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# THE $\{-2,-1\}$-SELFDUAL AND DECOMPOSABLE TOURNAMENTS

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**Abstract**

We only consider finite tournaments. The dual of a tournament is obtained by reversing all the arcs. A tournament is *selfdual* if it is isomorphic to its dual. Given a tournament $T$, a subset $X$ of $V(T)$ is a *module* of $T$ if each vertex outside $X$ dominates all the elements of $X$ or is dominated by all the elements of $X$. A tournament $T$ is *decomposable* if it admits a module $X$ such that $1 < |X| < |V(T)|$.

We characterize the decomposable tournaments whose subtournaments obtained by removing one or two vertices are selfdual. We deduce the following result. Let $T$ be a non decomposable tournament. If the subtournaments of $T$ obtained by removing two or three vertices are selfdual, then the subtournaments of $T$ obtained by removing a single vertex are not decomposable. Lastly, we provide two applications to tournaments reconstruction.

**Keywords:** tournament, decomposable, selfdual.

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## 1. Introduction

We only consider finite structures. We are interested in the notions of selfduality and decomposability for tournaments. The dual of a tournament is obtained by reversing all the arcs. A tournament is selfdual if it is isomorphic to its dual. The decomposabilty is introduced as follows. A module is a vertex subset whose elements cannot be distinguished by a vertex outside. The notion of module is a generalization of the usual notion of interval for linear orders. A tournament is decomposable if it admits a proper module with at least two elements. A tournament, with at least three vertices, is prime if it is not decomposable.

Our main result consists in characterizing the decomposable tournaments (with at least 7 vertices) whose subtournaments obtained by deleting one or two vertices are selfdual (see Theorem 7). Except two degenerate classes, these tournaments are very regular, and are decomposed into lexicographic products. We use two new tools. The first one is a study of strongly connected subtournaments of a prime tournament (see Section 3). In the second one, we examine the selfduality of a tournament by using the orbits of its automorphism group (see Proposition 40). The proof of Theorem 7 is detailed. It is deduced from six facts.

A first consequence of our main result follows (see Theorem 8). Let $T$ be a prime tournament (with at least 8 vertices). If the subtournaments of $T$ obtained by removing two or three vertices are selfdual, then the subtournaments of $T$ obtained by removing a single vertex are prime. The following result is an immediate consequence of Theorem 8 (see Corollary 10). It is a nice result in Pouzet's reconstruction of prime tournaments. Let $T$ be a prime tournament (with at least 8 vertices). If $T$ admits a vertex whose deletion yields a decomposable subtournament, then $T$ satisfies the following assertion (we say that $T$ is $\{-3,-2\}$-reconstructible). Consider a tournament $U$ with the same vertex set as $T$. Suppose that for any vertices $u$, $v$ and $w$ of $T$ such that $|\{u,v,w\}| = 2$ or 3, the subtournaments of $T$ and $U$ obtained by removing $u$, $v$ and $w$ are isomorphic. Then, $T$ and $U$ are isomorphic.

Lastly, we obtain the following result in Pouzet's reconstruction of decomposable tournaments (see Theorem 11). Its proof uses our main result. Let $T$ be a decomposable tournament (with at least 7 vertices). Consider a tournament $U$ with the same vertex set as $T$. Suppose that for vertices $u$ and $v$ of $T$, the subtournaments of $T$ and $U$ obtained by removing $u$ and $v$ are isomorphic. Suppose also that for distinct vertices $u$, $v$ and $w$ of $T$, the subtournaments of $T$ and $U$ induced by $\{u,v,w\}$ are isomorphic. Then, $T$ and $U$ are isomorphic.

At present, we formalize our presentation. For a tournament $T$, let $V(T)$ and $A(T)$ denote the vertex set and arc set (each arc is an ordered pair of distinct vertices). The cardinality of $V(T)$ is denoted by $v(T)$. Given distinct vertices $v$ and $w$ of $T$, $v \longrightarrow w$ means $vw \in A(T)$. Given $X \subseteq V(T)$, $T[X]$ denotes the subtournament of $T$ induced by $X$. For convenience, $T[V(T) \setminus X]$ is also denoted by $T - X$ and by $T - x$ when $X = \{x\}$.

For instance, the 3-*cycle* is the tournament $C_3 = (\{0, 1, 2\}, \{01, 12, 20\})$. A tournament is a *linear order* if it does not contain $C_3$ as a subtournament. Given $n \geq 2$, the usual linear order on $\{0, \ldots, n-1\}$ is the tournament $L_n = (\{0, \ldots, n-1\}, \{m(m+1) : 0 \leq m < n-1\})$. Given a tournament $T$ such that $v(T) \geq 3$, $T$ is a *circle* if it is obtained from a linear order by reversing the arc between its smallest vertex and its largest one.

### 1.1. Decomposability

Let $T$ be a tournament. A subset $X$ of $V(T)$ is a *module* [32] of $T$ if for any $x, y \in X$ and $v \in V(T)$, we have

$$\left.\begin{array}{r} xv \in A(T) \\ \text{and} \\ vy \in A(T) \end{array}\right\} \Rightarrow \ v \in X.$$

For linear orders, the notions of a module and of an interval coincide. They also share the same properties.

**Proposition 1.** *Given a tournament $T$, we have*

1. *$\emptyset$, $V(T)$ and $\{x\}$, where $x \in V(T)$, are modules of $T$;*
2. *given $W \subseteq V(T)$, if $X$ is a module of $T$, then $X \cap W$ is a module of $T[W]$;*
3. *if $X$ and $Y$ are modules of $T$, then $X \cap Y$ is a module of $T$;*
4. *if $X$ and $Y$ are modules of $T$ such that $X \cap Y \neq \emptyset$, then $X \cup Y$ is a module of $T$;*
5. *if $X$ and $Y$ are modules of $T$ such that $X \setminus Y \neq \emptyset$, then $Y \setminus X$ is a module of $T$;*
6. *if $X$ and $Y$ are modules of $T$ such that $X \cap Y = \emptyset$, then $xy \in A(T)$ for any $x \in X$ and $y \in Y$ or $yx \in A(T)$ for any $x \in X$ and $y \in Y$.*

Following the first assertion of Proposition 1, $\emptyset$, $V(T)$ and $\{x\}$, where $x \in V(T)$, are modules of a tournament $T$, called *trivial*. A tournament is *indecomposable* if all its modules are trivial, otherwise it is *decomposable*. Since every tournament with at most 2 vertices is indecomposable, we say that a tournament $T$ is *prime* if $T$ is indecomposable and $v(T) \geq 3$.

