph of H induced by

$V(H) - A_i$ has domination number one. Let $\{x\}$ be a minimum dominating set for the subgraph of H induced by $V(H) - A_i$. If $x \in A_j$ for $j \neq 0$, then the function $g = ((N[v] \cup \{v_j\}) - \{x\}, \{v_1, v_2, \ldots, v_k\} - \{v_i, v_j\}, \{x, v_i\})$ is a DRDF for G of weight less than $2n - 2\Delta(G) + 1$, a contradiction. Thus $x \in A_0$ and therefore the first condition for being an H-partition holds. Now we investigate the second condition for being an H-partition. Thus assume that $1 \leq \gamma(H) \leq 2$.

Assume that $\gamma(H) = 1$. If $x \in good(H)$ and $x \in A_i$ for some $i \neq 0$, then $\deg(x) = \Delta + 1$, a contradiction. Thus $good(H) \subseteq A_0$. Assume that $\gamma(H - good(H)) = 2$. Let $\{x, y\}$ be a $\gamma(H - good(H))$-set. If $\{x, y\} \subseteq A_i$ for some $i \in \{1, 2, \ldots, k\}$, then the function $g = ((N[v] - \{x, y\}) \cup \{v_i\}, \{x, y\} \cup V(G) - (N[v] \cup \{v_i\}), \emptyset)$ is a DRDF for G of weight less than $2n - 2\Delta(G) + 1$, a contradiction. If $x \in A_i$ and $y \in A_j$ for $i \neq j$, then the function $g = ((N[v] - \{x, y\}) \cup \{v_i, v_j\}, V(G) - (N[v] \cup \{v_i, v_j\}), \{x, y\})$ is a DRDF for G of weight less than $2n - 2\Delta(G) + 1$, a contradiction.

Next assume that $\gamma(H) = 2$. We show that $\bigcup_{i=1}^{i=k} A_i$ contains at most one vertex of a $\gamma(H)$-set, for $i = 1, 2, \ldots, k$, or $\gamma(H) \subseteq A_i$ for $i \in \{1, \ldots, k\}$ and $\gamma(H) \not\subseteq \bigcap_{u \in A_0} N(u)$. Assume that $\{x, y\} \subseteq V(H)$ is an arbitrary $\gamma(H)$-set. If $\{x, y\} \subseteq A_i$ for some $i = 1, 2, \ldots, k$ and $\gamma(H) \subseteq \bigcap_{u \in A_0} N(u)$, then the function $g = ((N[v] - \{x, y\}) \cup \{v_i\}, \{x, y\} \cup V(G) - (N[v] \cup \{v_i\}), \emptyset)$ is a DRDF for G of weight less than $w(f)$, a contradiction. Thus assume that $x \in A_i$ and $y \in A_j$ for some $i \neq j$. Then the function $g = ((N[v] - \{x, y\}) \cup \{v_i, v_j\}, V(G) - (N[v] \cup \{v_i, v_j\}), \{x, y\})$ is a DRDF for G of weight less than $2n - 2\Delta(G) + 1$, a contradiction.

We conclude that P is an H-partition. If $V(G) - N[v] = \emptyset$, then clearly $G \in \mathcal{G}$. Thus assume that $V(G) - N[v] \neq \emptyset$. Since $V(G) - N[v]$ is a 2-packing in G, it is an independent set in G as well. Consequently, $G \in \mathcal{G}$.

Conversely, let $G \in \mathcal{G}$ and $v \in V(G)$ be a vertex with $\deg(v) = \Delta(G)$. If $\deg(v) = n - 1$, then $\gamma_{dR}(G) = 3 = 2n - 2\Delta(G) + 1$. Thus assume that $\deg(v) < n - 1$. Let G be obtained from an arbitrary graph H with H-partition $A_0, A_1, \ldots, A_k$ by adding new vertices $v, v_1, v_2, \ldots, v_k$ as described in the construction of the family $\mathcal{G}$. We show that $\gamma_{dR}(G) \geq 2k + 3$. Let $f = (V_0, V_2, V_3)$ be a $\gamma_{dR}(G)$-function. Clearly $f(N[v_i]) \geq 2$ for $i = 1, 2, \ldots, k$, since f is a $\gamma_{dR}(G)$-function. If $f(v) = 3$, then $\gamma_{dR}(G) \geq 2k + 3$. Assume that $f(v) = 2$. If there exists a vertex $x \in A_0$ with $f(x) \geq 2$, then $w(f) > 2k + 3$, since $f(N[v_i]) \geq 2$ for $i = 1, 2, \ldots, k$. Thus assume that $A_0 \subseteq V_0$. Hence there exists a vertex $u \in N(v) - A_0$ with $f(u) \geq 2$, and we may assume that $u \in N[v_1]$. Then $f(N[v_1]) \geq 3$ and $f(N[v_i]) \geq 2$ for $i = 2, \ldots, k$ and thus $w(f) \geq 5 + 2(k - 1) = 2k + 3$.

