# NEW FORMULAE FOR THE DECYCLING NUMBER OF GRAPHS 

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#### Abstract

A set $S$ of vertices of a graph $G$ is called a decycling set if $G-S$ is acyclic. The minimum order of a decycling set is called the decycling number of $G$, and denoted by $\nabla(G)$. Our results include: (a) For any graph $G$, $$
\nabla(G)=n-\max _{T}\{\alpha(G-E(T))\},
$$ where $T$ is taken over all the spanning trees of $G$ and $\alpha(G-E(T))$ is the independence number of the co-tree $G-E(T)$. This formula implies that computing the decycling number of a graph $G$ is equivalent to finding a spanning tree in $G$ such that its co-tree has the largest independence number. Applying the formula, the lower bounds for the decycling number of some (dense) graphs may be obtained. (b) For any decycling set $S$ of a $k$-regular graph $G$, $$
|S|=\frac{1}{k-1}(\beta(G)+m(S)),
$$ where $\beta(G)=|E(G)|-|V(G)|+1$ and $m(S)=c+|E(S)|-1, c$ and $|E(S)|$ are, respectively, the number of components of $G-S$ and the number of


[^0]edges in $G[S]$. Hence $S$ is a $\nabla$-set if and only if $m(S)$ is minimum, where $\nabla$-set denotes a decycling set containing exactly $\nabla(G)$ vertices of $G$. This provides a new way to locate $\nabla(G)$ for $k$-regular graphs $G$. (c) 4-regular graphs $G$ with the decycling number $\nabla(G)=\left\lceil\frac{\beta(G)}{3}\right\rceil$ are determined.
Keywords: decycling number, independence number, cycle rank, margin number.
2010 Mathematics Subject Classification: 05C07, 05C38.

## 1. Introduction

Graphs throughout this paper are loopless and multiple edges are permitted. For general theoretic notations, we follow Diestel [4]. The minimum number of edges whose removal eliminates all cycles in a given graph has been known as the cycle rank of the graph, and this parameter has a simple expression $\beta(G)=$ $|E(G)|-|V(G)|+w$ (see $[9]$ ), where $w$ is the number of components of $G$. The corresponding problem of eliminating all cycles from a graph by means of deletion of vertices goes back at least to the work of Kirchhoff on spanning trees [10].

Let $G=(V, E)$ be a graph. We define a vertex set $S$ of $G$ to be a decycling set if $G-S$ is cycle-free. The cardinality of a minimum decycling set of $G$ is called the decycling number, and denoted by $\nabla(G)$ (or $\nabla$ for short). A decycling set containing exactly $\nabla(G)$ vertices of $G$ is called a $\nabla$-set. Vertices of a decycling set are labeled by " $\bullet$ " and the bold edges induce a spanning tree $T$ of a graph $G$ in the following figures. Let $m(S)=c+|E(S)|-1$ be the margin number of a decycling set $S$, where $c$ and $|E(S)|$ are, respectively, the number of components of $G-S$ and the number of edges in $G[S]$. Determining the decycling number is equivalent to finding the size of the largest induced forest of $G$ proposed first by Erdös [5]. If $S \subseteq V(G)$ is a $\nabla$-set, then $G-S$ is a largest induced forest of $G$. The problem of determining the decycling number of graphs have been proved to be NP-complete by Karp [11], even for general graphs such as bipartite graphs, planar graphs and perfect graphs, the decycling problem is very hard to solve. It is easy to see that $\nabla(G)=0$ if and only if $G$ is a forest, and $\nabla(G)=1$ if and only if $G$ has at least one cycle and a vertex on all of its cycles. One may see [2] as a brief survey.

In this paper, we consider the decycling problem from two new perspectives: the effects of (a) spanning trees and (b) the margin number, respectively, on the decycling number of graphs. Given a connected graph $G$ and a surface $P$, we say that $G$ can be embedded into $P$ if there exists a polyhedron $\sum$ on $P$ such that the 1-skeleton of $\sum$ has a subgraph homeomorphic to $G$. The components of $\sum-G$ are called the faces of the embedding. When each face is homeomorphic to an
open disc, the embedding is called a cellular. The maximum genus, denoted by $\gamma_{M}(G)$, of a connected graph $G$ is the largest genus of an orientable surface on which $G$ admits a cellular embedding. Let $T$ be a spanning tree of a connected graph $G$. The subgraph $G-E(T)$ of $G$ is called a co-tree of $G$. Note that the number of edges in any co-tree of $G$ is just the cycle $\operatorname{rank} \beta(G)$. The Betti deficiency of $G$, denoted by $\xi(G)$, is defined the minimum number of odd components (i.e., the components containing odd number of edges) among co-trees of $G$. We call $T$ a Xuong-tree if the number of odd components of $G-E(T)$ is $\xi(G)$. The following result of Xuong defines an edge-partition of a co-tree.

Lemma 1 [20]. Let $G$ be a connected graph and $T_{X}$ be a Xuong-tree of $G$. Then there exists an edge-partition of $E(G)-E\left(T_{X}\right)$ as follows:

$$
E(G)-E\left(T_{X}\right)=\left\{e_{1}, e_{2}\right\} \cup\left\{e_{3}, e_{4}\right\} \cup \cdots \cup\left\{e_{2 m-1}, e_{2 m}\right\} \cup\left\{f_{1}, f_{2}, \ldots, f_{s}\right\},
$$

where (1) $m=\gamma_{M}(G), s=\xi(G)$; (2) for any $i=1,2, \ldots, m, e_{2 i-1} \cap e_{2 i} \neq \emptyset$, and $\left\{f_{1}, f_{2}, \ldots, f_{s}\right\}$ is a matching of $G$.

Let $T_{X}$ be a Xuong-tree and the edge-partition of $E(G)-E\left(T_{X}\right)$ be as defined in Lemma 1. Consider a set

$$
S_{X}=\left\{u_{i} \mid u_{i} \in e_{2 i-1} \cap e_{2 i}, 1 \leq i \leq m\right\} \cup\left\{v_{j} \mid v_{j} \text { is an end of } f_{j}, 1 \leq j \leq s\right\} .
$$

Then $G-S_{X}$ contains no cycle (since removing $S_{X}$ from $G$ will eliminate all the possible fundamental cycles of $G$ ) and hence $S_{X}$ is a decycling set of $G$, that is, $\nabla(G) \leq\left|S_{X}\right|$.