We define the quotient of a tournament by considering a partition of its vertex set in modules. Precisely, let $T$ be a tournament. A partition $P$ of $V(T)$ is a *modular partition* of $T$ if all the elements of $P$ are modules of $T$. The last assertion of Proposition 1 justifies the following definition of the quotient. With each modular partition $P$ of $T$, associate the *quotient* $T/P$ of $T$ by $P$ defined on

$V(T/P) = P$ as follows. Given $X, Y \in P$ such that $X \neq Y$, $XY \in A(T/P)$ if $xy \in A(T)$, where $x \in X$ and $y \in Y$. The opposite operation of the quotient is the lexicographic sum defined as follows. Given a tournament $T$, with each vertex $v \in V(T)$ associate a tournament $T_v$. Suppose that the vertex sets $V(T_v)$ are nonempty and pairwise disjoint. Consider the function

$$\begin{array}{rcl} p : \bigcup_{v \in V(T)} V(T_v) & \longrightarrow & V(T) \\ x & \longmapsto & p(x), \text{ where } x \in V\big(T_{p(x)}\big). \end{array}$$

The *lexicographic sum* $\sum_T T_v$ of the tournaments $T_v$ over the tournament $T$ is defined on

$$V\left(\sum_T T_v\right) = \bigcup_{v \in V(T)} V(T_v)$$

as follows. Given $x, y \in \bigcup_{v \in V(T)} V(T_v)$,

$$xy \in A\left(\sum_T T_v\right) \text{ if } \begin{cases} p(x) = p(y) \text{ and } xy \in A(T_{p(x)}) \\ \text{or} \\ p(x) \neq p(y) \text{ and } p(x)p(y) \in A(T). \end{cases}$$

When all the tournaments $T_v$ are isomorphic to a same tournament $U$, we obtain the lexicographic product of $U$ by $T$. Precisely, the *lexicographic product* $T \circ U$ of $U$ by $T$ is defined on $V(T \circ U) = V(T) \times V(U)$ as follows. Given $(x, y), (u, v) \in V(T \circ U)$ such that $(x, y) \neq (u, v)$,

$$(x, y)(u, v) \in A(T \circ U) \text{ if } \begin{cases} x = u \text{ and } yv \in A(U) \\ \text{or} \\ x \neq u \text{ and } xu \in A(T). \end{cases}$$

### 1.2. Selfduality

With each tournament $T$, associate its *dual* $T^\star$ defined by $V(T^\star) = V(T)$ and $A(T^\star) = \{uv : vu \in A(T)\}$. A tournament is *selfdual* if it is isomorphic to its dual. A tournament $T$ such that $v(T) \leq 3$ is clearly selfdual. This is false when $v(T) = 4$. Consider the tournaments $\delta^- = (\{0, 1, 2, 3\}, \{01, 12, 20\} \cup \{30, 31, 32\})$ and $\delta^+ = (\{0, 1, 2, 3\}, \{01, 12, 20\} \cup \{03, 13, 23\})$. The dual of $\delta^-$ is isomorphic to $\delta^+$. Hence $\delta^-$ and $\delta^+$ are not selfdual. It is easy to verify that a tournament $T$ such that $v(T) = 4$ is selfdual if and only if $T$ is isomorphic neither to $\delta^-$ nor to $\delta^+$. The tournaments $\delta^-$ and $\delta^+$ are called *diamonds*.

A tournament $T$ is *strongly selfdual* if for each $X \subseteq V(T)$, $T[X]$ is selfdual. The characterization of strongly selfdual tournaments follows.

**Theorem 2** [29]**.** *Given a tournament $T$ such that $v(T) \geq 8$, $T$ is strongly selfdual if and only if $T$ is a linear order or a circle.*

Following Theorem 2, Boudabbous, Dammak and Ille [7] characterized the prime tournaments, all of whose prime and proper subtournaments are selfdual. We consider the following weakening of strong selfduality. Given a tournament $T$ and $\mathcal{F} \subseteq \mathbb{Z}$, $T$ is *$\mathcal{F}$-selfdual* if we have

1. for every $X \subseteq V(T)$, if $|X| \in \mathcal{F} \setminus \{0\}$, then $T[X]$ is selfdual;
2. for every $X \subseteq V(T)$, if $-|X| \in \mathcal{F} \setminus \{0\}$, then $T - X$ is selfdual;
3. if $0 \in \mathcal{F}$, then $T$ is selfdual.

As previously noted, $\delta^-$ and $\delta^+$ are the only non-selfdual tournaments on 4 vertices. Thus, a tournament $T$ is $\{4\}$-selfdual if and only if $T$ does contain neither $\delta^-$ nor $\delta^+$ as subtournaments. The characterization of $\{4\}$-selfdual tournaments uses the following tournament. Given $n \geq 1$, $T_{2n+1}$ is the tournament obtained from $L_{2n+1}$ by reversing all the arcs between even and odd vertices (see Figure 1).

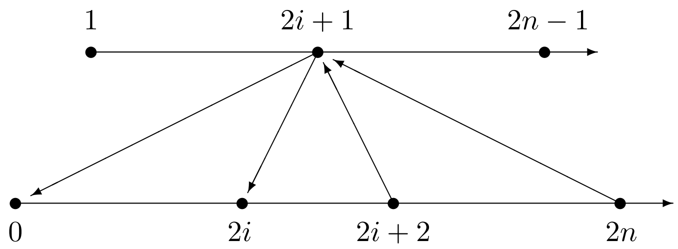


Figure 1. The tournament $T_{2n+1}$.

The characterization of $\{4\}$-selfdual tournaments follows.

**Theorem 3** [27]**.** *Given a tournament $T$, $T$ is $\{4\}$-selfdual if and only if $T$ is a linear order or $T$ is decomposed into a lexicographic sum of linear orders over $T_{2n+1}$, where $n \geq 1$.*

Let $T$ be an $\{n\}$-selfdual tournament, where $0 < n < v(T)$. As stated below (see Lemma 9), $T$ is $\{m\}$-selfdual for every $m > 0$ such that $m \leq \min(n, v(T) - n)$. Therefore, given $\mathcal{F} \subseteq \mathbb{Z}$, we can use Theorem 3 to characterize the $\mathcal{F}$-selfdual tournaments if there exists $n \in \mathcal{F}$ such that $|n| \geq 4$. For instance, Bouchaala and Boudabbous [6] obtained the following characterization of $\{-n\}$-selfdual tournaments, when $n \geq 4$ (compare with Theorem 2).