Assume next that $f(v) = 0$. If $A_0 \cap V_3 \neq \emptyset$ or $|A_0 \cap V_2| \geq 2$, then $w(f) \geq 2k+3$, since $f(N[v_i]) \geq 2$ for $i \geq 1$. Thus assume that $A_0 \cap V_3 = \emptyset$ and $|A_0 \cap V_2| \leq 1$. If $|A_0 \cap V_2| = 1$, then there exists a vertex $u \in N(v) - A_0$ with $f(u) \geq 2$, since $f(N(v)) \geq 3$. We may assume $u \in N[v_1]$. Then $f(N[v_1]) \geq 3$ and thus

$w(f) \geq 5 + 2(k-1) = 2k+3$, since $f(N[v_i]) \geq 2$ for $i = 2, \ldots, k$. Thus assume that $|A_0 \cap V_2| = 0$ and so $A_0 \subseteq V_0$.

We show that $V_3 \neq \emptyset$. Suppose to the contrary, that $V_3 = \emptyset$. Then v has at least two neighbors in V_2. Assume that $|N(v) \cap V_2| \geq 3$. If $v_i \in V_0$, for some i, where $1 \leq i \leq k$, then $|N(v_i) \cap V_2| \geq 2$. Hence $|\{v_1, v_2, \ldots, v_k\} \cap V_0| \leq \lfloor |N(v) \cap V_2|/2 \rfloor$. Then $w(f) = 2|N(v) \cap V_2| + 2(k - |\{v_1, v_2, \ldots, v_k\} \cap V_0|) \geq 2|N(v) \cap V_2| + 2(k - \lfloor |N(v) \cap V_2|/2 \rfloor) > 2k+3$, a contradiction. Thus assume that $v_i \notin V_0$, for each $i \in \{1, \ldots, k\}$. Then similarly we obtain a contradiction. We deduce that $|N(v) \cap V_2| = 2$. Let $N(v) \cap V_2 = \{x, y\}$. Then $\{x, y\}$ is a dominating set for H and $\{x, y\} \subseteq \bigcap_{u \in A_0} N(u)$. By the structure of $\mathcal{G}$, if $\gamma(H) = 1$, then $good(H) \subseteq A_0$ and $\gamma(H - good(H)) = 2$. Hence $\{x, y\}$ is a $\gamma(H - good(H))$-set, a contradiction, since every $\gamma(H - good(H))$-set has at most one common vertex with $\bigcup_{i=1}^{i=k} A_i$ whenever $\gamma(H - good(H)) = 2$. Thus we may assume that $\gamma(H) = 2$. Since $\{x, y\} \subseteq \bigcup_{i=1}^{i=k} A_i$, there exists $i \in \{1, \ldots, k\}$ such that $\{x, y\} \subseteq A_i$ and $\{x, y\} \not\subseteq \bigcap_{u \in A_0} N(u)$, a contradiction. Hence $V_3 \neq \emptyset$.

