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DECOMPOSITION OF THE PRODUCT OF CYCLES BASED ON DEGREE PARTITION

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Abstract

The Cartesian product of n cycles is a $2n$-regular, $2n$-connected and bipancyclic graph. Let G be the Cartesian product of n even cycles and let $2n = n_1 + n_2 + \cdots + n_k$ with $k \geq 2$ and $n_i \geq 2$ for each i. We prove that if $k = 2$, then G can be decomposed into two spanning subgraphs G_1 and G_2 such that each G_i is n_i-regular, n_i-connected, and bipancyclic or nearly bipancyclic. For $k > 2$, we establish that if all n_i in the partition of $2n$ are even, then G can be decomposed into k spanning subgraphs $G_1, G_2, \ldots, G_k$ such that each G_i is n_i-regular and n_i-connected. These results are analogous to the corresponding results for hypercubes.

Keywords: hypercube, Cartesian product, n-connected, regular, bipancyclic, spanning subgraph.

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1. Introduction

The graphs considered in this paper are simple, undirected and finite. The *Cartesian product* of two graphs G_1 and G_2 is the graph $G_1 \square G_2$ with vertex set $V(G_1) \times V(G_2)$ in which (u_1, u_2) is adjacent to (v_1, v_2) if and only if u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$, or u_2 is adjacent to v_2 in G_2 and $u_1 = v_1$. The n-dimensional hypercube Q_n is the Cartesian product of n copies the complete graph K_2. Therefore Q_n is the Cartesian product of $n/2$ copies of a cycle of length 4 when n is even. The Cartesian product of cycles and hypercubes are popular interconnection network topologies (see [6, 11]). The hypercube Q_n is an n-regular and n-connected graph whereas the Cartesian product of n cycles is a $2n$-regular and $2n$-connected graph (see [16]).

Unless mentioned otherwise, in the remainder of this paper product means the Cartesian product of graphs.

A cycle is *even* if its length is a positive even integer. A bipartite graph G is *bipancyclic* if G is either a cycle or has cycles of every even length from 4 to $|V(G)|$. A 3-regular graph is *nearly bipancyclic* if it has cycles of every even length from 4 to $|V(G)|$ except possibly for 4 and 8. The bipancyclicity property of a given network is an important factor in determining whether the network topology can simulate rings of various lengths.

Alspach *et al.* [1] proved that the product of cycles can be decomposed into Hamiltonian cycles. This result subsumes earlier results due to Kotzig [10] and Foregger [8] on Hamiltonian decomposition of the product of cycles. El-Zanati and Eynden [7] studied the decomposition of the product of cycles, each of length a power of 2, into non-spanning cycles. Borse *et al.* [4] proved that if $m \geq 2$ and m divides n, then the product of n even cycles can be decomposed into isomorphic, spanning, m-regular, m-connected subgraphs which are bipancyclic or nearly bipancyclic also. The analogous results for the class of hypercubes are obtained in [1, 4, 7].

Motivated by applications in parallel computing, Borse and Kandekar [3] considered the decomposition of the hypercube Q_n into two regular spanning subgraphs according to the partition of n into two parts and obtained the following result.

Theorem 1.1. *Let $n, n_1, n_2 \geq 2$ be integers such that $n = n_1 + n_2$. Then the hypercube Q_n can be decomposed into two spanning subgraphs G_1 and G_2 such that G_i is n_i-regular and n_i-connected for $i = 1, 2$. Moreover, G_i is bipancyclic if $n_i \neq 3$ and nearly bipancyclic if $n_i = 3$.*

We extend this result to the class of the product of even cycles as follows.

Theorem 1.2. *Let $n, n_1, n_2 \geq 2$ be integers such that $2n = n_1 + n_2$ and let G be the product of n even cycles. Then G can be decomposed into two spanning subgraphs G_1 and G_2 such that G_i is n_i-regular and n_i-connected for each $i = 1, 2$. Moreover, G_i is bipancyclic if $n_i \neq 3$ and nearly bipancyclic if $n_i = 3$.*

For the decomposition of Q_n according to the general partition of n, Sonawane and Borse [15] proved the following result.

Theorem 1.3 [15]**.** *Let $k, n_1, n_2, \ldots, n_k \geq 2$ be integers such that at most one n_i is odd and $n = n_1 + n_2 + \cdots + n_k$. Then Q_n can be decomposed into k spanning subgraphs $G_1, G_2, \ldots, G_k$ such that each G_i is n_i-regular and n_i-connected.*

We extend this result also to the class of the product of even cycles as follows.

Theorem 1.4. *Let $n, k \geq 2$ and $n_1, n_2, \ldots, n_k \geq 1$ be integers such that $n = n_1 + n_2 + \cdots + n_k$ and G be the product of n even cycles. Then G can be decomposed into k spanning subgraphs $G_1, G_2, \ldots, G_k$ such that each G_i is $2n_i$-regular and $2n_i$-connected.*

We prove Theorem 1.2 in Section 2. The proof of Theorem 1.4 is given in Section 3.

2. Decomposition Into Two Subgraphs

In this section, we prove Theorem 1.2. Firstly, we prove this theorem for the special cases $n_1 = 2$ and $n_1 = 3$. The general case follows from these two cases.

For $n \geq 1$, let $[n] = \{1, 2, \ldots, n\}$. We define a particular type of 3-regular graph below.

Definition 2.1. Let $r, s \geq 4$ be even integers and let W be the 3-regular graph with vertex set $V(W) = \{v_i^j : i \in [r];\ j \in [s]\}$ and the edge set $E(W) = \{v_i^j v_{i+1}^j : i \in [r]; j \in [s]\} \cup \{v_i^j v_i^{j+1} : i = 1, 3, 5, \ldots, r-1;\ j = 1, 3, 5, \ldots, s-1\} \cup \{v_i^j v_i^{j+1} : i = 2, 4, 6, \ldots, r;\ j = 2, 4, 6, \ldots, s\}$, where $v_i^{s+1} = v_i^1$ and $v_{r+1}^j = v_1^j$ (see Figure 1). The graph W is isomorphic to a honeycomb toroidal graph $H(s, r, 0)$ defined in [2].