**Theorem 4.** *Let $n \geq 4$. Given a tournament $T$ such that $v(T) \geq n + 6$, $T$ is $\{-n\}$-selfdual if and only if $T$ is strongly selfdual.*

Since every tournament is $\{1, 2, 3\}$-selfdual, it remains to study the $\mathcal{F}$-selfdual tournaments, when $\mathcal{F} \subseteq \{-3, -2, -1, 0\}$. Boussaïri [11] conjectured the following.

**Conjecture 5.** *The* $\{-3\}$*-selfduality and the strong selfduality are equivalent for tournaments with enough vertices.*

Achour, Boudabbous and Boussaïri [1] answered the conjecture positively in the decomposable case.

**Theorem 6.** *Given a tournament* $T$ *such that* $v(T) \geq 9$*,* $T$ *is decomposable and* $\{-3\}$*-selfdual if and only if* $T$ *is strongly selfdual.*

### 1.3. Main results

Conjecture 5 admits a negative answer if we replace the $\{-3\}$-selfduality by the $\{-2, -1\}$-selfduality. Indeed, for $n \geq 1$, the tournament $T_{2n+1}$ (see Figure 1) is prime and $\{-2, -1\}$-selfdual. Following Theorem 6, our main theorem provides a characterization of decomposable and $\{-2, -1\}$-selfdual tournaments. We need the following notation and definitions. Given a tournament $T$, $\mathrm{Aut}(T)$ denotes the automorphism group of $T$. A tournament $T$ is *vertex-transitive* if $\mathrm{Aut}(T)$ acts transitively on $V(T)$. More weakly, a tournament $T$ is *monomorphic* [16] if for any $u, v \in V(T)$, $T - u$ and $T - v$ are isomorphic. We introduce the following strengthening of vertex-transitivity. A tournament $T$ is *vertex-selfdual* if for any $u, v \in V(T)$, there exists an isomorphism from $T$ onto $T^\star$ that exchanges $u$ and $v$. For instance, for $n \geq 1$, the tournament $T_{2n+1}$ is vertex-selfdual (see Remark 52). The main result follows.

**Theorem 7.** *Given a tournament* $T$ *such that* $v(T) \geq 7$*,* $T$ *is decomposable and* $\{-2, -1\}$*-selfdual if and only if* $T$ *is a linear order or* $T$ *is a circle or* $T$ *is decomposed into a lexicographic product* $\mathbb{Q} \circ U$*, where* $\mathbb{Q}$ *is a prime and vertex-selfdual tournament, and* $U$ *is a monomorphic and* $\{-2, 0\}$*-selfdual tournament, with* $v(U) \geq 2$*.*

The second result follows from Theorem 7. It provides an important property of $\{-3, -2\}$-selfdual and prime tournaments. Note that such tournaments might not exist if Conjecture 5 admits a positive answer. We need the following definition. Given a prime tournament $T$, a vertex $v$ of $T$ is *critical* (in terms of primality) if $T - v$ is decomposable. The second result follows.

**Theorem 8.** *Given a prime tournament* $T$ *such that* $v(T) \geq 8$*, if* $T$ *is* $\{-3, -2\}$*-selfdual, then* $T$ *does not have any critical vertex.*

Lastly, we obtain two consequences of Theorem 7 in tournaments reconstruction. We begin by defining hypomorphic tournaments. Let $\mathcal{F} \subseteq \mathbb{Z} \setminus \{0\}$. Given tournaments $T$ and $U$ such that $V(T) = V(U)$, $T$ and $U$ are *$\mathcal{F}$-hypomorphic* if for every $X \subseteq V(T)$, we have

1. if $|X| \in \mathcal{F}$, then $T[X]$ and $U[X]$ are isomorphic;
2. if $-|X| \in \mathcal{F}$, then $T - X$ and $U - X$ are isomorphic.

Given $\mathcal{F} \subseteq \mathbb{Z} \setminus \{0\}$, a tournament $T$ is *$\mathcal{F}$-reconstructible* provided that for every tournament $U$ such that $V(U) = V(T)$, we have: if $T$ and $U$ are $\mathcal{F}$-hypomorphic, then $T$ and $U$ are isomorphic. We say that the tournaments are $\mathcal{F}$-reconstructible if there exists $n \geq 1$ such that every tournament $T$ is $\mathcal{F}$-reconstructible whenever $v(T) \geq n$. If the tournaments are $\mathcal{F}$-reconstructible, then the smallest of such integers $n$ is called the *$\mathcal{F}$-threshold* and is denoted by $t_{\mathcal{F}}$.

Ulam [34] introduced the problem of $\{-1\}$-reconstruction. Stockmeyer [33] showed that the tournaments are not $\{-1\}$-reconstructible. Precisely, for $n \geq 3$, he built a tournament $\tau$, with $v(\tau) = 2^n + 2$, such that $\tau$ is $\{-1\}$-selfdual and prime, but $\tau$ is not selfdual. Afterwards, Fraïssé proposed the problem of the $\{1, \ldots, k\}$-reconstructibility of tournaments (and more generally of relations). Lopez [23, 24] proved that the tournaments are $\{2, \ldots, 6\}$-reconstructible, and $t_{\{2,\ldots,6\}} = 7$. Reid and Thomassen [29] obtained independently the $\{2, \ldots, 6\}$-reconstructibility of tournaments. Lastly, Pouzet proposed the problem of the $\{-k\}$-reconstructibility of tournaments (and more generally of relations) for $k \geq 2$ (see [4, Problem 24]). The following lemma is useful to translate results on Fraïssé's reconstruction in terms of Pouzet's reconstruction.