Corollary 2. $\nabla(G) \leq\left|S_{X}\right| \leq \gamma_{M}(G)+\xi(G)$ holds for every graph $G$.
It is easy to see that the bound $\left|S_{X}\right|$ heavily depends on the choice of Xuongtree $T_{X}$ (since different $T_{X}$ may lead to quiet different value of $\left|S_{X}\right|$ ). For instance, the wheel graph $W_{1, n}=K_{1} \vee C_{n}$ with $n$ spokes has $\nabla\left(W_{1, n}\right)=2$. If one chooses a Xuong-tree $K_{1, n}$ as a spanning tree of $W_{1, n}$, then the corresponding $\left|S_{X}\right|=\left\lceil\frac{n}{2}\right\rceil$; meanwhile, a Hamilton path in $W_{1, n}$ will determine another $S_{X}$ whose number of elements reaches the best value $\nabla\left(W_{1, n}\right)=2$. Therefore, how to find a set $S_{X} \subseteq V(G)$ with the smallest size is a key to determine $\nabla(G)$.

This paper is organized as follows.
In Section 2, we show that $\nabla(G)=n-\max _{T}\{\alpha(G-E(T))\}$ holds for any graph $G$. This implies that determining the decycling number is equivalent to finding the largest independence number of a co-tree. So, finding the decycling number $\nabla(G)$, determining the largest independence number of a co-tree $G$ $E(T)$ and finding the size of a largest induced forest in a graph $G$ are mutually equivalent. In this sense, finding the decycling number of a graph is very hard. Applying this formula, we may obtain lower bounds for the decycling number of
some (dense) graphs. As an example, we prove that $\nabla\left(K_{n}-\left(E\left(T_{1}\right) \cup E\left(T_{2}\right) \cup\right.\right.$ $\left.\left.\cdots \cup E\left(T_{k-1}\right)\right)\right) \geq n-2 k$, where $T_{1}, T_{2}, \ldots, T_{k-1}$ are $k-1$ edge-disjoint spanning trees of a complete graph $K_{n}$. Many examples and applications are presented to show how to apply trees into identifying the decycling number of a graph.

In Section 3, we obtain an another formula to compute the decycling number of regular graphs. For any decycling set $S$ of a $k$-regular graph $G$, we get that $|S|=\frac{1}{k-1}(\beta(G)+m(S))$. Obviously, $S$ is a $\nabla$-set if and only if $m(S)$ is minimum. Therefore, lower bounds for the decycling number of some (dense) graphs can be obtained. For a $k$-regular graph $G$, if $m(S) \geq 0$ for any $\nabla$-set $S$ of $G$, then $\nabla(G) \geq \frac{\beta(G)}{k-1}$. In many cases, these lower bounds may be tight (i.e., best possible) (see $[3,8,12,14-16,18,19]$ ). Observe that for some (4-regular) graphs $G$ of order $n$, there exists a decycling set $S$ such that the margin number $m(S)$ is a linear function of $n$. For instance, a toroidal 4-regular graph $G$ contains $n$ disjoint $K_{5}-e$ 's (see Figure 4) whose decycling number of $G$ is $2 n+1$. It is easy to see that the margin nuber $m(S)=n+2$ for any $\nabla$-set $S$ of $G$. Moreover, we discuss some relationships between the decycling number and the large genus embeddings of graphs, and show the effects of spanning trees on such topics. In particular, we give a new and direct proof of a result due to Speckenmeyer [17] and thus solve an open problem of Speckenmeyer searching for an efficient algorithm to compute $Z(G)$, the cardinality of the maximum nonseparating independent set of $G$.

In Section 4, we investigate the extremal 4-regular graphs $G$ with the decycling number $\nabla(G)=\left\lceil\frac{\beta(G)}{3}\right\rceil$. Our conclusion is that for any $\nabla$-set $S$ of graph $G$, there exists a spanning tree $T$ in $G$ such that elements of $S$ are taken from the leaves of $T$ with at most two exceptions (from the 2 or 3 -degree vertices of $T)$. Finally, we extend this result to general case.

## 2. A Formula Between the Decycling Number and the Independence Number

In this section, $\alpha(G)$ and $a(G)$ denote, respectively, the independence number and the number of vertices in a largest induced forest of a graph $G$.

Theorem 3. Let $G$ be a connected graph of order $n$. Then

$$
\nabla(G)=n-\max _{T}\{\alpha(G-E(T))\}
$$

where $T$ is taken over all spanning trees of $G$.
The above result reveals a relation among the decycling number, the independence number and the spanning trees in a graph and gives a new way to investigate these numbers.

Lemma 4. Let $G$ be a connected graph of order $n$. Then

$$
a(G)=\max _{T}\{\alpha(G-E(T))\},
$$

where $T$ is taken over all spanning trees of $G$.
Proof. Let $F$ be a largest induced forest of $G$ with $|F|=a(G)$. Then $V(G)-$ $V(F)$ is a decycling set of $G$. Since $G$ is connected, there exists a spanning tree $T$ of $G$ such that $F \subseteq T$, and $\alpha(G-E(T)) \geq|F|$. Hence,

$$
a(G) \leq \max _{T}\{\alpha(G-E(T))\} .
$$

Conversely, let $T_{1}$ be a spanning tree of $G$ such that

$$
\max _{T}\{\alpha(G-E(T))\}=\alpha\left(G-E\left(T_{1}\right),\right.
$$

and suppose that $A$ is the largest independent set of $G-E\left(T_{1}\right)$, that is, $|A|=$ $\alpha\left(G-E\left(T_{1}\right)\right)$. When we recover the edges of $T_{1}$ into $A$, it induces a forest, and $G-A$ is a decycling set. Hence $|A| \leq a(G)$. That is,