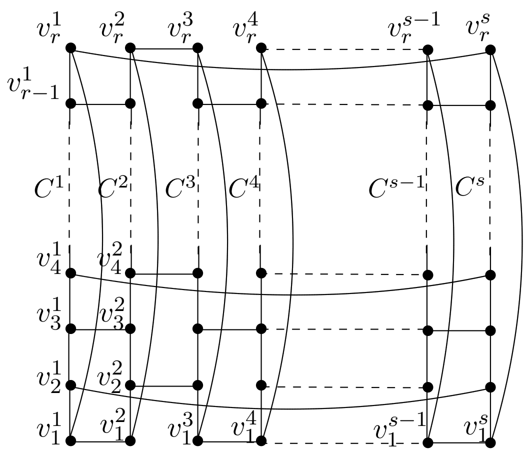

Figure 1. The graph W.

We need the following lemmas.

Lemma 2.2 [4]**.** *The graph W defined above is 3-regular, 3-connected and nearly bipancyclic.*

Lemma 2.3 [16]. *Let G_i be an m_i-regular and m_i-connected graph for $i = 1, 2$. Then the graph $G_1 \square G_2$ is $(m_1 + m_2)$-regular and $(m_1 + m_2)$-connected.*

We prove the special case $n_1 = 2$ of Theorem 1.2 in the following proposition.

Proposition 2.4. *Let $n \geq 2$ and let G be the product of n even cycles. Then G has a Hamiltonian cycle C such that $G - E(C)$ is a spanning, $(2n-2)$-regular, $(2n-2)$-connected and bipancyclic subgraph of G.*

Proof. We prove the result by induction on n. The result holds for $n = 2$ as, by [1], the product of two cycles can be decomposed into two Hamiltonian cycles. Suppose $n \geq 3$. Let $G = C_1 \square C_2 \square \cdots \square C_n$, where $C_1, C_2, \ldots, C_n$ are even cycles. Let $H = C_1 \square C_2 \square \cdots \square C_{n-1}$, $|V(H)| = r$ and $|V(C_n)| = s$. Then $G = H \square C_n$ and further, r and s are even integers such that $r = |C_1||C_2| \cdots |C_{n-1}| \geq 4^{n-1} \geq 16$ and $s \geq 4$. Label the vertices of the cycle C_n by $\{1, 2, \ldots, s\}$ so that j is adjacent to $j + 1$ modulo s.

By induction, H has a Hamiltonian cycle, say Z, such that $H - E(Z)$ is a spanning, $(2n-4)$-regular, $(2n-4)$-connected and bipancyclic subgraph of H. Label the vertices of the cycle Z by the set $\{v_1, v_2, \ldots, v_r\}$ so that v_p is adjacent to $v_{p+1 (\text{mod } r)}$. For compactness, let v_p^j denote the vertex (v_p, j) of $H \square C_n$, let superscripts be computed modulo s with representative in $[s]$ and subscripts be modulo r with representative in $[r]$. For $j \in [s]$, let H^j be the copy of H induced by the set $\{v_p^j : p \in [r]\}$ and let Z^j be the copy of Z in H^j. Let F be the set of edges of $H \square C_n$ between the graphs H^j, that is, $F = \{v_p^j v_p^{j+1} : p \in [r],\ j \in [s]\}$. Then $G = H^1 \cup H^2 \cup \cdots \cup H^s \cup F$.

We now construct a Hamiltonian cycle C by deleting one edge from Z^j and adding one edge of F between H^j and H^{j+1} for all j. Let $M = \{v_2^1 v_2^2, v_1^2 v_1^3, v_2^3 v_2^4, v_1^4 v_1^5, \ldots, v_2^{s-1} v_2^s, v_1^s v_1^1\}$ and let $C = \left(\bigcup_{j=1}^{s} (Z^j - v_1^j v_2^j)\right) \cup M$. Clearly, C is a Hamiltonian cycle in G.

Let $K = G - E(C)$. Then K is a spanning $(2n-2)$-regular subgraph of G. Further, $K = \left(\bigcup_{j=1}^{s} \left((H^j - Z^j) \cup \{v_1^j v_2^j\}\right)\right) \cup (F - M)$. We prove that K is $(2n-2)$-connected and bipancyclic.

Claim 1. *K is bipancyclic.*

Proof. We prove the claim by constructing a spanning bipancyclic subgraph of K. Since $H^j - E(Z^j)$ is bipancyclic, it has a Hamiltonian cycle X^j for $j \in [s]$. Therefore $V(X^j) = V(Z^j) = \{v_p^j : p \in [r]\}$. Let $J = (F - M) \cup (X^1 \cup X^2 \cup \cdots \cup X^s)$. Then J is a spanning subgraph of K. Note that the edge $v_1^j v_2^j$ of Z^j is a chord of X^j in H^j and so, a subpath of X^j from v_1^j to v_2^j has odd length. Obtain a 3-regular spanning subgraph W of J by deleting alternate edges of F between X^j and X^{j+1} starting from the edge $v_2^j v_2^{j+1}$ when j is odd, and starting from the edge $v_1^j v_1^{j+1}$ when j is even. It is easy to see that W is isomorphic to the graph in Figure 1. By Lemma 2.2, W is nearly bipancyclic. Therefore W and hence J contains cycles of every even length from 10 to $rs = |V(J)|$. The ladder graph in J formed by the paths $X^1 - v_2^1$ and $X^2 - v_2^2$ contains an l-cycle for any $l \in \{4, 6, 8\}$. Thus J contains cycle of every even length from 4 to $|V(J)|$ and so J is bipancyclic. As the graph J spans K, the graph K is also bipancyclic. □

Claim 2. *K is $(2n-2)$-connected.*

Proof. Let $D^j = (H^j - E(Z^j)) \cup \{v_1^j v_2^j\}$ for $j \in [s]$. Then $K = \left(\bigcup_{j=1}^s D^j\right) \cup (F - M)$. Since $H^j - E(Z^j)$ is $(2n-4)$-regular and $(2n-4)$-connected, D^j is $(2n-4)$-connected, and the degree of each of v_1^j and v_2^j in D^j is $2n-3$ and the degree of each of the remaining vertices in D^j is $2n-4$. For any j, D^{j+1} and D^{j-1} are the two neighbouring subgraphs of D^j in the graph K. Note that in K every vertex of D^j except v_1^j and v_2^j has neighbours in both D^{j+1} and D^{j-1}, whereas each of v_1^j and v_2^j has a neighbour in exactly one of D^{j+1} and D^{j-1}.

