Discussiones Mathematicae
Graph Theory 39 (2019) 117–123
doi:10.7151/dmgt.2062

HAMILTON CYCLES IN DOUBLE GENERALIZED PETERSEN GRAPHS

Yutaro Sakamoto

Department of Informatics
The University of Electro-Communications, Japan

e-mail: y-sakamoto@uec.ac.jp

Abstract

Coxeter referred to generalizing the Petersen graph. Zhou and Feng modified the graphs and introduced the double generalized Petersen graphs (DGPGs). Kutnar and Petecki proved that DGPGs are Hamiltonian in special cases and conjectured that all DGPGs are Hamiltonian. In this paper, we prove the conjecture by constructing Hamilton cycles in any given DGPG.

Keywords: double generalized Petersen graph, Hamilton cycle.

2010 Mathematics Subject Classification: 05C45, 05C38.

1. Introduction

A Hamilton cycle in a graph is a cycle which contains all vertices of the graph exactly once. In graph theory, the existence of Hamilton cycles is one of basic properties of graphs and it has been researched for many years. In general, it is very difficult to determine if a given graph has Hamilton cycles. Karp [5] proved that the problem is NP-complete.

Now we introduce the history of double generalized Petersen graphs (DGPGs) in brief.

Generalized Petersen graphs first appeared in work by Coxeter in 1950 on self-dual configurations [3]. They were reintroduced and termed generalized Petersen graphs (GPGs) by Watkins [7], who studied their Tait coloring. Castagna and Prins [2] proved Watkins' conjecture that all GPGs except the Petersen graph have a Tait coloring. Further properties of GPGs have been studied. For instance, Alspach [1] completed the determination of which GPGs have a Hamilton cycle. Fu, Yang and Jiang [4] studied the domination number of GPGs.

Zhou and Feng [8] extended the notion of GPGs to DGPGs. Let $\mathbb{Z}_n$ denote the set of integers $\mathbb{Z}/n\mathbb{Z}$ throughout this paper.

Definition 1 [8]**.** Let n and t be integers that satisfy $n \geq 3$ and $2 \leq 2t < n$. The double generalized Petersen graph $\mathrm{DP}(n, t)$ is an undirected simple graph with vertex set V and edge set E, where

$V = \{x_i, u_i, v_i, y_i \mid i \in \mathbb{Z}_n\}$,

$E = \{x_i x_{i+1}, y_i y_{i+1}, x_i u_i, y_i v_i, u_i v_{i+t}, v_i u_{i+t} \mid i \in \mathbb{Z}_n\}$.

Both GPGs and DGPGs are cubic graphs.

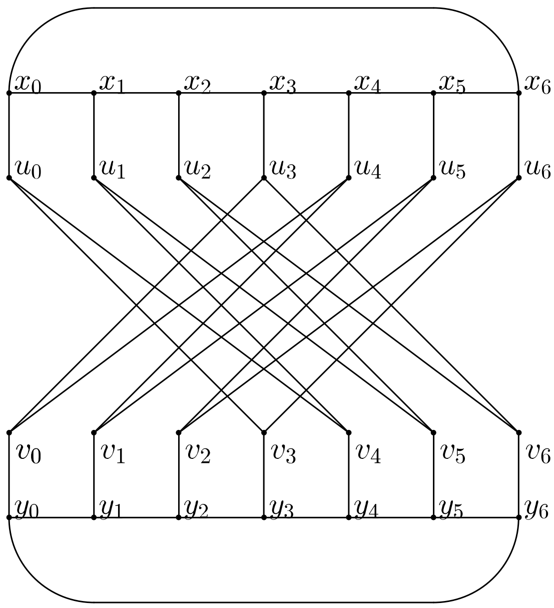

Figure 1. $\mathrm{DP}(7, 3)$.

Zhou and Feng [9] determined all non-Cayley vertex-transitive graphs and all vertex-transitive graphs among DGPGs.

Theorem 2 [9]**.** *DP(n, t) is vertex-transitive if and only if either $n = 5$ and $t = 2$, or $n = 2k$ and $t^2 \equiv \pm 1 (\mathrm{mod\ k})$. For the former case, DP$(5, 2)$ is isomorphic to Dodecahedron GP$(10, 2)$, which is a non-Cayley graph, and for the latter case, if $t^2 \equiv 1 (\mathrm{mod\ k})$, then DP$(n, t)$ is a Cayley graph, and if $t^2 \equiv -1 (\mathrm{mod\ k})$, then DP$(n, t)$ is a non-Cayley graph.*

Using this result, Kutnar and Petecki [6] gave the complete classification of automorphism groups of DGPGs. They also tried to find Hamilton cycles in DGPGs.