**Lemma 9** [28]. *Consider tournaments $T$ and $U$ such that $V(T) = V(U)$. Given $0 < p < v(T)$, if $T$ and $U$ are $\{p\}$-hypomorphic, then $T$ and $U$ are $\{q\}$-hypomorphic for each $q \geq 1$ such that $q \leq p$ and $q \leq v(T) - p$.*

For instance, given $k \geq 6$, since the tournaments are $\{2, \ldots, 6\}$-reconstructible and $t_{\{2,\ldots,6\}} = 7$ (see Lopez [24]), it follows from Lemma 9 that for every $k \geq 6$, the tournaments are $\{-k\}$-reconstructible and $t_{\{-k\}} \leq k+6$. Afterwards, Ille [19] proved that the tournaments are $\{-5\}$-reconstructible and $t_{\{-5\}} \leq 11$. Lastly, Lopez and Rauzy [25] showed that the tournaments are $\{-4\}$-reconstructible and $t_{\{-4\}} \leq 10$. Following these results, we are interested in the study of the $\mathcal{F}$-reconstruction of tournaments when $\mathcal{F} \subseteq \{-3, -2, -1\}$. Achour, Boudabbous and Boussaïri [1] proved that a decomposable tournament $T$ (with at least 9 vertices) is $\{-3\}$-reconstructible when it does not admit a module $M$ such that $|V(T) \setminus M| = 1$ or 2, and $T[M]$ is prime. The third result follows. It is an immediate consequence of Theorem 8 and [12, Corollary 1].

**Corollary 10.** *Given a prime tournament $T$ such that $v(T) \geq 8$, if $T$ possesses a critical vertex, then $T$ is $\{-3, -2\}$-reconstructible.*

Corollary 10 is the first positive result on $\mathcal{F}$-reconstruction of prime tournaments when $\mathcal{F} \subseteq \{-3, -2, -1\}$. If Conjecture 5 admits a positive answer, then it follows directly from Theorem 20 that a prime tournament (with enough vertices) is $\{-3\}$-reconstructible. Finally, we obtain the following result.

**Theorem 11.** *Given a decomposable tournament $T$, if $v(T) \geq 7$, then $T$ is $\{-2,-1,3\}$-reconstructible.*

We do not know if the decomposable tournaments are $\{-2,3\}$-reconstructible or $\{-1,3\}$-reconstructible.

## 2. Preliminaries

### 2.1. Gallai's decomposition of tournaments

We need the following strengthening of the notion of module to obtain an uniform decomposition theorem. Given a tournament $T$, a subset $X$ of $V(T)$ is a *strong module* [13] of $T$ provided that $X$ is a module of $T$, and for every module $Y$ of $T$, we have: if $X \cap Y \neq \emptyset$, then $X \subseteq Y$ or $Y \subseteq X$. With each tournament $T$, with $v(T) \geq 2$, associate the set $\Pi(T)$ of the maximal strong modules of $T$ under inclusion amongst all the proper and strong modules of $T$. Gallai's decomposition follows.

**Theorem 12** [17, 26]**.** *Given a tournament $T$ such that $v(T) \geq 2$, $\Pi(T)$ is a modular partition of $T$, and $T/\Pi(T)$ is a linear order or a prime tournament.*

The next remark provides observations on Theorem 12 that are very useful in the sequel.

**Remark 13.** Given a tournament $T$ such that $v(T) \geq 2$, the following assertions hold

1. $T$ is strongly connected if and only if $T/\Pi(T)$ is prime;
2. if $T$ is not strongly connected, then $T/\Pi(T)$ is a linear order, and $\Pi(T)$ is the set of the vertex sets of the strongly connected components of $T$;
3. if $P$ is a modular partition of $T$ such that $T/P$ is prime, then $P = \Pi(T)$;
4. if $T$ is strongly connected, then $\Pi(T)$ is the set of the maximal proper modules of $T$;
5. if $T$ is vertex-transitive, then $T/\Pi(T)$ is prime, and $T$ is isomorphic to the lexicographic product $(T/\Pi(T)) \circ T[X]$, where $X \in \Pi(T)$.

The next two remarks follow from Remark 13.

**Remark 14.** Let $T$ be a strongly connected tournament such that $v(T) \geq 3$. Consider $W \subseteq V(T)$ such that for every $X \in \Pi(T)$, $X \setminus W \neq \emptyset$. Set

$$\Pi(T) - W = \{X \setminus W : X \in \Pi(T)\}.$$

By Proposition 1, $\Pi(T) - W$ is a modular partition of $T - W$. Furthermore, the bijection

$$\begin{aligned} \pi_W : \Pi(T) &\longrightarrow \Pi(T) - W \\ X &\longmapsto X \setminus W \end{aligned}$$

is an isomorphism from $T/\Pi(T)$ onto $(T - W)/(\Pi(T) - W)$. Since $T$ is strongly connected, it follows from the first assertion of Remark 13 that $T/\Pi(T)$ is prime. Thus $(T - W)/(\Pi(T) - W)$ is prime. By the third assertion of Remark 13, we obtain

$$\Pi(T - W) = \Pi(T) - W.$$

Moreover, by the first assertion of Remark 13, $T - W$ is strongly connected.

**Remark 15.** Let $T$ be a tournament such that $v(T) \geq 2$. Consider $P \subseteq \Pi(T)$ such that $|P| \geq 3$ and $(T/\Pi(T))[P]$ is strongly connected. For convenience, set

$$\tau = T/\Pi(T).$$

Moreover, for each $Q \subseteq \Pi(T)$, set

(1) $$\cup Q = \bigcup_{X \in Q} X.$$

Note that if $Q = \{X\}$, where $X \in \Pi(T)$, then $\cup Q = X$. We verify that

$$T[\cup P] \text{ is strongly connected and } \Pi(T[\cup P]) = \{\cup \xi : \xi \in \Pi(\tau[P])\}.$$

Indeed, for each $\xi \in \Pi(\tau[P])$, $\cup \xi$ is a module of $T[\cup P]$. It follows that

$$\{\cup \xi : \xi \in \Pi(\tau[P])\}$$

is a modular partition of $T[\cup P]$. Furthermore, the bijection

$$\begin{aligned} \Pi(\tau[P]) &\longrightarrow \{\cup \xi : \xi \in \Pi(\tau[P])\} \\ \xi &\longmapsto \cup \xi, \end{aligned}$$

is an isomorphism from $\tau[P]/\Pi(\tau[P])$ onto $(T[\cup P])/\{\cup \xi : \xi \in \Pi(\tau[P])\}$. By the first assertion of Remark 13, $\tau[P]/\Pi(\tau[P])$ is prime. Thus $(T[\cup P])/\{\cup \xi : \xi \in \Pi(\tau[P])\}$ is prime. By the third assertion of Remark 13,

(2) $$\Pi(T[\cup P]) = \{\cup \xi : \xi \in \Pi(\tau[P])\}.$$

By the first assertion of Remark 13, $T[\cup P]$ is strongly connected.