Let $S \subset V(K)$ such that $|S| \leq 2n-3$. To prove the claim, it suffices to prove that $K - S$ is connected. As $V(D^j) = V(H^j)$ and $n \geq 3$, $|V(D^j)| \geq 4^{n-1} \geq 2n+1 \geq |S|+4$. Hence there are at least three edges between $D^j - S$ and $D^{j+1} - S$ in $K - S$ for any $j \in [s]$. Therefore, if $D^j - S$ is connected for all $j \in [s]$, then $K - S$ is connected. Suppose $D^j - S$ is not connected for some j. Without loss of generality, we may assume that $D^1 - S$ is not connected. Then D^1 contains at least $2n-4$ vertices from S. If $S \subset V(D^1)$, then every vertex of $D^1 - S$ has a neighbour in the connected graph $K - V(D^1)$ and so $K - S$ is connected. Suppose S is not a subset of $V(D^1)$. Then $|S| = 2n-3$ and $|V(D^1) \cap S| = 2n-4$, and so only one vertex, say x, from S is in $V(K) - V(D^1)$. Let $W = K - (V(D^1) \cup \{x\})$. Then W is connected. Every vertex of $D^1 - S$ except possibly v_1^1 and v_2^1 has a neighbour in W. Since the degree of each of v_1^1 and v_2^1 is $2n-3$ in D^1, these vertices cannot be isolated in $D^1 - S$. Hence the component of $D^1 - S$ containing v_1^1 or v_2^1 has a neighbour in W. This implies that $K - S$ is connected. □

Thus, C is a Hamiltonian cycle in G such that $G - E(C) = K$ is a spanning, $(2n-2)$-regular, $(2n-2)$-connected and bipancyclic subgraph of G. ■

Remark 2.5. By Lemma 2.3, the product G of n cycles is $2n$-regular and $2n$-connected. If C is a cycle in G, then the minimum degree of $G - E(C)$ is $2n-2$ and hence $G - E(C)$ cannot be k-connected for $k = 2n-1$ or $k = 2n$. However, the above proposition guarantees the existence of a cycle in G such that $G - E(C)$ is $(2n-2)$-connected. Such a cycle is *removable* in G. This result can be compared with an older theorem of Mader [13] which states that if H is a simple n-connected graph with minimum degree $n+2$, then there is a cycle C in H such that $H - E(C)$ is n-connected (also see [5, 9]).

We now prove the special case $n_1 = 3$ of Theorem 1.2.

Proposition 2.6. *Let $n \geq 3$ and let G be the product of n even cycles. Then G has a spanning, 3-regular, 3-connected and nearly bipancyclic subgraph W such that $G - E(W)$ is a spanning, $(2n-3)$-regular, $(2n-3)$-connected subgraph of G. Moreover, $G - E(W)$ is bipancyclic if $n \neq 3$, and it is nearly bipancyclic otherwise.*

Proof. Let $C_1, C_2, \ldots, C_n$ be even cycles and let $G = C_1 \square C_2 \square \cdots \square C_n$. Write G as $G = H \square C_n$, where $H = C_1 \square C_2 \square \cdots \square C_{n-1}$. It follows from Proposition 2.4 that H has a decomposition into two subgraphs C and D, where C is a Hamiltonian cycle, and D is spanning, $(2n-4)$-regular, $(2n-4)$-connected and bipancyclic. Obviously, $H = D \cup C$. Let $|V(C_n)| = s$ and $|V(H)| = r$. Then, as in the proof of Proposition 2.4, we have $G = H^1 \cup H^2 \cup \cdots \cup H^s \cup F$ with $V(H^j) = \{v_p^j : p \in [r]\}$ and $F = \{v_i^j v_i^{j+1} : i \in [r],\ j \in [s]\}$, where H^j is the copy of H corresponding to jth vertex of the cycle C_n. Further, C^j is the copy of C in H^j with vertices $v_1^j, v_2^j, \ldots, v_r^j, v_1^j$ in order. Partition the edge set F into two parts F_1 and F_2, where $F_1 = \{v_i^j v_i^{j+1} : i = 1, 3, 5, \ldots, r-1;\ j = 1, 3, 5, \ldots, s-1\} \cup \{v_i^j v_i^{j+1} : i = 2, 4, 6, \ldots, r;\ j = 2, 4, 6, \ldots, s\}$ and $F_2 = F - F_1$.

We now construct a 3-regular subgraph W of G as required. Let $W = C^1 \cup C^2 \cup \cdots \cup C^s \cup F_1$. Then W is isomorphic to the graph in Figure 1. By Lemma 2.2, W is a 3-regular, 3-connected and nearly bipancyclic subgraph of G. Let $W' = G - E(W)$. Clearly, $W' = D^1 \cup D^2 \cup \cdots \cup D^s \cup F_2$, where D^j is the copy of D in H^j. Further, W' is a spanning and $(2n-3)$-regular subgraph of G.

To complete the proof, it suffices to prove that W' is bipancyclic and $(2n-3)$-connected.

Let Y^j be a Hamiltonian cycle in D^j for $j \in [s]$. Let $W'' = Y^1 \cup Y^2 \cup \cdots \cup Y^s \cup F_2$. Then W'' is a spanning subgraph of W'. Observe that W'' is isomorphic to W and so it is nearly bipancyclic. Hence W' is nearly bipancyclic. Suppose $n \geq 4$. Then the cycles of lengths 4 and 8 exist in the graph D^j and so in W'. Hence W' contains cycles of every even length from 4 to $|V(W')|$. Thus W' is bipancyclic in this case.