Theorem 3 [6]**.** *DP*(n,t) *is Hamiltonian if* n *is even.*

Theorem 4 [6]**.** *DP*(n,t) *is Hamiltonian if* n *is odd and the greatest common divisor of* n *and* t *equals to* 1*.*

Theorems 2, 3 and 4 give some specific examples of the open question of whether all Cayley graphs of finite groups are Hamiltonian.

Theorem 5 [6]**.** *DP*(n,t) *is Hamiltonian if* $n \leq 31$*.*

Kutnar and Petecki used computers to prove Theorem 5. From the above theorems, they proposed the following conjecture.

Conjecture 6 [6]**.** *All DP*(n,t) *are Hamiltonian.*

In this paper, we shall prove the conjecture by constructing Hamilton cycles in any given DGPG.

Theorem 7. *All DP*(n,t) *are Hamiltonian.*

2. PRELIMINARIES

As mentioned in the previous section, $\mathbb{Z}_n$ denotes the set of integers $\mathbb{Z}/n\mathbb{Z}$. A sequence of vertices $w_0w_1w_2\cdots w_n$ denote a path in a graph. A path whose end points are the same vertex is called a cycle. Let $V(G)$ denotes the vertex set of a graph G. Let G be an arbitrary subgraph of $\mathrm{DP}(n,t)$. We define functions V_x, V_y, V_u, V_v as follows:

$$\begin{aligned}
V_x(G) &= V(G) \cap \{x_i \mid i \in \mathbb{Z}_n\},\\
V_y(G) &= V(G) \cap \{y_i \mid i \in \mathbb{Z}_n\},\\
V_u(G) &= V(G) \cap \{u_i \mid i \in \mathbb{Z}_n\},\\
V_v(G) &= V(G) \cap \{v_i \mid i \in \mathbb{Z}_n\}.
\end{aligned}$$

3. THE CONSTRUCTION OF HAMILTON CYCLES IN DP(n,t)

We assume that n is even. In this case, Kutnar and Petecki [6] showed that all $\mathrm{DP}(n,t)$ are Hamiltonian. We define paths X_i for each $i \in \mathbb{Z}_{n/2}$ as follows

$$X_i : u_{2i}x_{2i}x_{2i+1}u_{2i+1}v_{2i+1-t}y_{2i+1-t}y_{2i+2-t}v_{2i+2-t}u_{2(i+1)}.$$

Joining all of the paths gives a Hamilton cycle in $\mathrm{DP}(n,t)$.

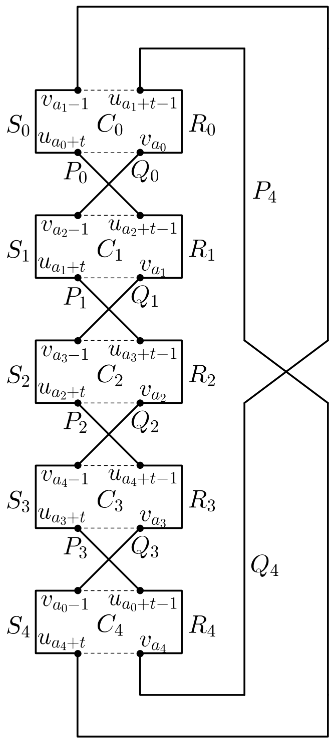

Figure 2. A Hamilton cycle in $\mathrm{DP}(n,t)$ $(2k+1=5)$.

We assume that n is odd. Let $2k+1$ be the greatest common divisor of n and t. In order to construct a Hamilton cycle in $\mathrm{DP}(n,t)$, we define paths P_i, Q_i, R_i, S_i for each $i \in \mathbb{Z}_{2k+1}$ as follows

$$\begin{aligned}
P_i &\colon u_{a_i+t}x_{a_i+t}x_{a_i+t+1}x_{a_i+t+2}\cdots x_{a_{i+2}+t-1}u_{a_{i+2}+t-1},\\
Q_i &\colon v_{a_i}y_{a_i}y_{a_i+1}y_{a_i+2}\cdots y_{a_{i+2}-1}v_{a_{i+2}-1},\\
R_i &\colon u_{a_{i+1}+t-1}v_{a_{i+1}+2t-1}u_{a_{i+1}+3t-1}\cdots v_{a_i},\\
S_i &\colon v_{a_{i+1}-1}u_{a_{i+1}-t-1}v_{a_{i+1}-2t-1}\cdots u_{a_i+t}
\end{aligned}$$

where $a_0, a_1, a_2, \ldots, a_{2k} \in \mathbb{Z}_n$ satisfy the following conditions:

$$\begin{aligned}
&a_i \equiv i \pmod{2\mathrm{k}+1} \text{ for each } \mathrm{i} \in \mathbb{Z}_{2\mathrm{k}+1},\\
&0 \le a_0 < a_2 < a_4 < \cdots < a_{2k} < a_1 < a_3 < a_5 < \cdots < a_{2k-1} < n.
\end{aligned}$$