$$
\max _{T}\{\alpha(G-E(T))\} \leq a(G) .
$$

Proof of Theorem 3. By Lemma 4 and $a(G)+\nabla(G)=n$, the theorem follows.
From the proof of Theorem 3, one may see that if $T$ is a spanning tree of $G$ and $A$ is the maximum independent set of the co-tree $G-E(T)$ (i.e., $|A|=$ $\alpha(G-E(T)))$, then $S=V(G)-A$ is a decycling set, and so $|V(G)-A|+|A|=n$, which means that $|A|$ is the largest among all spanning trees of $G$ if and only if the corresponding decycling set $S=V(G)-A$ is minimum. Therefore, how to find a spanning tree $T$ of $G$ such that $\alpha(G-E(T))$ is the maximum is very crucial to computing the decycling number $\nabla(G)$ of $G$. In the following, we shall present several applications and examples to show the effects of the spanning trees on searching for the value $\nabla(G)$ of a graph $G$.
Example 1. Let $T$ be a Hamilton path of a complete graph $K_{n}$. Then $T$ is a spanning tree of $K_{n}$ (see Figure 1(a)), and so $\alpha\left(K_{n}-E(T)\right) \geq 2$. By Theorem $3, \nabla\left(K_{n}\right) \leq n-2$. It deduces that $\nabla\left(K_{n}\right)=n-2$ because of $\nabla\left(K_{n}\right) \geq n-2$ (if we remove at most $n-3$ vertices of $K_{n}$, then the resultant graph will contain a cycle).
Example 2. For a complete bipartite graph $K_{m, n}$, let $V\left(K_{m, n}\right)=V=X \cup$ $Y$, where $X=\left\{x_{1}, x_{2}, \ldots, x_{m}\right\}, Y=\left\{y_{1}, y_{2}, \ldots, y_{n}\right\}$. Assume that $m \leq n$. We construct a spanning tree $T$ of $K_{m, n}$ as follows: $E(T)=\left\{x_{m} y_{j}, x_{i} y_{i} \mid i=\right.$ $1,2, \ldots, m-1, j=1,2, \ldots, n\}$ (see Figure 1(b)). Then $\alpha\left(K_{m, n}-E(T)\right) \geq n+1$.

By Theorem 3, $\nabla\left(K_{m, n}\right) \leq m-1$. Since $\nabla\left(K_{m, n}\right) \geq m-1$ (otherwise, there is a cycle by removing at most $m-2$ vertices of $K_{m, n}$ in the resultant graph), we have $\nabla\left(K_{m, n}\right)=m-1$.

(a)

(b)

Figure 1. (a) $\nabla\left(K_{5}\right)=5-2=3$; (b) $\nabla\left(K_{3,4}\right)=3-1=2$.

Example 3. For a complete $k$-partite graph $K_{m_{1}, m_{2}, \ldots, m_{k}}$, let $V\left(K_{m_{1}, m_{2}, \ldots, m_{k}}\right)=$ $V=X_{1} \cup X_{2} \cup \cdots \cup X_{k}$. Then $\left|X_{i}\right|=m_{i}, i=1,2, \ldots, k$. Without loss of generality, suppose that $m_{1} \leq m_{2} \leq \cdots \leq m_{k}$. We construct a spanning tree $T$ of $K_{m_{1}, m_{2}, \ldots, m_{k}}$ as follows: $E(T)=\left\{x_{1 j} y, x_{11} z\right\}$, where $y \in V \backslash X_{1}, z$ takes over the elements of $V \backslash X_{1}, x_{11}, x_{1 j} \in X_{1}, j=2, \ldots, m_{1}$. By Theorem 3, $\nabla\left(K_{m_{1}, m_{2}, \ldots, m_{k}}-\right.$ $E(T)) \leq \sum_{i=1}^{k} m_{i}-m_{k}-1$. Since $\nabla\left(K_{m_{1}, m_{2}, \ldots, m_{k}}-E(T)\right) \geq \sum_{i=1}^{k} m_{i}-m_{k}-1$ (otherwise, removing at most $\sum_{i=1}^{k} m_{i}-m_{k}-2$ vertices of $K_{m_{1}, m_{2}, \ldots, m_{k}}$ will leave a cycle in the resultant graph $), \nabla\left(K_{m_{1}, m_{2}, \ldots, m_{k}}\right)=\sum_{i=1}^{k} m_{i}-m_{k}-1$.

Example 4. Ren [16] proved that $\nabla(G)=\gamma_{M}(G)+\xi(G)$ for $G$ being a cubic graph, where $\gamma_{M}(G)$ and $\xi(G)$ are the maximum genus and Betti deficiency of $G$, respectively. We consider a Xuong-tree $T_{X}$ of a cubic graph $G$ and an edgepartition of its co-tree $G-E\left(T_{X}\right)$ as defined in Lemma 1. Then the set $S_{X}$ (as defined in Corollary 2) is a $\nabla$-set since $G-S_{X}$ contains no cycle and $\left|S_{X}\right|=$ $\gamma_{M}(G)+\xi(G)$. Now $V(G)-S_{X}$ is an independent set whose cardinality is $n-$ $\max _{T}\{\alpha(G-E(T))\}$. Conversely, let $T_{1}$ be a spanning tree of $G$ such that $\alpha\left(G-E\left(T_{1}\right)\right)=\max _{T}\{\alpha(G-E(T))\}$. Then there exists an independent set $S$ of $G-E\left(T_{1}\right)$ with $|S|=\alpha\left(G-E\left(T_{1}\right)\right)$ such that $G-E\left(T_{1}\right)$ contains an independent set $A$ with $|A|=\alpha\left(G-E\left(T_{1}\right)\right)=\max _{T}\{\alpha(G-E(T))\}$. It is clear that $G[A]$ is a largest induced forest of $G$ and $S=V(G)-A$ is a $\nabla$-set. As shown in [16], $T_{1}$ is a Xuong-tree of $G$ (since the number of odd components of $G-E\left(T_{1}\right)$ is $\xi(G)$ ).

Based on Theorem 3, many results and problems on the largest induced forests and the decycling set can be translated into one another. For instance, Albertson and Berman [1] posed the following conjecture.

Conjecture 5 [1]. Every planar graph has an induced forest with at least half the vertices.

The above conjecture can also be expressed into the following three forms.

Theorem 6. Let $G$ be a planar graph of order n. Then the following statements are mutually equivalent:
(a) $\nabla(G) \leq \frac{n}{2}$;
(b) $|F| \geq \frac{n}{2}$ holds for a largest induced forest $F$ of $G$;
(c) There exists a spanning tree $T$ in $G$ such that $\alpha(G-E(T)) \geq \frac{n}{2}$.