We now prove that W' is $(2n-3)$-connected. Let $S \subset W'$ with $|S| \leq 2n-4$. It is enough to prove that $W' - S$ is connected. Suppose $S \subset D^j$ for some j. Then each component of $D^j - S$ is joined to D^{j+1} or D^{j-1} and hence $W' - S$ is connected. Suppose S intersects at least two of $V(D^1), V(D^2), \ldots, V(D^s)$. Then $|S \cap V(D^j)| < (2n-4)$ and hence $D^j - S$ is connected as D^j is $(2n-4)$-connected for each $j \in [s]$. Through the edges of the matching F_2, half of the vertices of D^j have distinct neighbours in D^{j+1} and the remaining half have distinct neighbours in D^{j-1}. Therefore the connected graph $D^j - S$ is joined to each of $D^{j+1} - S$ and $D^{j-1} - S$ in W' by at least one edge. It implies that $W' - S$ is connected. Thus W' is $(2n-3)$-connected. This completes the proof. ■

The next lemma follows from the fact that the product of even cycles is bipancyclic (see [6]).

Lemma 2.7. *If G_1 and G_2 are bipartite Hamiltonian graphs, then $G_1 \Box G_2$ is bipancyclic.*

We now prove Theorem 1.2.

***Proof of Theorem* 1.2.** We may assume that $n_2 \geq n_1 \geq 2$. By Propositions 2.4 and 2.6, the result holds for $n_1 = 2$ and $n_1 = 3$. Therefore the result also holds for the cases $n = 2$ and $n = 3$. Suppose $n, n_1, n_2 \geq 4$. Assume that the result holds for all integers from 4 to $n-1$. Let G be the product of n even cycles $C_1, C_2, \dots, C_n$ and let $H = C_1 \Box C_2 \Box \cdots \Box C_{n-2}$. Then $G = H \Box (C_{n-1} \Box C_n)$. Since $2n = n_1 + n_2$, we can express $2(n-2)$ as $2(n-2) = (n_1 - 2) + (n_2 - 2)$. Note that $n_1 - 2 \geq 2$ and $n_2 - 2 \geq 2$. Hence, by induction, H has a decomposition into two spanning subgraphs W_1 and W_2 such that W_i is $(n_i - 2)$-regular, $(n_i - 2)$-connected, and bipancyclic or nearly bipancyclic for $i = 1, 2$. Therefore, each W_i contains a Hamiltonian cycle. By [1], the product of two cycles has a Hamiltonian decomposition. Hence $C_{n-1} \Box C_n$ can be decomposed into two Hamiltonian cycles, say Z_1 and Z_2. This implies that $G = (W_1 \cup W_2) \Box (Z_1 \cup Z_2) = (W_1 \Box Z_1) \cup (W_2 \Box Z_2) = G_1 \cup G_2$, where $G_1 = W_1 \Box Z_1$ and $G_2 = W_2 \Box Z_2$. Hence G_1 and G_2 are edge-disjoint spanning subgraphs of G with $G = G_1 \cup G_2$. By Lemma 2.7, G_i is bipancyclic and further, by Lemma 2.3, it is n_i-regular and n_i-connected for $i = 1, 2$. Thus G_1 and G_2 give a decomposition of G as required. ■

Remark 2.8. It is worth mentioning that Theorem 1.2 gives a partial solution to the following question due to Mader [14, p. 73].

Given any n-connected graph and $k \in \{1, 2, \dots, n\}$, is there always a k-connected subgraph H of G so that $G - E(H)$ is $(n-k)$-connected ?

3. DECOMPOSITION INTO k SUBGRAPHS

In this section, we prove Theorem 1.4. Firstly, we give a construction of obtaining I-connected spanning subgraph of $C_1 \Box C_2 \Box \cdots \Box C_n$ from the given I-connected spanning subgraph of $C_1 \Box C_2 \Box \cdots \Box C_{n-1}$.

Suppose $n \geq 2$. Let $C_1, C_2, \dots, C_n$ be even cycles, $H = C_1 \Box C_2 \Box \cdots \Box C_{n-1}$ and $G = H \Box C_n$. Let $|V(C_n)| = s$. Then $s \geq 4$. Let H^j be the copy of H in G corresponding to jth vertex of the cycle C_n. Then, as in the proof of Proposition 2.4, $G = H^1 \cup H^2 \cup \cdots \cup H^s \cup F$, where $F = \bigcup_{j=1}^{s} (\{xy \colon x \in V(H^j), y \in V(H^{j+1})\})$. By Lemma 2.3, H is $(2n-2)$-connected. Let l be an even integer such that $2 \leq l \leq 2n - 2$ and let K be a spanning l-connected subgraph of H in which $M = \{u_1 u_2, u_3 u_4, \dots, u_{l-1} u_l\}$ is a matching consisting of $l/2$ edges $u_i u_{i+1}$. Let K^j

be the corresponding copy of K in H^j and let $M^j = \{u_1^j u_2^j, u_3^j u_4^j, \dots, u_{l-1}^j u_l^j\}$ be the matching in K^j corresponding to the matching M. Then K^j is l-connected and $K^j - M^j$ is $l/2$-connected. In G, each vertex of K^j is adjacent to the corresponding vertex of K^{j+1} through an edge from F. Let $N = \{u_i^j u_i^{j+1} \colon j \in [s]$ and j odd; $i \in [l]$ and i odd$\} \cup \{u_i^j u_i^{j+1} \colon j \in [s]$ and j even; $i \in [l]$ and i even$\}$. Then $N \subseteq F$ and N is a matching in G. Also, $V(N) = V(M^1) \cup V(M^2) \cup \cdots \cup V(M^s)$.

Let $W = \left(\bigcup_{j=1}^{s} (K^j - M^j)\right) \cup N$ (see Figure 2). Then W is a spanning subgraph of G. We prove below that W is l-connected also.