Lastly, suppose that $(T/\Pi(T))[P]$, that is, $\tau[P]$ is prime. We clearly obtain that $\Pi(\tau[P]) = \{\{X\} : X \in P\}$. Therefore

(by (2)) $$\begin{aligned} \Pi(T[\cup P]) &= \{\cup \xi : \xi \in \Pi(\tau[P])\} \\ &= \{\cup \{X\} : X \in P\} = \{X : X \in P\} = P. \end{aligned}$$

### 2.2. Prime tournaments

We begin with an obvious remark. Let $T$ be a strongly connected tournament (with $v(T) \geq 3$). For every $v \in V(T)$, there exists $X \subseteq V(T)$ such that $v \in X$ and $T[X]$ is isomorphic to $C_3$. Since $C_3$ is prime, we obtain

(3) for every $v \in V(T)$, there exists $X \subseteq V(T)$ such that $\begin{cases} v \in X, \\ |X| = 3 \\ \text{and} \\ T[X] \text{ is prime.} \end{cases}$

Of course, (3) holds for prime tournaments. To construct prime subtournaments of a larger size in a prime tournament, we use the partition $p_{(T,X)}$ defined below. Let $T$ be a tournament. Given $X \subsetneq V(T)$ such that $T[X]$ is prime, consider the following subsets of $V(T) \setminus X$

- $\mathrm{Ext}_T(X)$ denotes the set of $v \in V(T) \setminus X$ such that $T[X \cup \{v\}]$ is prime;
- $\langle X \rangle_T$ denotes the set of $v \in V(T) \setminus X$ such that $X$ is a module of $T[X \cup \{v\}]$;
- for each $a \in X$, $X_T(a)$ denotes the set of $v \in V(T) \setminus X$ such that $\{a, v\}$ is a module of $T[X \cup \{v\}]$.

The set $\{\mathrm{Ext}_T(X), \langle X \rangle_T\} \cup \{X_T(a) : a \in X\}$ is denoted by $p_{(T,X)}$. The next lemma is basic and its proof is easy.

**Lemma 16.** *Given a tournament $T$, consider $X \subsetneq V(T)$ such that $T[X]$ is prime. The set $p_{(T,X)}$ is a partition of $V(T) \setminus X$. Moreover, the following assertions hold.*

1. *For $x \in \langle X \rangle_T$ and $y \in V(T) \setminus (X \cup \langle X \rangle_T)$, if $T[X \cup \{x, y\}]$ is decomposable, then $X \cup \{y\}$ is a module of $T[X \cup \{x, y\}]$.*
2. *Given $a \in X$, for $x \in X_T(a)$ and $y \in V(T) \setminus (X \cup X_T(a))$, if $T[X \cup \{x, y\}]$ is decomposable, then $\{a, x\}$ is a module of $T[X \cup \{x, y\}]$.*
3. *For $x, y \in \mathrm{Ext}_T(X)$ such that $x \neq y$, if $T[X \cup \{x, y\}]$ is decomposable, then $\{x, y\}$ is a module of $T[X \cup \{x, y\}]$.*

The next result follows from Lemma 16.

**Proposition 17.** *Given a prime tournament $T$, consider $X \subseteq V(T)$ such that $T[X]$ is prime. The following assertions hold.*

1. *If $\langle X \rangle_T \neq \emptyset$, then there exist $x \in \langle X \rangle_T$ and $y \in V(T) \setminus (X \cup \langle X \rangle_T)$ such that $T[X \cup \{x, y\}]$ is prime.*
2. *Given $a \in X$, if $X_T(a) \neq \emptyset$, then there exist $x \in X_T(a)$ and $y \in V(T) \setminus (X \cup X_T(a))$ such that $T[X \cup \{x, y\}]$ is prime.*

3. *If* $|V(T) \setminus X| \geq 2$ *and* $V(T) \setminus X = \mathrm{Ext}_T(X)$, *then there exist* $x, y \in \mathrm{Ext}_T(X)$ *such that* $x \neq y$ *and* $T[X \cup \{x, y\}]$ *is prime.*

The next result is a simple consequence of Proposition 17.

**Corollary 18** [14]**.** *Given a prime tournament* $T$, *consider* $X \subseteq V(T)$ *such that* $T[X]$ *is prime. If* $|V(T) \setminus X| \geq 2$, *then there exist* $v, w \in V(T) \setminus X$ *such that* $v \neq w$ *and* $T[X \cup \{v, w\}]$ *is prime.*

The next result follows from (3) by applying several times Corollary 18.

**Corollary 19.** *Given a prime tournament* $T$ *such that* $v(T) \geq 5$, *the following assertions hold.*

1. *If* $v(T)$ *is odd, then for each* $x \in V(T)$, *there exist* $v, w \in V(T) \setminus \{x\}$ *such that* $v \neq w$ *and* $T - \{v, w\}$ *is prime.*
2. *If* $v(T)$ *is even, then for each* $x \in V(T)$, *there exists* $v \in V(T) \setminus \{x\}$ *such that* $T - v$ *is prime.*

### 2.3. Primality and {3}-hypomorphy

The following theorem is fundamental in the study of prime and $\{3\}$-hypomorphic tournaments. It is a major tool in duality and reconstruction problems.

**Theorem 20** [12]**.** *For a prime tournament* $T$, $T$ *and* $T^\star$ *are the only tournaments that are* $\{3\}$*-hypomorphic to* $T$.

The next result follows from Remark 13 and Theorem 20.

**Corollary 21** [12]**.** *Let* $T$ *and* $U$ *be* $\{3\}$*-hypomorphic tournaments with* $v(T) \geq 3$.

1. $T$ *is strongly connected if and only if* $U$ *is strongly connected.*
2. $\Pi(T) = \Pi(U)$.
3. *If* $T$ *is strongly connected, then* $U/\Pi(U) = T/\Pi(T)$ *or* $(T/\Pi(T))^\star$.