We proceed according to the size of V_3. Assume that $|V_3| = 1$. Let $V_3 = \{x\}$. Assume first that $x \in \{v_1, v_2, \ldots, v_k\}$. Without loss of generality assume that $x = v_1$. Since $f(v) = 0$, there are at least two vertices $a, b \in N(v)$, with weight 2. If $\{a, b\} \cap A_1 \neq \emptyset$, then clearly $w(f) \geq 7 + 2(k-1) > 2k+3$. Hence assume that $\{a, b\} \cap A_1 = \emptyset$. If $\{a, b\} \subseteq A_i$ for some $2 \leq i \leq k$, then for $j \notin \{i, 1\}$, $f(N[v_j]) \geq 2$ and so $w(f) \geq 7 + 2(k-2) \geq 2k+3$. Assume that $a \in A_i$ and $b \in A_j$ for $i \neq j$. Then $f(N[v_i]) \geq 4$ and $f(N[v_j]) \geq 4$. Since $f(N[v_r]) \geq 2$ for $r \notin \{1, i, j\}$, we have $w(f) \geq 11 + 2(k-3) > 2k+3$. Thus we may assume $x \notin \{v_1, v_2, \ldots, v_k\}$. Since $A_0 \subseteq V_0$, we have $x \in A_j$, for some integer j, where $1 \leq j \leq k$. Assume that $\gamma(H) \geq 2$. Then there exists a vertex w such that $w \notin N(x)$. Hence $f(w) = 2$ or w has at least two neighbors x, y in V_2. By the structure of $\mathcal{G}$, $|\{x, y\} \cap \{v_1, v_2, \ldots, v_k\}| \leq 1$. Since $A_0 \subseteq V_0$, we have $\{x, y\} \cap \bigcup_{i=1}^{i=k} A_i \neq \emptyset$. Therefore there exists at least one vertex $u \in \bigcup_{i=1}^{i=k} A_i$ such that $f(u) = 2$. If $\gamma(H) = 1$, then $x \notin good(H)$ and $\gamma(H - good(H)) \geq 2$, and as before, there is a vertex $u \in \bigcup_{i=1}^{i=k} A_i$ such that $f(u) = 2$. Hence $\gamma_{dR}(G) \geq 2(k-1) + 3 + 2 \geq 2k+3$. Next assume that $|V_3| = 2$. Let $V_3 = \{x, y\}$. Assume that $\{x, y\} \subseteq V(G) - N[v]$. Then $|N(v) \cap V_2| \geq 2$. Clearly $|\{v_1, v_2, \ldots, v_k\} \cap V_0| \leq |N(v) \cap V_2|/2$. Thus $w(f) = 6 + 2|N(v) \cap V_2| + 2(k - 2 - |\{v_1, v_2, \ldots, v_k\} \cap V_0|) \geq 6 + 2|N(v) \cap V_2| + 2(k - 2 - |N(v) \cap V_2|/2) = 2k + |N(v) \cap V_2| + 2 > 2k+3$. Now assume that $|\{x, y\} \cap V(G) - N[v]| = 1$. We may assume that $x = v_1$. If $k = 1$, then $\gamma_{dR}(G) \geq 6 > 2k+3$. Thus assume that $k \geq 2$. By the structure of $\mathcal{G}$, there is a vertex $u \in \bigcup_{i=2}^{i=k} A_i$ such that $f(u) = 2$. Therefore, $\gamma_{dR}(G) \geq 2(k-2) + 8 > 2k+3$. Now we assume that $|\{x, y\} \cap V(G) - N[v]| = 0$. If $\{x, y\} \subseteq A_j$ for some integer $j \in \{1, 2, \ldots, k\}$, then $w(f) \geq 2(k-1) + 6 > 2k+3$. Thus assume that $\{x, y\} \not\subseteq A_j$ for any integer $j \in \{1, 2, \ldots, k\}$. Then there are integers r and s such that $x \in A_r$ and $y \in A_s$. By the construction of G, $\{x, y\}$ is not a dominating set for H, and so there is a vertex $u \in V(H)$ such that u is not dominated by $\{x, y\}$. If $f(u) = 0$ then $f(N[u]) \geq 4$, and we obtain that $\gamma_{dR}(G) \geq 2(k-2) + 6 + 4 > 2k + 3$. Thus assume that $f(u) = 2$. This time we obtain $\gamma_{dR}(G) \geq 2(k-2) + 6 + 2 > 2k+3$. It remains to assume that $|V_3| \geq 3$. Let $x, y, z \in V_3$. If $\{x, y, z\} \subseteq N[v_i]$ for some $i \in \{1, 2, \ldots, k\}$ then replacing $f(v)$ by 3, $f(v_i)$ by 2, and $f(u)$ by 0 for $u \in \{x, y, z\} - \{v_i\}$ yields a DRDF for G of weight less than $\gamma_{dR}(G)$, a contradiction. Assume that there are two integers r and s such that $x \in N[v_r]$ and $\{y, z\} \subseteq N[v_s]$. Then replacing $f(v)$ by 3, $f(v_r)$ and $f(v_s)$ by 2, and $f(u)$ by 0 for $u \in \{x, y, z\} - \{v_r, v_s\}$ yields a DRDF for G of weight less than $\gamma_{dR}(G)$, a contradiction. Thus, without loss of generality, assume that $x \in N[v_1]$, $y \in N[v_2]$ and $z \in N[v_3]$. Let g be defined on $V(G)$ by $g(v) = 3$, $g(u) = 2$ if $u \in \{v_1, v_2, v_3\}$, $g(u) = 0$ if $u \in \{x, y, z\}$ and $g(u) = f(u)$ if $u \in V(G) - \{v, v_1, v_2, v_3, x, y, z\}$. Then g is a $\gamma_{dR}(G)$-function with $g(v) = 3$, which has been considered formerly. Hence $\gamma_{dR}(G) \geq 2k + 3 = 2n - 2\Delta + 1$. Consequently, $\gamma_{dR}(G) = 2n - 2\Delta + 1$. ■

We are now ready to state the first result on the Nordhaus-Gaddum type inequalities of a graph.