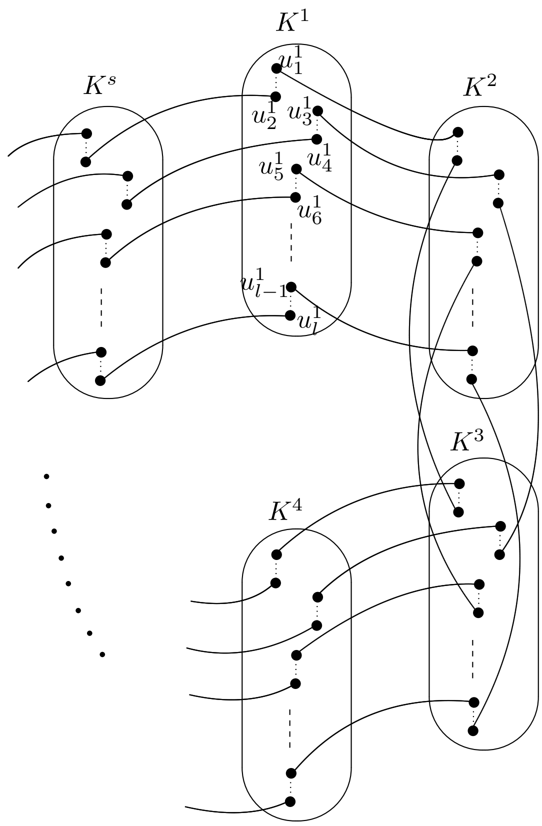

Figure 2. The spanning subgraph W of G.

Lemma 3.1. *The graph W defined above is l-connected.*

Proof. Let $S \subset V(W)$ with $|S| \leq l - 1$. It suffices to prove that $W - S$ is connected. As $V(W) = V(K^1) \cup V(K^2) \cup \cdots \cup V(K^s)$, $S \subset V(K^1) \cup V(K^2) \cup \cdots \cup V(K^s)$. Let $S^j = S \cap V(K^j)$ for $j \in [s]$. By $j+1$ and $j-1$, we mean $j + 1 (\bmod s)$ and $j - 1 (\bmod s)$, respectively.

Let $W^j = K^j - M^j$ for $j \in [s]$. Since K^j is l-connected and matching M^j contains $l/2$ edges, W^j is $l/2$-connected and further, it contains all l vertices of M^j half of which have neighbours in W^{j-1} while the remaining half have neighbours in W^{j+1}. Clearly, $W^j - S^j$ contains $l - |S^j| > |S| - |S^j| = (|S^1| + |S^2| + \cdots + |S^{j-1}| + |S^j| + |S^{j+1}| + \cdots + |S^s|) - |S^j| \geq |S^{j-1}| + |S^{j+1}|$ vertices of M^j. Therefore $W^j - S^j$ has at least one neighbour in $W^{j-1} - S^{j-1}$ or $W^{j+1} - S^{j+1}$. Further, if $W^j - S^j$ has no neighbour in $W^{j+1} - S^{j+1}$, then $|S^j| + |S^{j+1}| \geq l/2$.

We may assume that $|S^1| \geq |S^j|$ for all $j \in [s]$. Then $|S^j| < l/2$ for $j \neq 1$ as $|S| = \sum_{j=1}^{s} |S^j| < l$. Therefore, as W^j is $l/2$-connected, $W^j - S^j$ is connected for $j \neq 1$.

Suppose $|S^1| \geq l/2$. Then $\sum_{j \neq 1} |S^j| < l/2$. Therefore $W^j - S^j$ has a neighbour in $W^{j+1} - S^{j+1}$ for $2 \leq j \leq s-1$. Let $T = W - (S \cup V(K^1))$. Then T is a connected subgraph of W. We prove that every vertex of $W^1 - S^1$ has a neighbour in T. Let D be a component of $W^1 - S^1$. If $W^1 - S^1$ is connected, then $D = W^1 - S^1$. Note that $W^1 - S^1 = (K^1 - M^1) - S^1 = (K^1 - S^1) - M^1$. Since K^1 is l-connected, $K^1 - S^1$ is $(l - |S^1|)$-connected. Therefore, if $W^1 - S^1$ is not connected, then D contains at least $l - |S^1| > |S^2| + |S^s|$ vertices of the matching M^1. Each of these vertices has a neighbour in W^2 or W^s. In any case, D has a neighbour in $W^2 - S^2$ or $W^s - S^s$ and so is in the connected graph T. This implies that $W - S$ is connected.

Suppose $|S^1| < l/2$. Then $|S^j| < l/2$ for all j. Hence $W^j - S^j$ is connected. Suppose $W^j - S^j$ has no neighbour in $W^{j+1} - S^{j+1}$ for some j. Then $W^{j-1} - S^{j-1}$ contains a neighbour of $W^j - S^j$. By the same argument, $W^{j+2} - S^{j+2}$ contains a neighbour of $W^{j+1} - S^{j+1}$. Suppose $W^i - S^i$ has no neighbour in $W^{i+1} - S^{i+1}$ for some $i \neq j$. Then $i \neq j-1, i \neq j+1$ and further, $|S^i| + |S^{i+1}| \geq l/2$. Therefore $|S| \geq |S^j| + |S^{j+1}| + |S^i| + |S^{i+1}| \geq l/2 + l/2 = l$, a contradiction. Hence, for any $i \neq j$, $W^i - S^i$ has neighbours in $W^{j+1} - S^{i+1}$. This implies that $W - S$ is connected. ■

Definition 3.2. Let G be the product of n even cycles. Suppose the diameter of G is d. Let v_0 be an end-vertex of a path in G of length d. Fix v_0. Let $V_0 = \{v_0\}$ and let $V_i = \{v \in V(G) : d(v_0, v) = i\}$ for $i \in [d]$, where $d(v_0, v)$ denotes the distance between v_0 and v in G. Clearly, the sets $V_0, V_1, \ldots, V_d$ are mutually disjoint, non-empty and they partition the set $V(G)$. Let K be a spanning subgraph of G. For $i \in [d]$, let $E_i(K) = \{xy \in E(K) : x \in V_{i-1},\ y \in V_i\}$. Then the edge sets $E_1(K), E_2(K), \ldots, E_d(K)$ are non-empty and mutually disjoint (see Figure 3).