For instance, if $a_0 = 0, a_2 = 2, a_4 = 4, \ldots, a_{2k} = 2k, a_1 = 2k+2, a_3 = 2k+4, a_5 = 2k+6, \ldots, a_{2k-1} = 4k$, the above conditions are met. Join the paths in the following way, noting that the terminal and initial vertices of successive paths do coincide, including those of Q_{2k} and S_0.

$$\begin{aligned}
&((S_0 \text{ - } P_0) \text{ - } (R_1 \text{ - } Q_1) \text{ - } (S_2 \text{ - } P_2) \text{ - } (R_3 \text{ - } Q_3) \text{ - } \cdots \\
&\qquad \cdots \text{ - } (R_{2k\text{ - }1} \text{ - } Q_{2k\text{ - }1}) \text{ - } (S_{2k} \text{ - } P_{2k})) \text{ - } \\
&\text{ - } ((R_0 \text{ - } Q_0) \text{ - } (S_1 \text{ - } P_1) \text{ - } (R_2 \text{ - } Q_2) \text{ - } (S_3 \text{ - } P_3) \text{ - } \cdots \\
&\qquad \cdots \text{ - } (S_{2k\text{ - }1} \text{ - } P_{2k\text{ - }1}) \text{ - } (R_{2k} \text{ - } Q_{2k}))
\end{aligned}$$

We shall prove that the above walk is indeed a Hamilton cycle in the next section. An example of a Hamilton cycle in $\mathrm{DP}(n, t)$ is shown in Figure 2.

4. The Main Theorem

Theorem 7. *All DP(n, t) are Hamiltonian.*

For any odd integer $n \geq 3$, we shall prove that the walk described in the previous section contains all vertices of $\mathrm{DP}(n, t)$, and each vertex appears exactly once. This shows that the walk is a Hamilton cycle. Let G be $\mathrm{DP}(n, t)$ and $2k+1$ be the greatest common divisor of n and t.

Firstly, we note that paths $Q_0, Q_1, \ldots, Q_{2k}$ contain all of $V_y(G)$ since we have

$$\begin{aligned}
\bigcup_{i=0}^{2k} V_y(Q_i) &= \left(\bigcup_{i=0}^{k} V_y(Q_{2i})\right) \cup \left(\bigcup_{i=0}^{k-1} V_y(Q_{2i+1})\right) \\
&= \left(\bigcup_{i=0}^{k} \{y_j \mid a_{2i} \leq j < a_{2i+2}\}\right) \\
&\cup \left(\bigcup_{i=0}^{k-1} \{y_j \mid a_{2i+1} \leq j < a_{2i+3}\}\right) \\
&= \{y_j \mid j \in \mathbb{Z}_n\} = V_y(G).
\end{aligned}$$

In a similar manner, it is clear that $\bigcup_{i=0}^{2k} V_x(P_i) = V_x(G)$.

Secondly, we will prove that paths $R_0, R_1, \ldots, R_{2k}, S_0, S_1, \ldots, S_{2k}$ contain all of $V_u(G) \cup V_v(G)$. We define cycles C_i in $\mathrm{DP}(n, t)$ for all $i \in \mathbb{Z}_{2k+1}$ as follows:

$$C_i \colon u_i v_{i+t} u_{i+2t} v_{i+3t} \cdots u_{i+(p-1)t} v_i u_{i+t} v_{i+2t} u_{i+3t} \cdots v_{i+(p-1)t} u_i.$$

Let p and q be relatively prime odd integers that satisfy $n = p(2k+1)$ and $t = q(2k+1)$. For all $i \in \mathbb{Z}_{2k+1}$, C_i consists of paths $D_i \colon u_i v_{i+t} u_{i+2t} v_{i+3t} \cdots u_{i+(p-1)t}$ and $E_i \colon v_i u_{i+t} v_{i+2t} u_{i+3t} \cdots v_{i+(p-1)t}$. Since p is odd, the last vertex of D_i is indeed $u_{i+(p-1)t}$ and not $v_{i+(p-1)t}$. By symmetry, the last vertex of E_i is $v_{i+(p-1)t}$. In addition, $u_{i+(p-1)t}$ and $v_{i+(p-1)t}$ are respectively adjacent to v_i and u_i since $pt = pq(2k+1)$ is a multiple of n. Since pt is the least common multiple of

n and t, we note that the vertices $u_i, u_{i+t}, u_{i+2t}, u_{i+3t}, \ldots, u_{i+(p-1)t}$ are distinct. Similarly, the vertices $v_i, v_{i+t}, v_{i+2t}, v_{i+3t}, \ldots, v_{i+(p-1)t}$ are also distinct.