### 2.4. Criticality

We use the following notation.

**Notation 22.** Given a prime tournament $T$, recall that a vertex $v$ of $T$ is *critical* if $T - v$ is decomposable. The set of critical vertices of $T$ is denoted by $\mathscr{C}(T)$.

A prime tournament $T$ is *critical* if $\mathscr{C}(T) = V(T)$. Schmerl and Trotter [31] characterized the critical tournaments. They obtained the tournament $T_{2n+1}$ (see Figure 1), and the tournaments $U_{2n+1}$ and $W_{2n+1}$ defined on $\{0, \ldots, 2n\}$, where $n \geq 1$, as follows. The tournament $U_{2n+1}$ is obtained from $L_{2n+1}$ by reversing all the arcs between even vertices (see Figure 2).

The tournament $W_{2n+1}$ is obtained from $L_{2n+1}$ by reversing all the arcs between $2n$ and the even elements of $\{0, \ldots, 2n-1\}$ (see Figure 3).

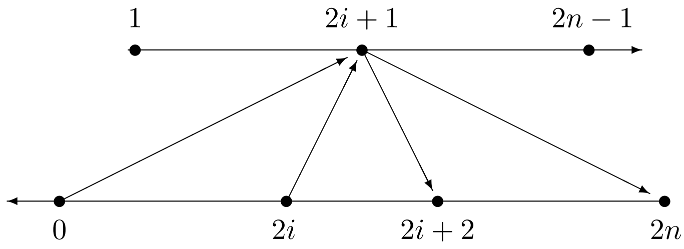


Figure 2. The tournament $U_{2n+1}$.

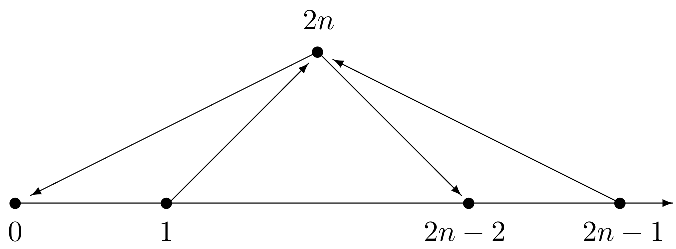


Figure 3. The tournament $W_{2n+1}$.

**Theorem 23** [31]**.** *Given a tournament $\tau$, with $v(\tau) \geq 5$, $\tau$ is critical if and only if $v(\tau)$ is odd, and $\tau$ is isomorphic to $T_{v(\tau)}$, $U_{v(\tau)}$ or $W_{v(\tau)}$.*

The following result is obtained from the characterization of critical tournaments.

**Theorem 24** [31]**.** *Given a prime tournament $T$, if $v(T) \geq 7$, then there exist $v, w \in V(T)$ such that $v \neq w$ and $T - \{v, w\}$ is prime.*

Theorem 24 is improved as follows.

**Theorem 25** [30]**.** *Given a prime tournament $T$, consider $X \subseteq V(T)$ such that $T[X]$ is prime. If $\mid V(T) \setminus X \mid \geq 4$, then there exist $v, w \in V(T) \setminus X$ such that $v \neq w$ and $T - \{v, w\}$ is prime.*

Theorem 24 leads Ille [20] to associate a graph with a prime tournament.

**Definition 26.** Let $T$ be a prime tournament. The *primality graph* $\mathscr{P}(T)$ of $T$ is defined on $V(T)$ as follows. Given distinct $v, w \in V(T)$,

$$vw \in E(\mathscr{P}(T)) \text{ if } T - \{v, w\} \text{ is prime.}$$

The basic properties of the primality graph follow. The next lemma is stated in [20] without a proof. For a proof, see [9, Lemma 10].

**Lemma 27** (Ille [20]). *Let $T$ be a prime tournament with $v(T) \geq 5$. For every $v \in \mathscr{C}(T)$, $d_{\mathscr{P}(T)}(v) \leq 2$. Moreover, the next two assertions hold.*

1. *Given $v \in \mathscr{C}(T)$, if $d_{\mathscr{P}(T)}(v) = 1$, then $V(T) \setminus (\{v\} \cup N_{\mathscr{P}(T)}(v))$ is a module of $T - v$.*
2. *Given $v \in \mathscr{C}(T)$, if $d_{\mathscr{P}(T)}(v) = 2$, then $N_{\mathscr{P}(T)}(v)$ is a module of $T - v$.*

Given a critical tournament $T$, it follows from Lemma 27 that the connected components of $\mathscr{P}(T)$ are paths or cycles. The next result is important in the study of non-critical and prime tournaments.

**Theorem 28** [9]. *Let $T$ be a non-critical and prime tournament with $v(T) \geq 7$. For every connected component $C$ of $\mathscr{P}(T)$, with $v(C) \geq 2$, we have $V(C) \setminus \mathscr{C}(T) \neq \emptyset$.*

Belkhechine *et al.* [3] characterized the prime tournaments admitting a single non-critical vertex. The next result follows from their characterization (see [3, Remark 2]).

**Proposition 29.** *Let $T$ be a prime tournament. If $T$ possesses a unique non-critical vertex $u$, then $v(T) \geq 7$ and $\mathscr{P}(T)$ admits a connected component $C$ satisfying the following two assertions*

1. *$v(T) - v(C) \leq 2$, and each element of $V(T) \setminus V(C)$ is an isolated vertex of $\mathscr{P}(T)$;*
2. *$u \in V(C)$, $C$ is a path and $d_{\mathscr{P}(T)}(u) = 2$.*

The next result is an easy consequence of Lemma 27 and Proposition 29.

**Corollary 30.** *Let $T$ be a prime tournament. If $T$ possesses a unique non-critical vertex $u$, then there exist $v, w \in V(T) \setminus \{u\}$ such that $v \neq w$, $vw \in E(\mathscr{P}(T))$ and $V(T) \setminus \{v, w\}$ is a module of $T - v$.*

## 3. THE STRONGLY CONNECTED SUBTOURNAMENTS OF A PRIME TOURNAMENT

Let $T$ be a tournament. Consider $X \subsetneq V(T)$ such that $T[X]$ is strongly connected and $|X| \geq 3$. As in Subsection 2.2 when $T[X]$ is prime, we consider the following subsets of $V(T) \setminus X$

- $\text{Ext}_T(X)$ is the set of $v \in V(T) \setminus X$ such that $T[X \cup \{v\}]$ is strongly connected and $\{v\} \in \Pi(T[X \cup \{v\}])$;
- $\langle X \rangle_T$ is the set of $v \in V(T) \setminus X$ such that $X$ is a module of $T[X \cup \{v\}]$;

- for each $M \in \Pi(T[X])$, $X_T(M)$ is the set of $v \in V(T) \setminus X$ such that $T[X \cup \{v\}]$ is strongly connected and $M \cup \{v\} \in \Pi(T[X \cup \{v\}])$.