Lemma 3.3. *Let G, K and $E_i(K)$ be as in Definition 3.2. Then $E_1(K), E_2(K), \ldots, E_d(K)$ partition the edge set $E(K)$ of the graph K.*

Proof. By definition of V_i, there is no edge in G with one end-vertex in V_j and the other in $V_{j'}$ when $|j - j'| \neq 1$. Suppose two vertices x and y of some

V_i are adjacent. Let P_x and P_y be shortest paths in G from v_0 to x, and v_0 to y respectively. Then each of P_x and P_y takes exactly one vertex from each of $V_0, V_1, \ldots, V_i$. Therefore $P_x \cup P_y \cup \{xy\}$ contains an odd cycle in G, a contradiction to the fact that G is bipartite. Thus each V_i is independent. This implies that $E(K) = E_1(K) \cup E_2(K) \cup \cdots \cup E_d(K)$. ■

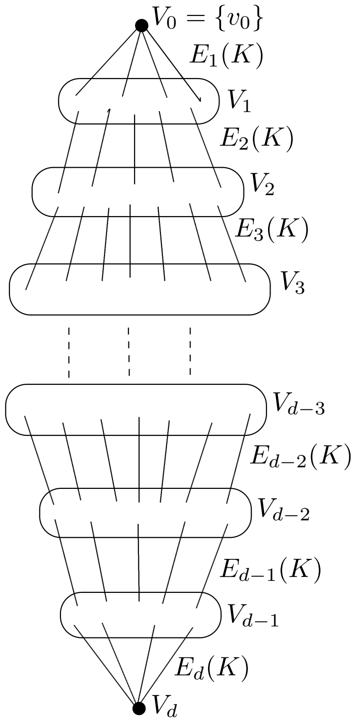

Figure 3. A decomposition of G.

We need the following result.

Lemma 3.4 [16]**.** *Let G_i be a graph with diameter d_i for $i = 1, 2, \ldots, k$. Then the diameter of the graph $G_1 \Box G_2 \Box \cdots \Box G_k$ is $d_1 + d_2 + \cdots + d_k$.*

We are all set to prove Theorem 1.4. This theorem is restated below for convenience.

Theorem 3.5. *Let G be the product of n even cycles and let $n = n_1 + n_2 + \cdots + n_k$ with $k \geq 2$ and $n_i \geq 1$ for $i \in [k]$. Then G can be decomposed into k spanning subgraphs $G_1, G_2, \ldots, G_k$ such that each G_i is $2n_i$-regular and $2n_i$-connected.*

Proof. We prove the result by induction on n. Obviously, $n \geq k$. If $n = k$, then G is the product of k cycles and hence, by [1], G can be decomposed into k Hamiltonian cycles. Thus the result holds for $n = k$.

Suppose $n \geq k+1$. Then $n_i \geq 2$ for some $i \in [k]$. Without loss of generality, we may assume that $n_k \geq 2$. Assume that the result holds for $n-1$. Consider $n-1 = n_1 + n_2 + \cdots + n_{k-1} + (n_k - 1)$. Let $G = C_1 \Box C_2 \Box \cdots \Box C_n$, where $C_1, C_2, \ldots, C_n$ are even cycles. Let $|C_n| = s$ and let $H = C_1 \Box C_2 \Box \cdots \Box C_{n-1}$. Then, as in the proof of Proposition 2.4, $G = H \Box C_n = H^1 \cup H^2 \cup \cdots \cup H^s \cup F$, where H^j is a copy of H and $F = \bigcup_{j=1}^{s}(\{xy: x \in V(H^j),\ y \in V(H^{j+1})\})$ with $H^{s+1} = H^1$.

By induction, H can be decomposed into k spanning subgraphs $H_1, H_2, \ldots, H_k$ such that H_i is $2n_i$-regular and $2n_i$-connected for $i \in [k-1]$, and H_k is $2(n_k-1)$-regular and $2(n_k-1)$-connected.

Let d be the diameter of H. Since each C_i is an even cycle, the diameter of C_i is $|C_i|/2 \geq 2$. Therefore, by Lemma 3.4, $d = \frac{|C_1|+|C_2|+\cdots+|C_{n-1}|}{2} \geq 2(n-1) = 2n-2 \geq 2n-2n_k$. Let u_0 be an end-vertex of a path in H of length d. As in the Definition 3.2, we partition the vertex set $V(H)$ of H into the sets $V_0(H), V_1(H), \ldots, V_d(H)$, where $V_0(H) = \{u_0\}$ and $V_i(H) = \{u \in V(H): d(u, u_0) = i\}$ for $i \in [d]$. Since H_i for $i \in [k]$ is a spanning subgraph of H, it follows from Lemma 3.3 that the edge set $E(H_i)$ of H_i can be partitioned into the sets $E_1(H_i), E_2(H_i), \ldots, E_d(H_i)$, where $E_t(H_i) = \{xy \in E(H_i): x \in V_{t-1},\ y \in V_t\}$ for $t \in [d]$. Note that if $e \in E_t(H_i)$ and $f \in E_{t'}(H_i)$ with $t' \geq t+2$, then e and f are vertex-disjoint (see Figure 3).

For $i \in [k-1]$, we obtain a matching M_i of H_i by choosing one edge from n_i consecutive sets $E_{2t-1}(H_i)$ as follows.

Choose one edge from each of the sets $E_1(H_1), E_3(H_1), \ldots, E_{2n_1-1}(H_1)$ to get M_1. Thus, we let $M_1 = \{u_{t-1}u_t \in E_t(H_1): t = 1, 3, 5, \ldots, 2n_1 - 1\}$. In general, we define $M_i = \{u_{t-1}u_t \in E_t(H_i): t = 2p_i + 1, 2p_i + 3, \ldots, 2p_i + 2n_i - 1\}$, where $p_1 = 0$ and $p_i = n_1 + n_2 + n_3 + \cdots + n_{i-1}$ for $2 \leq i \leq k-1$.