Theorem 4. *If G is a graph of order $n \geq 2$, then $7 \leq \gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 2n+3$. Equality holds for the lower bound if and only if G or $\overline{G}$ is K_2, and equality holds for the upper bound if and only if G or $\overline{G}$ is a complete graph.*

Proof. Clearly $\gamma_{dR}(G) \geq 3$, since G has $n \geq 2$ vertices. Furthermore, $\gamma_{dR}(G) = 3$ if and only if G has a dominating vertex. Since a graph and its complement cannot both have dominating vertices, $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \geq 7$. Assume that the equality holds. Without loss of generality, assume that $\gamma_{dR}(G) = 3$ and $\gamma_{dR}(\overline{G}) = 4$. As noted, G has a dominating vertex, say x. Since x is an isolated vertex in $\overline{G}$, we find that $n = 2$, and consequently, $G = K_2$. For the upper bound, Theorem 3 yields

$$\begin{aligned}\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) &\leq (2n - 2\Delta(G) + 1) + (2n - 2\Delta(\overline{G}) + 1) \\ &= 2n - 2\Delta(G) + 2\delta(G) + 4 \leq 2n + 4.\end{aligned}$$

If $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) = 2n+4$, then equality holds throughout the calculation, and $\delta(G) = \Delta(G)$. Hence G is k-regular for some k. Moreover, $\gamma_{dR}(G) = 2n-2k+1$ and $\gamma_{dR}(\overline{G}) = 2k+3$. Let $v \in V(G)$. If some vertex u outside $N[v]$ has a neighbors outside $N[v]$, then the DRDF $(N(u) \cup N(v), V(G) - N[u] - N[v], \{u, v\})$ has weight at most $2n - 2k$, a contradiction. If there is a vertex u such that $u \notin N[v]$, then we have $N(u) = N(v)$ and the DRDF $(N(u), V(G) - N(v), \emptyset)$ has weight $2n-2k$, a contradiction. Hence $k = \deg(v) = n-1$, and so G is a complete graph. But then $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) = 2n + 3$, a contradiction. We conclude that $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 2n + 3$. We next prove the equality part. Assume that $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) = 2n + 3$. If $\Delta(G) \geq \delta(G) + 1$, then we can easily see that

$\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) < 2n+3$, a contradiction. Thus $\delta(G) = \Delta(G)$, and so G is k-regular for some integer k. If $\gamma_{dR}(G) \leq 2n - 2\Delta(G)$ and $\gamma_{dR}(\overline{G}) \leq 2n - 2\Delta(\overline{G})$, then we see that $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) < 2n+3$, a contradiction. Thus without loss of generality, assume that $\gamma_{dR}(G) = 2n - 2\Delta(G) + 1$ and so $\gamma_{dR}(\overline{G}) = 2\Delta(G) + 2$. Let $v \in V(G)$. If a vertex u outside $N[v]$ has neighbors outside $N[v]$, then the function $f = (N(u) \cup N(v), V(G) - N[u] - N[v], \{u, v\})$ is a DRDF for graph G. Hence $\gamma_{dR}(G) \leq w(f) = 6 + 2(n - \Delta(G) - 1 - |N[u] \cap (V(G) - N[v])|) \leq 6 + 2(n - k - 3) = 2n - 2k$, a contradiction. If there is a vertex u such that $u \notin N[v]$, then we have $N(u) = N(v)$ and the DRDF $(N(u), V(G) - N(v), \emptyset)$ has weight $2n - 2k$, a contradiction. Hence $k = deg(v) = n - 1$, and so G is a complete graph. ■

As an immediate consequence of Theorem 4, if G is a graph of order $n \geq 2$ and G is not complete, then $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 2n+2$. We next characterize all graphs achieving equality for this bound.

Proposition 5. *If G is a graph of order $n \geq 2$, then $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) = 2n+2$ if and only if G is C_5, P_4 or $K_n - e$ (a complete graph minus an edge).*

Proof. Assume first that $G \in \mathcal{G}$ (described before Theorem 3). Thus G is obtained from an arbitrary graph H with an H-partition $A_0, A_1, \ldots, A_k$ by adding new vertices $v, v_1, \ldots, v_k$, joining v to all of the vertices of H, and joining v_i to all of the vertices of A_i for $i = 1, 2, \ldots, k$.