For a plane triangulation $G$, a plane with all faces are triangles. By Theorem 6 (c), we may first find a spanning tree $T$ of $G$ to determine the independence number of its co-tree $G-E(T)$, and further to solve the decycling number of $G$. Since the number of edges of $G-E(T)$ is $2 n-5$, a natural idea is that the problem of the computation of the decycling number of a plane triangulation $G$ can be put into the following problem.

Problem 7. Determine the independence number for a planar graph $G$ of order $n$ with at most $2 n-5$ edges.

In the literature, there are many results on the decycling number for sparse graphs such as 3 (or 4 )-regular graphs, see $[3,8,12,14-16,18,19]$, but little is known for those with many edges (i.e., dense graphs). Theorem 3 offers a way to estimate the lower bounds for the decycling number of dense graphs. The following result is an application.

Theorem 8. Let $T_{1}, T_{2}, \ldots, T_{k-1}$ be $k-1$ edge-disjoint spanning trees of a complete graph $K_{n}$. Then

$$
\nabla\left(K_{n}-\left(E\left(T_{1}\right) \cup E\left(T_{2}\right) \cup \cdots \cup E\left(T_{k-1}\right)\right)\right) \geq n-2 k .
$$

And the equality holds if and only if $K_{n}-\left(E\left(T_{1}\right) \cup E\left(T_{2}\right) \cup \cdots \cup E\left(T_{k-1}\right)\right)$ contains a spanning tree $T_{k}$ such that the graph $T_{1} \cup T_{2} \cup \cdots \cup T_{k-1} \cup T_{k}$ contains $K_{2 k}$.

Proof. Let $T_{1}, T_{2}, \ldots, T_{k-1}$ be $k-1$ edge-disjoint spanning trees of a complete graph $K_{n}$. Assume (reductio ad absurdum) that $T_{k}$ is a spanning tree of $K_{n}-$ $\left(E\left(T_{1}\right) \cup E\left(T_{2}\right) \cup \cdots \cup E\left(T_{k-1}\right)\right)$ such that $\alpha\left(K_{n}-\left(E\left(T_{1}\right) \cup E\left(T_{2}\right) \cup \cdots \cup E\left(T_{k-1}\right)\right.\right.$ $\left.\left.\cup E\left(T_{k}\right)\right)\right) \geq 2 k+1$. Then $T_{1} \cup T_{2} \cup \cdots \cup T_{k-1} \cup T_{k}$ contains $K_{2 k+1}$, and hence

$$
\left|\left\{E\left(T_{1}\right) \cup E\left(T_{2}\right) \cup \cdots \cup E\left(T_{k-1}\right) \cup E\left(T_{k}\right)\right\} \cap E\left(K_{2 k+1}\right)\right| \geq k(2 k+1) .
$$

Color the edges of $K_{2 k+1}$ with $k$ different colors, then the number of edges with the same color is not more than $2 k$. Otherwise, there will exist a subgraph (induced by these edges) containing a cycle, which contraries to the number of edges of $K_{2 k+1}$. By Theorem 3,

$$
\begin{aligned}
& \nabla\left(K_{n}-\left(E\left(T_{1}\right) \cup E\left(T_{2}\right) \cup \cdots \cup E\left(T_{k-1}\right)\right)\right) \\
& =n-\max _{T_{k}}\left\{\alpha\left(K_{n}-\left(E\left(T_{1}\right) \cup E\left(T_{2}\right) \cup \cdots \cup E\left(T_{k-1}\right) \cup E\left(T_{k}\right)\right)\right)\right\},
\end{aligned}
$$

that is,

$$
\max _{T_{k}}\left\{\alpha\left(K_{n}-\left(E\left(T_{1}\right) \cup E\left(T_{2}\right) \cup \cdots \cup E\left(T_{k-1}\right) \cup E\left(T_{k}\right)\right)\right)\right\} \leq 2 k .
$$

Let $\nabla\left(K_{n}-\left(E\left(T_{1}\right) \cup E\left(T_{2}\right) \cup \cdots \cup E\left(T_{k-1}\right)\right)\right)=n-2 k$. By Theorem 3, there exists a spanning tree $T_{k}$ of $K_{n}-\left(E\left(T_{1}\right) \cup E\left(T_{2}\right) \cup \cdots \cup E\left(T_{k-1}\right)\right)$ such that $\alpha\left(K_{n}-\left(E\left(T_{1}\right) \cup E\left(T_{2}\right) \cup \cdots \cup E\left(T_{k-1}\right) \cup E\left(T_{k}\right)\right)=2 k\right.$. The edges of each $T_{i}(1 \leq i \leq k)$ in $K_{2 k}$ form a spanning tree of $K_{2 k}$. Conversely, for $k-1$ edge-disjoint spanning trees $T_{1}, T_{2}, \ldots, T_{k-1}$ of $K_{n}$, if there is a spanning tree $T_{k}$ of $K_{n}-\left(E\left(T_{1}\right) \cup E\left(T_{2}\right) \cup \cdots \cup E\left(T_{k-1}\right)\right)$ such that the subgraphs of $K_{n}$ determined by these trees $T_{1}, T_{2}, \ldots, T_{k-1}, T_{k}$ containing a complete graph $K_{2 k}$, then $\alpha\left(K_{n}-\left(E\left(T_{1}\right) \cup E\left(T_{2}\right) \cup \cdots \cup E\left(T_{k-1}\right) \cup E\left(T_{k}\right)\right)\right) \geq 2 k$.

## 3. A Formula Between the Decycling Number and the Margin Number

Let $E(S, G-S)$ be the set of edges such that each edge has one vertex in $S$ and another one in $G-S . d_{G}(x)$ and $\Delta(G)$ (or $\Delta$ for short) represent the degree of vertex $x$ and the maximum degree of $G$, respectively. In this section, we present another formula for the decycling number $\nabla(G)$ of a $k$-regular graph $G$.
Theorem 9. Let $S$ be a decycling set of $G$. Then

$$
\sum_{x \in S}\left(d_{G}(x)-1\right)=\beta(G)+m(S) .
$$

Proof. Let $S=\left\{x_{1}, x_{2}, \ldots, x_{|S|}\right\}$ be a decycling set of $G$. Then

$$
\begin{aligned}
q-\sum_{i=1}^{|S|} d_{G}\left(x_{i}\right) & =q-|E(S, G-S)|-2|E(S)| \\
& =p-|S|-c-|E(S)|,
\end{aligned}
$$

where $p=|V(G)|, q=|E(G)|, c$ and $|E(S)|$ are, respectively, the number of components of $G-S$ and the number of edges of $G[S]$.