For $j \in [s]$, the graph H^j is a copy of H. Let H_i^j be the subgraph of H^j corresponding to the subgraph H_i of H for $i \in [k]$. Therefore the graphs $H_1^j, H_2^j, \ldots, H_k^j$ decompose the graph H^j. Further, the edge set $E(H_i^j)$ has a partition into non-empty sets $E_1(H_i^j), E_2(H_i^j), \ldots, E_d(H_i^j)$. For $i \in [k-1]$, let M_i^j be the matching in H_i^j corresponding to the matching M_i of H and let u_t^j be the vertex of H^j corresponding to the vertex u_t of H. Then $M_i^j = \{u_{t-1}^j u_t^j \in E_t(H_i^j): t = 2p_i+1, 2p_i+3, \ldots, 2p_i+2n_i-1\}$. Let M^j be the union of these $k-1$ matchings M_i^j. Therefore $M^j = \bigcup_{i=1}^{k-1} M_i^j = \left\{u_0^j u_1^j, u_2^j u_3^j, \ldots, u_{2n_1+\cdots+2n_{k-1}-2}^j u_{2n_1+\cdots+2n_{k-1}-1}^j\right\}$. Clearly, M^j is a matching in H^j.

We now construct the subgraphs $G_1, G_2, \ldots, G_k$ of G which give a decomposition of G, as required.

Construction of the graphs G_i for $i \in [k]$.

Let $i \in [k-1]$. We obtain G_i from $H_i^1 \cup \cdots \cup H_i^s$ by deleting the matching M_i^j from H_i^j for each j and then adding a matching D_i consisting of edges from the set

F having one end in M_i^j and the other end in M_i^{j+1} or M_i^{j-1}. More precisely, let $D_i = \{u_t^j u_t^{j+1} : j = 1, 3, \ldots, s-1;\ t = 2p_i, 2p_i+2, \ldots, 2p_i+2n_i-2\} \cup \{u_t^j u_t^{j+1} : j = 2, 4, \ldots, s;\ t = 2p_i+1, 2p_i+3, \ldots, 2p_i+2n_i-1\}$.

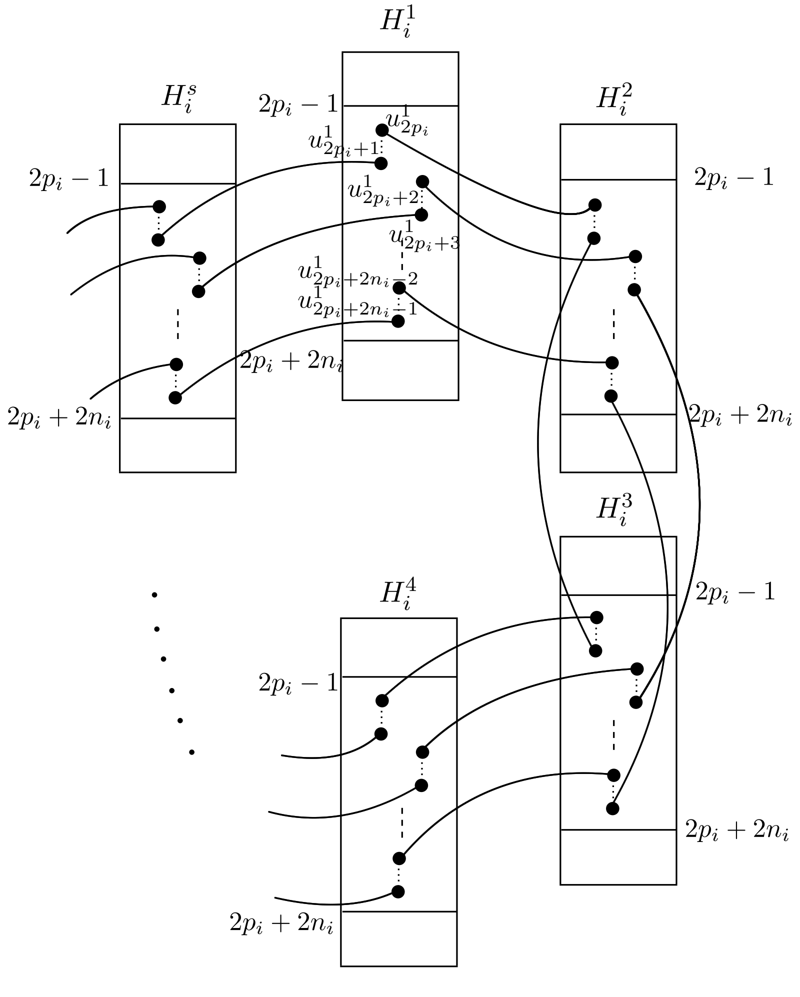

Figure 4. The graph G_i.

For $i \in [k-1]$, let $G_i = \left(\bigcup_{j=1}^{s} (H_i^j - M_i^j)\right) \cup D_i$ (see Figure 4). Note that D_i is a matching consisting of n_i edges between H^j and H^{j+1} for each $j \in [s]$ and so the total number of edges in D_i is sn_i.

For any $i \in [k-1]$ and $i' \in [k-1]$ with $i \neq i'$, the graphs H_i^j and $H_{i'}^j$ are edge-disjoint for each j. This implies that $G_1, G_2, \ldots, G_{k-1}$ are mutually edge-disjoint subgraphs of G. Since H_i^j is a $2n_i$-regular and spanning subgraph of H^j, G_i is also a $2n_i$-regular and spanning subgraph of G. Further, as H_i^j is $2n_i$-connected, Lemma 3.1 implies that G_i is also $2n_i$-connected.

Let $G_k = G - E(G_1 \cup G_2 \cup \cdots \cup G_{k-1})$. The graph G_k is shown in Figure 5. It is easy to see that $G_k = \left(\bigcup_{j=1}^{s}(H_k^j \cup M^j)\right) \cup (F - D)$, where $D = \bigcup_{i=1}^{k-1} D_i$. The edges of the matching M^j are shown by the bold edges in Figure 5. Clearly, D is a matching in G consisting of $s(n_1 + n_2 + \cdots + n_{k-1}) = s(n - n_k)$ edges of F.