Assume that $n - \Delta(G) - 1 \geq 2$. Then the construction of G implies that $\Delta(G) \geq 3$. If $\{x, y\} \subseteq V(G) - N[v]$, then $\{x, y\}$ is a dominating set for $\overline{G}$ and so $\gamma_{dR}(\overline{G}) \leq 6$. Thus $2n + 2 = \gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq (2n - 2\Delta(G) + 1) + 6 = 2n - 2\Delta(G) + 7$, and so $\Delta(G) \leq 2$, a contradiction. Thus $n - \Delta(G) - 1 \leq 1$. Assume that $n - \Delta(G) - 1 = 0$. Then $\gamma_{dR}(G) = 3$, and so $\gamma_{dR}(\overline{G}) = 2n - 1$, and it can be easily seen that $\overline{G}$ has one component K_2 and $|V(G)| - 2$ components K_1. This implies that $G = K_n - e$. Next assume that $n - \Delta(G) - 1 = 1$. Then $\gamma_{dR}(G) = 5$. If $|A_0| \geq 2$, then the function $g = (A_0 \cup \{v\}, V(G) - (A_0 \cup \{v, v_1\}), \{v_1\})$ is a DRDF on $\overline{G}$, and so $\gamma_{dR}(\overline{G}) \leq 3 + 2(n - |A_0| - 2) \leq 2n - 5$. Therefore, $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 2n$, a contradiction. Thus $|A_0| = 1$. Let $A_0 = \{v_0\}$. If there are two vertices $\{x, y\} \subseteq A_1$ such that $e = xy \in E(\overline{G})$, then the function $g = (\{y, v, v_0\}, V(G) - \{y, v, v_0, x, v_1\}), \{x, v_1\})$ is a DRDF for $\overline{G}$, and so $\gamma_{dR}(\overline{G}) \leq 6 + 2(n-5) = 2n - 4$, hence $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 2n + 1$, a contradiction. Hence the subgraph H induced by A_1 is a complete graph. If there is a vertex $x \in N(v_0) \cap A_1$, then $\{x\}$ is a dominating set for G and so $\gamma_{dR}(G) = 3$, a contradiction. Thus $N(v_0) \cap A_1 = \emptyset$. If $|A_1| \geq 2$ and $\{x, y\} \subseteq A_1$, then the function $g = (\{y, v, x, y\}, V(G) - \{y, v, v_0, x, v_1\}), \{v_0, v_1\})$ is a DRDF for $\overline{G}$, and so $\gamma_{dR}(\overline{G}) \leq 6 + 2(n-5) = 2n - 4$, hence $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 2n + 1$, a contradiction. Hence $|A_1| = 1$ and so $G = P_4$.

Now assume that $G, \overline{G} \notin \mathcal{G}$. By Theorem 3, $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq (2n - 2\Delta(G)) + (2n - 2\Delta(\overline{G})) = 2n - 2\Delta(G) + 2\delta(G) + 2 \leq 2n + 2$. If $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) = 2n + 2$, then equality holds throughout the calculation, and $\delta(G) = \Delta(G)$. Hence G is k-regular for some k. We may assume that $k \leq (n-1)/2$, since the argument is symmetric in G and $\overline{G}$. Since the equality holds, we have $\gamma_{dR}(G) = 2n - 2k$ and $\gamma_{dR}(\overline{G}) = 2k + 2$. Let $v \in V(G)$. If a vertex u outside $N[v]$ has at least two neighbors outside $N[v]$, then the function $f = (N(u) \cup N(v), V(G) - N[u] - N[v], \{u, v\})$ is a DRDF for graph G. Hence $\gamma_{dR}(G) \leq w(f) = 6 + 2(n - \Delta(G) - 1 - |N[u] \cap (V(G) - N[v])|) \leq 6 + 2(n - k - 4) = 2n - 2k - 2$, a contradiction. Therefore, every vertex outside $N[v]$ has at least $k - 1$ neighbors in $N(v)$. Now assume that some vertex w in $N(v)$ has at least three neighbors outside $N[v]$. Then function $f' = (N(u) \cup N(v), V(G) - N[u] - N[v], \{w, v\})$ is a DRDF for graph G, and so $\gamma_{dR}(G) \leq w(f') = 6 + 2(n - \Delta(G) - 1 - |N(w) \cap (V(G) - N[v])|) \leq 6 + 2(n - k - 4) = 2n - 2k - 2$ a contradiction. Thus every vertex in $N(v)$ has at most two neighbors outside $N[v]$. Counting the edges joining $N(v)$ and $V(G) - N[v]$ from both sides yields $(k-1)(n-k-1) \leq 2k$, implying that $n \leq k + 3 + \frac{2}{k-1}$ for $k > 1$. Since $n \geq 2k + 1$, we have $k^2 \leq 3k$, which implies that $k \leq 3$. If $k = 3$, then $n = 7$, a contradiction, since there is no 3-regular 7-vertex graph. If $k = 0$, then the only graph G is $\bar{K_n}$, and we observe that the equality does not hold, a contradiction. Thus $k = 2$. Now we find that $5 = 2k + 1 \leq n \leq k + 3 + \frac{2}{k-1} = 7$. If $n = 6$, then we have $\gamma_{dR}(G) = 6 < 2n - 2k$, and if $n = 7$, then we have $\gamma_{dR}(G) = 8 < 2n - 2k$, both of which is a contradiction. Thus $n = 5$ and so $G = C_5$. ■

The following theorem provides an upper bound for the double Roman domination number. The method of proof is in similar lines with those presented for domination number and Roman domination number, [1, 7].