The next remark develops the last item above.

**Remark 31.** Given a tournament $T$, consider $X \subsetneq V(T)$ such that $T[X]$ is strongly connected and $|X| \geq 3$. Let $M \in \Pi(T[X])$. For each $v \in X_T(M)$, we have $\Pi(T[X \cup \{v\}]) = (\Pi(T[X]) \setminus \{M\}) \cup \{M \cup \{v\}\}$.

Given a tournament $T$, consider $X \subsetneq V(T)$ such that $T[X]$ is strongly connected and $|X| \geq 3$. The set $\{\mathrm{Ext}_T(X), \langle X \rangle_T\} \cup \{X_T(M) : M \in \Pi(T[X])\}$ is denoted by $q_{(T,X)}$.

**Proposition 32** [10]**.** *Given a tournament $T$, consider $X \subsetneq V(T)$ such that $T[X]$ is strongly connected and $|X| \geq 3$. The set $q_{(T,X)}$ is a partition of $V(T) \setminus X$.*

An analogue of Proposition 17 and Corollary 18 follows.

**Theorem 33** [10]**.** *Given a prime tournament $T$, consider $X \subsetneq V(T)$ such that $T[X]$ is strongly connected and $|X| \geq 3$. Then, there exist $v, w \in V(T) \setminus X$ such that $T[X \cup \{v, w\}]$ is strongly connected and $\{v\}, \{w\} \in \Pi(T[X \cup \{v, w\}])$. More precisely, the following two assertions hold.*

1. *If $\langle X \rangle_T \neq \emptyset$, then there exist $v \in \langle X \rangle_T$ and $w \in V(T) \setminus (X \cup \langle X \rangle_T)$ such that $T[X \cup \{v, w\}]$ is strongly connected and $\{v\}, \{w\} \in \Pi(T[X \cup \{v, w\}])$.*
2. *Suppose that $\mathrm{Ext}_T(X) = \emptyset$. For each $M \in \Pi(T[X])$, if $|M \cup X_T(M)| \geq 2$, then there exist $v \in X_T(M)$ and $w \in V(T) \setminus (X \cup X_T(M))$ such that $T[X \cup \{v, w\}]$ is strongly connected and $\{v\}, \{w\} \in \Pi(T[X \cup \{v, w\}])$.*

The next remark enlarges on Theorem 33.

**Remark 34.** Given a prime tournament $T$, consider $X \subsetneq V(T)$ such that $T[X]$ is strongly connected and $|X| \geq 3$. Let $Y$ be a nonempty subset of $V(T) \setminus X$ such that $T[X \cup Y]$ is strongly connected, and for each $y \in Y$, $\{y\} \in \Pi(T[X \cup Y])$. For each $x \in X$,

$$\text{if } \{x\} \in \Pi(T[X]), \text{ then } \{x\} \in \Pi(T[X \cup Y]).$$

Indeed, let $x \in X$ such that $\{x\} \in \Pi(T[X])$. There exists $M \in \Pi(T[X \cup Y])$ such that $x \in M$. Since $\{y\} \in \Pi(T[X \cup Y])$ for each $y \in Y$, we obtain that $M \cap Y = \emptyset$. By the second assertion of Proposition 1, $M$ is a module of $T[X]$. Since $T[X]$ is strongly connected, it follows from the fourth assertion of Remark 13 that there exists $N \in \Pi(T[X])$ such that $M \subseteq N$. Since $\{x\} \in \Pi(T[X])$, we have $\{x\} = N$, and hence $\{x\} = M$. Therefore $\{x\} \in \Pi(T[X \cup Y])$.

## 4. Selfdual Tournaments

Let $T$ be a tournament. As $T$ and $T^\star$ share the same modules, they also share the same strong modules. It follows that $\Pi(T) = \Pi(T^\star)$. We obtain

$$T^\star/\Pi(T^\star) = T^\star/\Pi(T) = (T/\Pi(T))^\star.$$

Given a selfdual tournament $T$, consider an isomorphism $f$ from $T$ onto $T^\star$. For every $X \in \Pi(T)$, $f(X) \in \Pi(T^\star)$ and hence $f(X) \in \Pi(T)$. Furthermore, the permutation $f/\Pi(T)$ of $\Pi(T)$ defined by

$$\begin{array}{rcl} \Pi(T) & \longrightarrow & \Pi(T) \\ X & \longmapsto & f(X), \end{array} \tag{4}$$

is an isomorphism from $T/\Pi(T)$ onto $(T/\Pi(T))^\star$. Thus, $T/\Pi(T)$ is selfdual.

We use the following notation. Given a permutation group $\Gamma$ of a set $S$, the set of the orbits of $\Gamma$ is denoted by $S/\Gamma$. When $\Gamma$ is generated by a permutation $f$, $S/\Gamma$ is also denoted by $S/f$.

The next lemma follows from simple observations made in [15, Section 1].

**Lemma 35.** *Given a selfdual tournament $T$, every isomorphism $f$ from $T$ onto $T^\star$ satisfies the following three assertions.*

1. *For each $O \in V(T)/f$ such that $|O| \geq 2$, $|O|$ is even and $|O|/2$ is odd.*
2. *There exists a vertex $x$ of $T$ such that $f(x) = x$ if and only if $v(T)$ is odd. (Such a vertex is unique.)*
3. *There exists an odd integer $k \geq 1$ such that $f^k$ is an involutive isomorphism from $T$ onto $T^\star$.*

In the following remark, we consider the case of selfdual and non strongly connected tournaments.