It follows that the graph G_k is a spanning and $2n_k$-regular subgraph of G. Thus the graph G decomposes into the spanning subgraphs $G_1, G_2, \ldots, G_k$.

It only remains to prove that the graph G_k is $2n_k$-connected.

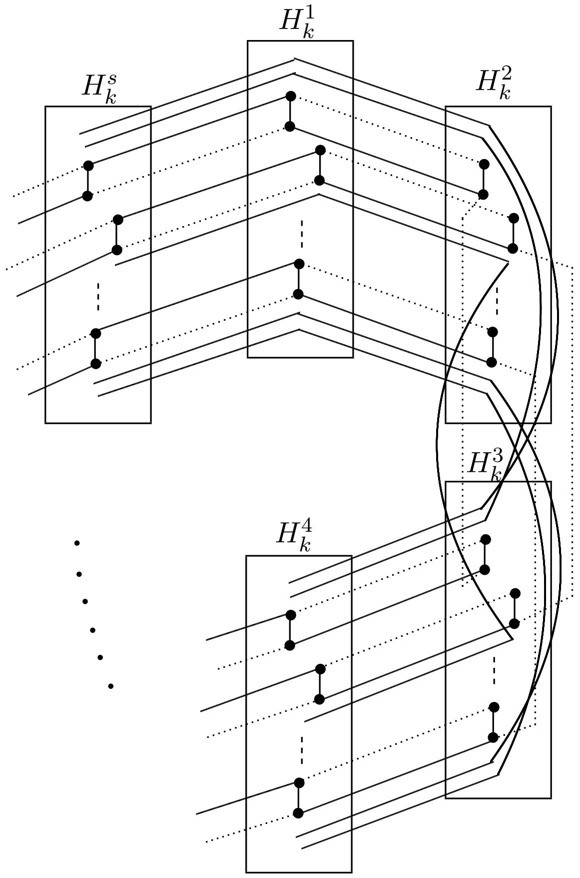

Figure 5. The graph G_k.

Claim. *G_k is $2n_k$-connected.*

Proof. Let $S \subset V(G_k) = \bigcup_{j=1}^{s} V(H^j)$ such that $0 < |S| \leq 2n_k - 1$. It suffices to prove that $G_k - S$ is connected. Let $S^j = V(H_k^j) \cap S$ for $j \in [s]$. Since $V(H_k^j) = V(H^j), |V(H_k^j)| = |V(H^j)| = r = |C_1||C_2| \cdots |C_{n-1}| \geq 4^{n-1} = 2^{2n-2} =$

$2^{2n-2n_k+2n_k-2} = 2^{2n-2n_k} \cdot 2^{2n_k-2} > (2n-2n_k)(2n_k-1) \geq 2(n-n_k)|S| \geq (n-n_k)+|S|$. Hence, there are at least $r-(n-n_k)-|S| > 0$ edges between $H_k^j - S^j$ and $H_k^{j+1} - S^{j+1}$ in $G_k - S$.

Obviously at least one S^j is non-empty. We may assume that $S^1 \neq \emptyset$ and $|S^1| \geq |S^j|$ for $j \in [s]$. Suppose two more S^j are non-empty. Then $|S^j| < 2n_k - 2$ for $j \in [s]$ as $|S| \leq 2n_k - 1$. Hence each $H_k^j - S^j$ is connected as H_k^j is $(2n_k-2)$-connected. Further, $H_k^j - S^j$ is connected to $H_k^{j+1} - S^{j+1}$ by edges of $F - D$. This implies that $G_k - S$ is connected.

Suppose $S^j = \emptyset$ for all $j \neq 1$. Therefore $H_k^j - S^j = H_k^j$ is connected for all $j \neq 1$. Obviously, each vertex of $H_k^1 - S^1$ has a neighbour in H_k^2 or H_k^s. Hence $G_k - S$ is connected.

Suppose only one S^j other than S^1 is nonempty. Suppose $S^2 \neq \emptyset$. Then $S = S^1 \cup S^2$. If $H_k^1 - S^1$ and $H_k^2 - S^2$ are connected, then they are connected to each other by an edge of $F - D$ and so $G_k - S$ is connected. Suppose $H_k^1 - S^1$ is not connected. Then $|S^1| = 2n_k - 2$ and $|S^2| = 1$ as $|S| \leq 2n_k - 1$ and H_k^1 is $(2n_k - 2)$-connected. This implies that $H_k^j - S^j$ for any $j \neq 1$ is connected. Let $T = G_k - (V(H_k^1) \cup S) = G_k - (V(H_k^1) \cup S^2)$. Then T is connected. It suffices to prove that every component of $H_k^1 - S^1$ has a neighbour in T. Let W be a component of $H_k^1 - S^1$ and let v be a vertex of W. If v has a neighbour in H_k^s, then we are through. Suppose v has no neighbour in H_k^s. Then v has a neighbour v' in H_k^2. If $v' \notin S^2$, then also we are through. Suppose $v' \in S^2$. Then $S^2 = \{v'\}$. Also, v is an end-vertex of an edge of the matching M^1. Therefore the degree of v in H_k^1 is $2n_k - 1$. Hence v has a neighbour u in $H_k^1 - S^1$. Obviously, u is in W. Further, u has a neighbour in the subgraphs H_k^s or $H_k^2 - S^2 = H_k^2 - \{v'\}$ of T. Thus W has a neighbour in the connected graph T. Hence $G_k - S$ is connected.

Similarly, $G_k - S$ is connected when $S^s \neq \emptyset$.

Suppose $S^j \neq \emptyset$ for some $j \notin \{1, 2, s\}$. Then every component of $H_k^1 - S^1$ has a neighbour in H_k^2 or H_k^s. It follows that $G_k - S$ is connected. This proves the claim. □

Thus, the graph G decomposes into the spanning subgraphs $G_1, G_2, \ldots, G_k$, where G_i is $2n_i$-regular and $2n_i$-connected for $i = 1, 2, \ldots, k$. This completes the proof. ■

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