As $\beta(G)=q-p+1$,

$$
\sum_{i=1}^{|S|}\left(d_{G}\left(x_{i}\right)-1\right)=\beta(G)+c+|E(S)|-1 .
$$

And for $m(S)=c+|E(S)|-1$, we have

$$
\sum_{i=1}^{|S|}\left(d_{G}\left(x_{i}\right)-1\right)=\beta(G)+m(S)
$$

This completes the proof.

Remark 1. Observe that if all the vertices of $S$ have degree $k$, then $|S|=$ $\frac{1}{k-1}(\beta(G)+m(S))$. In particular, if $G$ is a $k$-regular graph, then $S$ is a $\nabla$-set if and only if $m(S)$ is minimum among all the decycling set $S$ of $G$.

Although there exists some uncertain parameter like $m(S)$, this result provides a way to locate the value of $\nabla(G)$ : once we find a decycling set $S$ such that $m(S)$ reaches the minimum, then $S$ is a $\nabla$-set of $G$. We will show its applications in the discussion to come.

A simple corollary of Theorem 9 is:
Corollary 10. Let $G$ be a graph with maximum degree $\Delta$ which has the $\nabla$-set such that each vertex of this set has degree $\Delta$. Then

$$
\begin{equation*}
\nabla(G) \geq \frac{\beta(G)}{\Delta-1} \tag{1}
\end{equation*}
$$

Remark 2. (i) $\nabla(G)=\frac{\beta(G)}{\Delta-1}$ if and only if $m(S)=0$ which means that for a $\nabla$-set $S$ of $G, G-S$ is a tree $T_{0}$ and $G[S]$ is an empty subgraph. In this case, $\nabla(G)$ has a strong combinatorial characterization: for any vertex $x \in S$ incident to a vertex $y \in V\left(T_{0}\right)$, insert the edge $x y$ into $T_{0}$. This procedure determines a spanning tree $T$ (it is in fact a Xuong-tree) of $G$ (such that $\xi(G)=0$ ) if we add $|S|$ edges into $T_{0}$. Therefore, deleting a vertex $x$ of $S$ will destroy $d_{G}(x)-1$ fundamental cycles of $G$ and deleting $\nabla(G)$ vertices of $S$ will destroy all fundamental cycles of $G$;
(ii) the inequality (1) may be tight, see $[3,8,12,14-16,18,19]$.

The following examples show the formula of Theorem 9 applying on some types of regular graphs.

Example 5. Let $S$ be a decycling set of a hypercube $Q_{n}$ (a graph contains $2^{n}$ n-tuples of 0 's and 1's as vertices with two vertices adjacent if they differ in exactly one position). Then

$$
\begin{equation*}
2^{n-1}-\frac{2^{n-1}-1}{n-1} \leq \nabla\left(Q_{n}\right) \leq 2^{n-1}-\frac{2^{n-1}-m(S)-1}{n-1} \tag{2}
\end{equation*}
$$

The inequalities in (2) are equalities for $n=3,4[3]$ (see Figure 2).
Proof. By the definition of $Q_{n}, \beta\left(Q_{n}\right)=(n-2) 2^{n-1}+1$. By Corollary 10, $\nabla\left(Q_{n}\right) \geq 2^{n-1}-\frac{2^{n-1}-1}{n-1}$. Let $S$ be a decycling set of $Q_{n}$. Then $|S|=2^{n-1}-$ $\frac{2^{n-1}-m(S)-1}{n-1}$, thus $\nabla\left(Q_{n}\right) \leq|S|=2^{n-1}-\frac{2^{n-1}-m(S)-1}{n-1}$.

Remark 3. (i) Two spanning trees $T_{1}$ and $T_{2}$ which induced by the bold edges in Figure 2(a) and (b) of $Q_{n}$ satisfying that $Q_{n}-E\left(T_{i}\right)(i=1,2)$ has the largest independence number, respectively.
(ii) Focardi [6] proved that $2^{n-1}-\frac{2^{n-1}-1}{n-1} \leq \nabla\left(Q_{n}\right) \leq 2^{n-1}-\frac{2^{n-1}}{2(n-1)}$. The lower bound was also proved by Beineke and Vandell [3]. In fact, if the upper bound of Focardi's result is best possible, then $m(S)=2^{n-2}-1$ for any $\nabla$-set $S$. Therefore, determining the decycling number of $Q_{n}$ for larger $n$ is very difficult.


Figure 2. (a) $Q_{3}$ with $m(S)=1$; (b) $Q_{4}$ with $m(S)=1$.

Example 6. For any two cycles $C_{m}$ and $C_{n}$, their Cartesian product is the graph $C_{m} \times C_{n}$ with vertex set $V\left(C_{m} \times C_{n}\right)=\left\{w_{i j} \mid i=1,2, \ldots, m, j=1,2, \ldots, n\right\}$ and edge set $E\left(C_{m} \times C_{n}\right)=\left\{w_{i j} w_{r s} \mid i=r, v_{j} v_{s} \in E\left(C_{n}\right)\right.$ or $\left.j=s, u_{i} u_{r} \in E\left(C_{m}\right)\right\}$. Clearly, $C_{m} \times C_{n}$ is a 4-regular graph, see Figure 3. For any decycling set $S$ of $C_{m} \times C_{n}$, we have

$$
\begin{equation*}
\frac{m n+1}{3} \leq \nabla\left(C_{m} \times C_{n}\right) \leq \frac{m n+m(S)+1}{3}, m, n \geq 3 \tag{3}
\end{equation*}
$$

In particular, the bounds of $\nabla\left(C_{m} \times C_{n}\right)$ in (3) are sharp for $m=3$ (or $n=3$ ) (see Figure 3).