Theorem 6. *For a graph G on n vertices,*

$$\gamma_{dR}(G) \leq 3n \frac{\ln 2(1 + \delta) - \ln 3 + 1}{1 + \delta}.$$

Proof. Given a graph G, select a set of vertices A, where each vertex is selected independently with probability p (with p to be defined later). The expected size of A is np. Let $B = V - N[A]$. Clearly $f = (V - (A \cup B), B, A)$ is an *DRDF* for G. We now compute the expected size of B. The probability that v is in B is equal to the probability that v is not in A and that no vertex in A is the neighbor of v. This probability is $(1-p)^{1+\deg(v)}$. Since $e^{-x} \geq 1 - x$ for any $x \geq 0$, and $\deg(v) \geq \delta(G)$, we can conclude that $Pr(v \in B) \leq e^{-p(1+\delta(G))}$. Thus, the expected size of B is at most $ne^{-p(1+\delta(G))}$, and the expected weight of f, denoted $E[f(V)]$, is at most $3np + 2ne^{-p(1+\delta(G))}$. The upper bound for $E[f(V)]$ is minimized when $p = \ln(2(1 + \delta(G))/2)/(1 + \delta(G))$ and substituting this value for p gives

$$E[f(V)] \leq 3n\frac{\ln 2(1+\delta) - \ln 3 + 1}{1+\delta}. \tag{1}$$

Since the expected weight of $f(V)$ is at most $3n\frac{\ln 2(1+\delta)-\ln 3+1}{1+\delta}$, there must be some DRDF with at most this weight. ∎

The technique used in the following theorem is in similar lines to those presented in [5] for Roman domination.

Theorem 7. *If G is a graph of order $n \geq 240$ with $diam(G) = diam(\overline{G}) = 2$, then*

$$\gamma_{dR}(G)\gamma_{dR}(\overline{G}) < \frac{15}{2}n.$$

Proof. Let G be a graph of order $n \geq 240$ with $diam(G) = diam(\overline{G}) = 2$, and let v be a vertex of minimum degree in G. If $\deg(v) \leq 2$, then the diameter constraint implies that $(V(G) - N(v), \emptyset, N(v))$ is a DRDF of G and $(V(G) - N[v], \emptyset, N(v), v)$ is a DRDF of $\overline{G}$, and so $\gamma_{dR}(G)\gamma_{dR}(\overline{G}) \leq (3|N(v)|)(3 + 2|N(v)|) \leq 6 \times 7 = 42 < \frac{15}{2}n$. Hence we may assume that $\deg(v) \geq 3$. Let $R = V(G) - N_G[v]$. We choose a family of disjoint subsets of $N_G(v)$ dominating R as follows. Initialize $B_1 = N_G(v)$; note that B_1 dominates R, since $diam(G) = 2$. If B_i dominates R, then let A_i be a minimal subset of B_i dominating R, and let $B_{i+1} = B_i - A_i$. If B_{i+1} does not dominate R, then stop, setting $q = i$ and $A^* = B_{q+1}$. Otherwise, increment i. Note that $A_1, \ldots, A_q$ is a partition of $N_G(v) - A^*$, with each A_i being a minimal set that dominates R.

Since A_i is a minimal dominating set for R, there is a vertex $r_i \in R$ having only one neighbor in A_i. Let a_i be this neighbor. Since A^* does not dominate R, there exists $w \in R$ such that $A^* \subseteq N_{\overline{G}}(w)$. Let $S = \{r_1, \ldots, r_q\} \cup \{v, w\}$ and $T = \{a_1, \ldots, a_q\}$. Now $(V(G) - (S \cup T), T, S)$ is a DRDF for $\overline{G}$, since v dominates R, w dominates A^*, and r_i dominates $A_i - \{a_i\}$. Thus $\gamma_{dR}(\overline{G}) \leq 5q + 6$, which reduces to $5q + 3$ if $A^* = \emptyset$.