We know that cycles $C_0, C_1, \ldots, C_{2k}$ contain all of $V_u(G)$ since

$$\begin{aligned}\bigcup_{i=0}^{2k} V_u(C_i) &= \bigcup_{i=0}^{2k} \{u_{i+jt} \mid 0 \le j < p\}\\ &= \bigcup_{i=0}^{2k} \{u_{i+jq(2k+1)} \mid 0 \le j < p\}\\ &= \bigcup_{i=0}^{2k} \{u_{i+j(2k+1)} \mid 0 \le j < p\}\\ &= \bigcup_{i=0}^{2k} \{u_m \mid m \in \mathbb{Z}_n, m \equiv i \pmod{2\mathrm{k}+1}\}\\ &= \{u_m \mid m \in \mathbb{Z}_n\}.\end{aligned}$$

By symmetry, we have

$$\begin{aligned}\left(\bigcup_{i=0}^{2k} V_u(C_i)\right) \cup \left(\bigcup_{i=0}^{2k} V_v(C_i)\right) &= \{u_m \mid m \in \mathbb{Z}_n\} \cup \{v_m \mid m \in \mathbb{Z}_n\}\\ &= V_u(G) \cup V_v(G).\end{aligned}$$

Therefore cycles $C_0, C_1, \ldots, C_{2k}$ contain all of $V_u(G) \cup V_v(G)$.

Observe that both R_i and S_i are subgraphs of C_i. For all $i \in \mathbb{Z}_{2k+1}$, R_i and S_i share no vertex and contain all vertices in C_i since the first vertex of R_i and the last vertex of R_i are respectively adjacent to the first vertex of S_i and the last vertex of S_i. Therefore, paths $R_0, R_1, \ldots, R_{2k}, S_0, S_1, \ldots, S_{2k}$ contain all of $V_u(G) \cup V_v(G)$. This completes the proof of Theorem 7.

Acknowledgments

The author would like to thank the referees for their helpful comments and suggestions.

References

[1] B. Alspach, *The classification of Hamiltonian generalized Petersen graphs*, J. Combin. Theory Ser. B **34** (1983) 293–312.
doi:10.1016/0095-8956(83)90042-4

[2] F. Castagna and G. Prins, *Every generalized Petersen graph has a Tait coloring*, Pacific J. Math. **40** (1972) 53–58.
doi:10.2140/pjm.1972.40.53

[3] H.S.M. Coxeter, *Self-dual configurations and regular graphs*, Bull. Amer. Math. Soc. **56** (1950) 413–455.
doi:10.1090/S0002-9904-1950-09407-5

[4] X. Fu, Y. Yang and B. Jiang, *On the domination number of generalized Petersen graphs* $P(n, 2)$, Discrete Math. **309** (2009) 2445–2451.
doi:10.1016/j.disc.2008.05.026

[5] R.M. Karp, *Reducibility among combinatorial problems* in: R.E. Miller and J.W. Thatcher (Eds.), (Plenum Press, New York, 1972) 85–103
doi:10.1007/978-1-4684-2001-2_9.

[6] K. Kutnar and P. Petecki, *On automorphisms and structural properties of double generalized Petersen graphs*, Discrete Math. **339** (2016) 2861–2870.
doi:10.1016/j.disc.2016.05.032

[7] M.E. Watkins, *A theorem on Tait colorings with an application to the generalized Petersen graphs*, J. Combin. Theory **6** (1969) 152–164.
doi:10.1016/S0021-9800(69)80116-X

[8] J.-X. Zhou and Y.-Q. Feng, *Cubic vertex-transitive non-Cayley graphs of order* $8p$, Electron. J. Combin. **19** (2012) #P53.

[9] J.-X. Zhou and Y.-Q. Feng, *Cubic bi-Cayley graphs over Abelian groups*, European J. Combin. **36** (2014) 679–693.
doi:10.1016/j.ejc.2013.10.005

Received 25 November 2016
Revised 10 June 2017
Accepted 10 June 2017