Proof. It is easy to see that $\beta\left(C_{m} \times C_{n}\right)=m n+1$. By Corollary 10, $\nabla\left(C_{m} \times\right.$ $\left.C_{n}\right) \geq \frac{1}{3}(m n+1)$. Let $S$ be a decycling set of $C_{m} \times C_{n}$. Then $\nabla\left(C_{m} \times C_{n}\right) \leq$ $|S|=\frac{1}{3}(m n+m(S)+1)$. When $m=3, \frac{3 n+1}{3} \leq \nabla\left(C_{3} \times C_{n}\right) \leq \frac{3 n+m(S)+1}{3}$, that is, $n+1 \leq \nabla\left(C_{3} \times C_{n}\right) \leq n+\frac{m(S)+1}{3}$, we can find a decycling set $S$ such that $m(S)=2\left(\right.$ see Figure 3), then $\nabla\left(C_{3} \times C_{n}\right)=n+1$.


Figure 3. A drawing of $C_{m} \times C_{n}$ on the torus.

Remark 4. (i) The spanning trees $T$ of $C_{3} \times C_{n}$ in Figure 3(a) and (b) satisfy that $C_{3} \times C_{n}-E(T)$ has the largest independence number, respectively.
(ii) Our result shows that $\nabla\left(C_{m} \times C_{n}\right)=\frac{m n+m(S)+1}{3}$ for $S$ being a $\nabla$-set of $C_{m} \times C_{n}$, which equals to Pike's result $\nabla\left(C_{m} \times C_{n}\right)=\left\lceil\frac{m n+2}{3}\right\rceil(m, n \neq 4)$ when $m(S) \leq 1[13]$. Therefore, this provides a way to locate the exact value of $\nabla(G)$ (to find a decycling set $S$ with the minimum $m(S)$ ).

The formula of Theorem 9 also has some applications in topological graph theory.

A vertex set $S$ is called a nonseparating independent set of $G$ if $S$ is an independent set of $G$ and $G-S$ is connected. The cardinality of a maximum nonseparating independent set of $G$ is denoted by $Z(G)$. The following result shows a close relation between nonseparating independence number $Z(G)$ and the maximum genus $\gamma_{M}(G)$ of a cubic graph $G$ and makes an extention of a result due to Speckenmeyer [17], we give a new and direct proof of it via trees.

Theorem 11. Let $G$ be a cubic graph. Then
(a) $Z(G)=\gamma_{M}(G)$;
(b) for every maximum nonseparating independent set $S$ of $G, G-S$ contains no two cycles sharing a vertex in common. Moreover, there exists a Xuong-tree $T_{X}$ such that the elements in $S$ are leaves of $T_{X}$.
Proof. Let $T_{X}$ be a Xuong-tree of $G$ with an edge-partition of $G-E(T)$ as defined in Lemma 1. Then for $1 \leq i \leq \gamma_{M}(G), e_{2 i-1} \cap e_{2 i}=\left\{u_{i}\right\}$ forms a set of independent vertices of $G$ (which are leaves of $T_{X}$ ). Hence $\left\{u_{1}, u_{2}, \ldots, u_{\gamma_{M}(G)}\right\}$ is a nonseparating independent set of $G$. Therefore, $Z(G) \geq \gamma_{M}(G)$. To see the converse inequality, we consider a nonseparating independent set $S$. Then $G-S$ is connected. We may suppose further that $G-S$ is a tree $T_{0}$ and $T$ is a spanning tree built in Remark 2(i). Then elements of $S$ are leaves of $T$. After repeating the argument in Remark 2(i), we may see that $|S| \leq \gamma_{M}(G)$ which means that $Z(G) \leq \gamma_{M}(G)$. This proves (a).

Suppose that $G-S$ contains two cycles with one vertex in common. Then by Lemma $1, \gamma_{M}(G-S) \geq 1$, which together with the construction of a largest genus embedding stated in the proof of (a), $\gamma_{M}(G) \geq Z(G)+1\left(=\gamma_{M}(G)+1\right)$, a contradiction. Therefore, cycles of $G-S$ are independent. In fact, the number of cycles in $G-S$ is $\xi(G)$. In addition, any spanning tree $T_{0}$ of $G-S$ is a subgraph of a Xuong-tree $T_{X}$ of $G$ (as stated in the proof of (a)). This proves (b).

By Theorem 11, we obtain a result which has been proved by Speckenmeyer in [17] as follows.
Corollary 12. Let $G$ be a cubic graph of order $n$. Then $\nabla(G)+Z(G)=\frac{n+2}{2}$.
Furthermore, the formula (a) in Theorem 11, together with a result of Furst [7], provide an efficient way to compute the value $Z(G)$ for cubic graphs and solves an open problem raised in [17] searching for a polynomial time algorithm to decide $Z(G)$ and $\nabla(G)$.

Theorem 13. Let $S$ be $a \nabla$-set of a $k$-regular graph $G$.
(a) If $m(S)=0$, then for every $\nabla$-set $S$, there exists a spanning tree $T$ in $G$ such that all vertices of $S$ are leaves of $T$.
(b) If $m(S)=0$, and $k \equiv 1(\bmod 2)$, then $\nabla(G)=\frac{2}{k-1} \gamma_{M}(G)$, and each spanning tree $T$ in (a) is a Xuong-tree of $G$.

Proof. Let $S=\left\{x_{1}, x_{2}, \ldots, x_{\nabla}\right\}$ be a $\nabla$-set of a $k$-regular graph $G$ with $m(S)=$ 0 . Then $G[S]$ has no cycle and $G-S$ is a tree $T_{0}$. Suppose that $y_{i}$ is a neighbor of $x_{i}$ in $T_{0}(i=1,2, \ldots, \nabla)$. Then $T=T_{0}+\left\{e_{i}=x_{i} y_{i} \mid 1 \leq i \leq \nabla\right\}$ is a spanning tree of $G$ and $S$ is a subset of leaves of $T$. If $k \equiv 1(\bmod 2)$, then we arrange the left $k-1$ edges (other than $\left.x_{i} y_{i}\right)$ into $\frac{k-1}{2}$ pairs for each $i(1 \leq i \leq \nabla)$ and thus it gives rise to an edge-partition of $G-E(T)$ with $\xi(G)=0$. Notice that in Xuong's construction of the maximum genus embedding [20], each pair of adjacent edges in $G-E(T)$ will contribute a genus, the graph $G$ may be embedded into an orientable surface with $\frac{k-1}{2} \nabla=\gamma_{M}(G)$ handles. This ends the proof of (b). As for (a), it follows from the discussion used in the proof of (a) of Theorem 11.