Let $U = A_j \cup \{v\}$, where $|A_j| = \min_i |A_i|$. Note that U is a dominating set of G. If $|U| = 2$, then $\gamma_{dR}(G) \leq 6$. Since $\overline{G}$ is connected and $n \geq 3$, Theorem 1 yields $\gamma_{dR}(\overline{G}) \leq \frac{5n}{4}$. If $\gamma_{dR}(\overline{G}) < \frac{5n}{4}$, then clearly $\gamma_{dR}(G)\gamma_{dR}(\overline{G}) < \frac{15}{2}n$. Now assume that $\gamma_{dR}(\overline{G}) = \frac{5n}{4}$. Then by Theorem 1, $\overline{G} \in \mathcal{H}$ and so $\gamma_{dR}(G) \leq 5$. Therefore, $\gamma_{dR}(G)\gamma_{dR}(\overline{G}) \leq \frac{25n}{4} < \frac{15}{2}n$. Hence we may assume that $|U| > 2$, which requires $q \leq \delta(G)/2$. If $q = 1$, then $\gamma_{dR}(\overline{G}) \leq 11$ and $\gamma_{dR}(G) \leq 3|U| \leq 3(\delta(G) + 1)$, and so $\gamma_{dR}(G)\gamma_{dR}(\overline{G}) \leq 33(\delta(G) + 1)$. Hence we may assume in this case that $\delta(G) \geq 5n/22 - 1$, but now Theorem 6 yields $\gamma_{dR}(G) \leq 66/5\left(\ln \frac{5}{33}n + 1\right)$. Using Calculus (or MATLAB) it can be seen that $11 \cdot 66/5\left(\ln \frac{5}{33}n + 1\right) < \frac{15n}{2}$, when $n \geq 21$, and thus $\gamma_{dR}(G)\gamma_{dR}(\overline{G}) < \frac{15n}{2}$.

Hence we may assume that $2 \leq q \leq \delta(G)/2$. Using the DRDF $(V(D) - U, \emptyset, U)$, we have $\gamma_{dR}(G)\gamma_{dR}(\overline{G}) \leq \left(\frac{3\delta(G)}{q} + 3\right) \cdot (5q + 6) \leq (15\delta(G) + 12) + \left(10q + \frac{6\delta(G)}{q}\right) \leq 21\delta(G) + 21$. (Note that since $q \leq \delta/2 \leq 5\delta/9$, we have $9(q-1) \leq 5\delta(q-1)/q$, and therefore $9q + 5\delta/q \leq 5\delta + 9$. On the other hand $q + \delta/q \leq \delta$, hence $10q + 6\delta/q \leq 6\delta + 9$.)

Since $21\delta(G) + 21 < \frac{15n}{2}$ when $\delta(G) < \frac{15n}{42} - 1$, we may assume that $\delta(G) \geq \frac{15n}{42} - 1$, and similarly $\delta(\overline{G}) \geq \frac{15n}{42} - 1$. By Theorem 6, $\max\left\{\gamma_{dR}(G), \gamma_{dR}(\overline{G})\right\} \leq \frac{42}{5}(\ln 5n - \ln 21 + 1)$. Hence $\gamma_{dR}(G)\gamma_{dR}(\overline{G}) \leq \left(\frac{42}{5}(\ln 5n - \ln 21 + 1)\right)^2$. Using Calculus (or MATLAB) it can be seen that for $n \geq 240$, this bound is less than $\frac{15n}{2}$. ■

Theorem 8. *If G is a graph of order $n \geq 3$ with $diam(G) \geq 3$, then*

$$\gamma_{dR}(G)\gamma_{dR}(\overline{G}) < \frac{15}{2}n.$$

Proof. If G has an isolated vertex or edge, then it is easily seen that $\gamma_{dR}(\overline{G}) \leq 5$, and so $\gamma_{dR}(G)\gamma_{dR}(\overline{G}) \leq 5n < \frac{15}{2}n$. Thus we may assume that each component of G has at least three vertices. Applying Theorem 1 to each component now yields $\gamma_{dR}(G) \leq 5n/4$. Since $diam(G) \geq 3$, G has two vertices u and v with no common neighbor. Then $\{u, v\}$ is a dominating set in $\overline{G}$ and $\gamma_{dR}(\overline{G}) \leq 6$. If $\gamma_{dR}(\overline{G}) < \frac{5n}{4}$, then clearly $\gamma_{dR}(G)\gamma_{dR}(\overline{G}) < \frac{15}{2}n$. Now assume that $\gamma_{dR}(\overline{G}) = \frac{5n}{4}$. By Theorem 1, $\overline{G} \in \mathcal{H}$ and so $\gamma_{dR}(G) \leq 5$. Hence $\gamma_{dR}(G)\gamma_{dR}(\overline{G}) \leq \frac{25n}{4} < \frac{15}{2}n$. ■

We next improve Theorems 7 and 8 for graphs with minimum degree one.