One case may appear if there is a vertex $x$ of a decycling set $S$ such that $d_{G}(x)<\Delta$. Then the formula of Theorem 9 will be invalid for this case. For instance, a grid of paths $P_{m}$ and $P_{n}$ is the graph $P_{m} \times P_{n}$ with vertex set $\left.V\left(P_{m} \times P_{n}\right)=\left\{w_{i j} \mid i=1,2, \ldots, m, j=1,2, \ldots, n\right\}\right)$ and edge set $E\left(P_{m} \times P_{n}\right)=$ $\left\{w_{i j} w_{r s} \mid i=r, v_{j} v_{s} \in E\left(P_{n}\right)\right.$ or $\left.j=s, u_{i} u_{r} \in E\left(P_{m}\right)\right\}$. We have to choose at least one vertex of degree 3 to eliminate the boundary cycle of $P_{m} \times P_{n}$. It is clear that any $\nabla$-set $S$ of the grid $P_{m} \times P_{n}$ does not need to contain a 2-degree vertex (since subdividing an edge of a graph does not change its decycling number). Therefore, we may only consider the $\nabla$-set $S$ whose vertices are of degree 4 or 3 . Here, we slightly extend the formula of Theorem 9 as follows.

Theorem 14. Let $G$ be a non-regular graph with maximum degree $\Delta$ and $S$ a decycling set of $G$. Suppose that $d(d<\Delta)$ is a fixed natural number with $S=S_{\alpha} \cup S_{\beta}, S_{\alpha}=\left\{x \mid d_{G}(x)=\Delta, x \in S\right\}, S_{\beta}=\left\{x \mid d_{G}(x)=d<\Delta, x \in S\right\}$. Then $|S|=\frac{1}{\Delta-1}\left(\beta(G)+(\Delta-d)\left|S_{\beta}\right|+m(S)\right)$.
Proof. Let $S$ be a decycling set of a graph $G$. Similar to the proof of Theorem 9,

$$
(\Delta-1)\left(|S|-\left|S_{\beta}\right|\right)+(d-1)\left|S_{\beta}\right|=\beta(G)+c+|E(S)|-1
$$

i.e.,

$$
(\Delta-1)|S|-(\Delta-d)\left|S_{\beta}\right|=\beta(G)+c+|E(S)|-1
$$

and then

$$
|S|=\frac{1}{\Delta-1}\left(\beta(G)+(\Delta-d)\left|S_{\beta}\right|+m(S)\right)
$$

since $m(S)=c+|E(S)|-1$.

Corollary 15. Let $G$ be a non-regular graph with maximum degree $\Delta$ which has the $\nabla$-set such that each vertex of this set has degree $\Delta$. Then

$$
\nabla(G) \geq \frac{1}{\Delta-1}(\beta(G)+\Delta-d),
$$

where $d(d<\Delta)$ is the degree of some vertices of a $\nabla$-set of $G$.
Example 7 (Cartesian product of two paths). For any decycling set $S$ of $P_{m} \times P_{n}$,

$$
\begin{equation*}
\frac{m n-m-n+2}{3} \leq \nabla\left(P_{m} \times P_{n}\right) \leq \frac{m n-m-n+\left|S_{\beta}\right|+m(S)+1}{3} \tag{4}
\end{equation*}
$$

for $m, n \geq 3$.
The bounds of $\nabla\left(P_{m} \times P_{n}\right)$ in (4) are best possible for $n=4,6,7[3]$.
Proof. By the definition of $P_{m} \times P_{n}, \beta\left(P_{m} \times P_{n}\right)=m n-m-n+1$. By Corollary 15 , it follows that $\nabla\left(P_{m} \times P_{n}\right) \geq \frac{1}{3}(m n-m-n+2)$. Let $S$ be a decycling set of $P_{m} \times P_{n}$. Then $\nabla\left(P_{m} \times P_{n}\right) \leq|S|=\frac{1}{3}\left(m n-m-n+\left|S_{\beta}\right|+m(S)+1\right)$.

Remark 5. Some nonregular graphs may also have a $\nabla$-set with a large margin number, such as the grid $P_{5} \times P_{n}$. Beineke [3] proved that $\nabla\left(P_{5} \times P_{n}\right)=\left\lfloor\frac{3 n}{2}\right\rfloor-$ $\left\lfloor\frac{n}{8}\right\rfloor-1$. Together with Theorem 14, we get $m(S)+\left|S_{\beta}\right|=3\left(\left\lfloor\frac{3 n}{2}\right\rfloor-\left\lfloor\frac{n}{8}\right\rfloor\right)-4 n+1$, which tends to infinite as $n \rightarrow \infty$.

The above discussions imply that the margin number $m(S)$ of a decycling set $S$ may be arbitrarily large for some regular graphs. For some (4-regular) graphs $G$ of order $n$, there exists a decycling set $S$ such that the margin number $m(S)$ is a linear function of $n$. For instance, a toroidal 4 -regular graph $G$ containing $n$ disjoint $K_{5}-e$ 's (see Figure 4) whose decycling number of $G$ is $2 n+1$, and by formula (1), its margin number $m(S)=n+2$.


Figure 4. A toroidal 4-regular graph with $m(S)=n+2$.
On the other hand, suppose that $S$ is a decycling set of a regular graph $G$, for any vertex $x \in S$, adding an edge to join $x$ and $G-S$, this procedure determines a Xuong-tree $T_{X}$ since $G-E\left(T_{X}\right)$ has no odd components (i.e., $\xi(G)=0$ ), which means that the elements of $S$ are taken from the leaves of a Xuong-tree. This may be extended to the $\nabla$-set $S$ with the margin number $m(S)$ are of relative
small, that is, when the margin number is small enough, the elements of $S$ are taken from the leaves of a spanning tree $T$ with few exceptions. We shall discuss this situation in Section 4.

## 4. 4-Regular Graphs

In this section we concentrate on studying the combinatorial structure of 4-regular graphs $G$ with the decycling number $\nabla(G)=\left\lceil\frac{\beta(G)}{3}\right\rceil$.