Theorem 9. *If G is a graph of order $n \geq 3$ with $\delta(G) = 1$, then*

$$\gamma_{dR}(G)\gamma_{dR}(\overline{G}) \leq \frac{25}{4}n,$$

with equality only if and only if G or $\overline{G}$ belongs to $\mathcal{H}$.

Proof. If G has an isolated edge, then $\gamma_{dR}(\overline{G}) = 5$, and so $\gamma_{dR}(G)\gamma_{dR}(\overline{G}) \leq 5n < \frac{25n}{4}$. Thus we may assume that each component of G has at least three vertices. Applying Theorem 1 to each component now yields that $\gamma_{dR}(G) \leq 5n/4$. Assume that $\deg(u) = 1$ and $N(u) = \{w\}$, then the function $f = (V(G) - \{u, w\}, \{w\}, \{u\})$ is a DRDF on $\overline{G}$ and so $\gamma_{dR}(\overline{G}) \leq 5$. Thus $\gamma_{dR}(G)\gamma_{dR}(\overline{G}) \leq \frac{25n}{4}$. Assume that $\gamma_{dR}(G)\gamma_{dR}(\overline{G}) = \frac{25n}{4}$. Then we may assume that $\gamma_{dR}(\overline{G}) = 5$ and $\gamma_{dR}(G) = \frac{5n}{4}$. By Theorem 1, $G \in \mathcal{H}$. ■

3. Complexity

In this section we show that the double Roman domination problem is NP-complete for bipartite graphs and chordal graphs. Consider the following decision problem. Note that a *chordal* graph is a graph with no induced cycle of length at least four.

> **Double Roman domination problem** (LRDP).
> **Instance**: Graph $G = (V, E)$, and an integer k.
> **Question**: Does G have a DRDF of weight at most k?

We shall prove the NP-completeness results by reducing the following Roman domination problem, which is known to be NP-complete.

> **Roman domination problem** (RDP).
> **Instance**: Graph $G = (V, E)$, and an integer k.
> **Question**: Does G have an RDF of weight at most k?

Theorem 10 (Liu and Chang, [10])**.** *The RDP is NP-complete for bipartite graphs and chordal graphs.*

Theorem 11. *The DRDP is NP-complete for bipartite graphs and chordal graphs.*

Proof. It is clear that DRDP belongs to NP. Let G be a bipartite (or chordal) graph with $V(G) = \{v_1, \ldots, v_n\}$. Form a graph H from G by joining each vertex v_i of G to the central vertex x_3^i of a path $P_5 : x_1^i x_2^i x_3^i x_4^i x_5^i$. (Note that $|V(H)| = 6|V(G)|$.) Any $\gamma_R(G)$-function can be extended to a DRDF for H by assigning 2 to x_1^i, x_3^i and x_5^i, and 0 to x_2^i and x_4^i for $i = 1, 2, \ldots, n$. Thus $\gamma_{dR}(H) \le \gamma_R(G) + 6n$. Let f be a $\gamma_{dR}(H)$-function. Clearly for $i = 1, 2, \ldots, n$, $6 \le f(x_1^i) + f(x_2^i) + f(x_3^i) + f(x_4^i) + f(x_5^i) \le 7$. If $f(x_1^i) + f(x_2^i) + f(x_3^i) + f(x_4^i) + f(x_5^i) = 7$ for some integer i then we may assume that $f(v_i) = 0$, and then we replace $f(x_1^i)$, $f(x_3^i)$ and $f(x_5^i)$ by 2, $f(x_2^i)$ and $f(x_4^i)$ by 0, and $f(v_i)$ by 1. Thus we may assume that $f(x_1^i) + f(x_2^i) + f(x_3^i) + f(x_4^i) + f(x_5^i) = 6$ for each $i = 1, 2, \ldots, n$. It follows that $f(x_1^i) = f(x_3^i) = f(x_5^i) = 2$, and $f(x_2^i) = f(x_4^i) = 0$ for each $i = 1, 2, \ldots, n$. Since f is a DRDF for H, any vertex $v \in V(G)$ with $f(v) = 0$ is adjacent to at least one vertex $u \in V(G)$ with $f(u) = 2$. Thus $f|_{V(G)}$ is a Roman dominating function for G of weight $\gamma_{dR}(H) - 6n$. We conclude that $\gamma_{dR}(H) = \gamma_R(G) + 6n$. Hence the NP-completeness of the double Roman domination problem in bipartite graphs or chordal graphs follows from that of the Roman domination problem. ∎

Acknowledgements

We would like to thank both referees for their helpful comments.

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Received 31 October 2016
Revised 6 May 2017
Accepted 8 May 2017