Theorem 16. Let $G$ be a 4 -regular graph with $\nabla(G)=\left\lceil\frac{\beta(G)}{3}\right\rceil$. Then there exists a spanning tree $T$ in $G$ such that elements of any $\nabla$-set of $G$ are simply the leaves of $T$ with at most two exceptions.

Proof. Let $S$ be a $\nabla$-set of a 4-regular graph $G$. Assume that $\beta(G)=3 m+r$, $0 \leq r \leq 2, m$ is a nonnegative integer. Then three claims arise.
Claim 1. If $r=0$, then $S$ is a $\nabla$-set if and only if $m(S)=0$ and vertices of $S$ are leaves of a spanning tree $T$ of $G$.

Proof. The first part follows from Theorem 9. Now suppose that $m(S)=0$. Then $c=1$ and $|E(S)|=0$. We can construct a spanning tree $T$ as we have reasoned in the proof of Theorem 9. It is clear that the elements of $S$ are leaves of $T$.

Claim 2. If $r=1$, then $S$ is $a \nabla$-set if and only if $m(S)=2$ and vertices of $S$ are leaves of a spanning tree $T$ of $G$ with at most two exceptions.

Proof. The first part follows from Theorem 9. It is clear that $c+|E(S)|=3$ since $m(S)=2$. We construct a spanning tree $T$ of $G$ satisfying the above condition. There are three cases according to the values of $c$ and $|E(S)|$.

Case 1. $\quad c=1$ and $|E(S)|=2$. Since $G$ is connected, there exists two edges, say $e_{1}=a b$ and $e_{2}=c d$ (possibly $b=c$ ) in $G[S]$. For each vertex $x \in S-\{a, b, c, d\}$, add edges $e=x y, e_{3}=b y$ and $e_{4}=c y$ (prescribe $e_{3}=e_{4}$ when $b=c$ ), into $G-S$, where $y \in G-S$. After this, we obtain a spanning tree $T$ of $G$ containing $G-S$ as its subgraph. Which satisfies the condition of Claim 2 (i.e., when $b \neq c, S$ has two vertices $b, c$ which are not leaves of $T$; if $b=c$, then the only exception of $S$ is $b=c$ ).

Case 2. $c=2$ and $|E(S)|=1$. Without loss of generality, let $Q_{1}, Q_{2}$ be the two components of $G-S$ and $E(S)=\{x y\}, x, y \in S$. Since $G$ is connected, there exists an edge $e=x y$ (possibly $x=y$ ) in $G[S]$. Two situations will appear to construct a spanning tree $T$ of $G$ : (a) If $x \neq y$, then (i) $x$ and $y$ join $Q_{1}$ and $Q_{2}$,
respectively. Let $x_{1} \in Q_{1}$ and $x_{2} \in Q_{2}$ be such that $x x_{1}, y x_{2} \in E(G)$. The edges $x y, x x_{1}, y x_{2}, Q_{1}$ and $Q_{2}$ form a tree $T_{1}$ containing $Q_{1}, Q_{2}$; (ii) $x$ joins $Q_{1}$ and $Q_{2}$. Let $x_{0} \in Q_{1}$ and $y_{0} \in Q_{2}$ be such that $x x_{0}, x y_{0} \in E(G)$. Then the edges $x y, x x_{0}, x y_{0}, Q_{1}$ and $Q_{2}$ form a tree $T_{1}$ containing $Q_{1}, Q_{2}$. For other vertices $z \in S-\{x, y\}$, we add an edge join $z$ with $Q_{1} \cup Q_{2}$. It is clear that such edges and $T_{1}$ form a spanning tree $T$ of $G$. (b) If $x=y$, then there is an edge $e_{0}=f g$ in $G[S]$ such that $x$ joins $Q_{1}$ and $Q_{2}, f$ joins $Q_{1} \cup Q_{2}$, the remaining vertices of $S-\{x, f, g\}$ as did in the case of $x \neq y$, so we may construct a spanning tree $T$ of $G$. The above spanning trees also satisfy the condition of Claim 2 .

Case 3. $c=3$ and $|E(S)|=0$. Suppose that $Q_{1}, Q_{2}$ and $Q_{3}$ are three components of $G-S$. Then a spanning tree $T$ of $G$ will be constructed as follows. Since $G$ is connected, there exist two vertices, say $x$ and $y$ (possibly $x=y$ ), in $S$ such that $x$ joins $Q_{1}$ and $Q_{2}, y$ joins $Q_{2}$ and $Q_{3}$. This time we may also construct a spanning tree $T$ of $G$ which contains $Q_{1}, Q_{2}$ and $Q_{3}$ as we did in Case 2. And hence, when $x \neq y, x$ and $y$ are the only two vertices in $S$ which are not leaves of $T$; if $x=y$, then $x$ is the only vertex in $S$ is not the leaf of $T$.

Claim 3. If $r=2$, then $S$ is a $\nabla$-set if and only if $m(S)=1$. Meanwhile, there exists a spanning tree $T$ of $G$ such that all (but at most one) vertices of $S$ are leaves of $T$.

Proof. The proof of Claim 3 is analogous to Claims 1 and 2, we omit its proof.
Now the entire proof of the theorem is complete.
We give three examples of 4-regular graphs with $m(S)=0,1,2$, respectively. See Figure 5(a), Figure 5(b) and Figure 2(b).


Figure 5. (a) 4-regular graph with $m(S)=0$; (b) 4-regular graph with $m(S)=2$.

After a similar discussion in 4-regular graphs, we may extend Theorem 16 to general case.
Theorem 17. Let $G$ be a $k$-regular graph with $\nabla(G)=\left\lceil\frac{\beta(G)}{k-1}\right\rceil$ and $\beta(G)=$ $m(k-1)+r, 0 \leq r \leq k-2, m$ is a nonnegative integer. Then $S$ is $a \nabla$-set of $G$
if and only if

$$
m(S)= \begin{cases}0, & \text { for } r=0 \\ k-r-1, & \text { for otherwise }\end{cases}
$$

Moreover, there exists a spanning tree $T$ in $G$ such that elements of $S$ are simply the leaves of $T$ with at most $m(S)$ exceptions.

## Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments and helpful suggestions. This research was supported by the National Natural Science Foundation of China under Grant No. 11171114 and No. 11401576, Science and Technology Commission of Shanghai Municipality under Grant No. 13dz2260400.

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