**Remark 36.** Let $T$ be a selfdual and non strongly connected tournament (with $v(T) \geq 2$). By the second assertion of Remark 13, $T/\Pi(T)$ is a linear order, and $\Pi(T)$ is the family of the vertex sets of the strongly connected components of $T$. The strongly connected components of $T$ can be indexed as $C_0, \ldots, C_n$ so that for any $i, j \in \{0, \ldots, n\}$, we have $V(C_i)V(C_j) \in A(T/\Pi(T))$ if and only if $i < j$. For every isomorphism $f$ from $T$ onto $T^\star$, we obtain that

$$(f/\Pi(T))(V(C_i)) = V(C_{n-i})$$

for each $i \in \{0, \ldots, n\}$.

We use the following notation.

**Notation 37.** Let $T$ be a tournament such that $v(T) \geq 2$. Given $i > 0$, we consider $\Pi_i(T) = \{X \in \Pi(T) : |X| = i\}$, and $\nu_i(T) = |\Pi_i(T)|$. Set $\Upsilon(T) = \{i > 0 : \nu_i(T) \neq 0\}$ and $\mu(T) = \max(\Upsilon(T))$. Furthermore, suppose that $T$ is strongly connected. Given $i > 0$, we consider $\Pi_{i,c}(T) = \Pi_i(T) \cap \mathscr{C}(T/\Pi(T))$ (see Notation 22), $\Pi_{i,\neg c}(T) = \Pi_i(T) \setminus \mathscr{C}(T/\Pi(T))$, $\nu_{i,c}(T) = |\Pi_{i,c}(T)|$ and $\nu_{i,\neg c}(T) = |\Pi_{i,\neg c}(T)|$.

In the next remark, we examine the selfduality in terms of Gallai's decomposition.

**Remark 38.** Let $T$ be a selfdual tournament such that $v(T) \geq 3$. Given an isomorphism $f$ from $T$ onto $T^\star$, consider the isomorphism $f/\Pi(T)$ from $T/\Pi(T)$ onto $(T/\Pi(T))^\star$ induced by $f$. The following assertions hold.

1. For each $i > 0$, we have $(f/\Pi(T))(\Pi_i(T)) = \Pi_i(T)$. By Lemma 35, if $\nu_i(T)$ is odd, then there is $X \in \Pi_i(T)$ such that $(f/\Pi(T))(X) = X$. Consequently
$$|\{i \in \Upsilon(T) : \nu_i(T) \text{ is odd}\}| \leq 1.$$
2. Suppose that $T$ is strongly connected. For each $i > 0$,
$$(f/\Pi(T))(\Pi_{i,c}(T)) = \Pi_{i,c}(T) \text{ and } (f/\Pi(T))(P_{i,\neg c}(T)) = \Pi_{i,\neg c}(T).$$
By Lemma 35, if $\nu_{i,c}(T)$ (respectively, $\nu_{i,\neg c}(T)$ ) is odd, then there exists $X \in \Pi_{i,c}(T)$ (respectively, $X \in \Pi_{i,\neg c}(T)$) such that $(f/\Pi(T))(X) = X$. Consequently
$$|\{i \in \Upsilon(T) : \nu_{i,c}(T) \text{ is odd}\} \cup \{i \in \Upsilon(T) : \nu_{i,\neg c}(T) \text{ is odd}\}| \leq 1.$$

The arguments presented in Remark 38 are well known in the study of selfdual and decomposable tournaments. Unfortunately, they lead to long and technical proofs. In the following proposition, we provide a new tool that allows us to synthesize our approach. We use the following notation.

**Notation 39.** Let $T$ be a tournament. Recall that $\mathrm{Aut}(T)$ denotes the automorphism group of $T$. For each $v \in V(T)$, $O_T(v)$ denotes the orbit of $v$ under $\mathrm{Aut}(T)$. Furthermore, suppose that $T$ is selfdual. We denote by $\mathrm{Fix}(T)$ the set of vertices $v$ of $T$ for which there exists an isomorphism $f$ from $T$ onto $T^\star$ such that $f(v) = v$.

**Proposition 40.** *Let $T$ be a selfdual tournament.*

1. *Let $f$ be an isomorphism from $T$ onto $T^\star$. For every $v \in V(T)$, we have $f(O_T(v)) = O_T(f(v))$. Thus $f$ induces a permutation $f_{\mathrm{Aut}(T)}$ of $V(T)/\mathrm{Aut}(T)$ defined by $f_{\mathrm{Aut}(T)}(O) = f(O)$ for every $O \in V(T)/\mathrm{Aut}(T)$.*

2. *The following three assertions are equivalent*
   - $v(T)$ *is odd;*
   - $\mathrm{Fix}(T) \neq \emptyset$*;*
   - $\mathrm{Fix}(T) \in V(T)/\mathrm{Aut}(T)$*.*

   *Furthermore, for each isomorphism $f$ from $T$ onto $T^\star$, we have*
   - *if $v(T)$ is odd, then* $f_{\mathrm{Aut}(T)}(\mathrm{Fix}(T)) = \mathrm{Fix}(T)$*;*
   - *for every $O \in V(T)/\mathrm{Aut}(T)$, if $f_{\mathrm{Aut}(T)}(O) = O$, then $O = \mathrm{Fix}(T)$.*
3. *For every isomorphism $f$ from $T$ onto $T^\star$, $f_{\mathrm{Aut}(T)}$ is involutive.*
4. *For any isomorphisms $f$ and $g$ from $T$ onto $T^\star$, we have*

$$f_{\mathrm{Aut}(T)} = g_{\mathrm{Aut}(T)}.$$

***Proof.*** For the first assertion, consider an isomorphism $f$ from $T$ onto $T^\star$. Let $v \in V(T)$. For every $w \in O_T(v)$, there exists $\varphi \in \mathrm{Aut}(T)$ such that $\varphi(v) = w$. We have $(f \circ \varphi \circ f^{-1})(f(v)) = f(w)$. Since $f \circ \varphi \circ f^{-1} \in \mathrm{Aut}(T)$, $f(w) \in O_T(f(v))$. It follows that $f(O_T(v)) \subseteq O_T(f(v))$. Similarly, since $f^{-1}$ is an isomorphism from $T$ onto $T^\star$, we obtain $f^{-1}(O_T(f(v))) \subseteq O_T(v)$. Thus $O_T(f(v)) \subseteq f(O_T(v))$. Therefore, for every $v \in V(T)$, $f(O_T(v)) = O_T(f(v))$. Consequently, the function

$$\begin{aligned} f/\mathrm{Aut}(T) : V(T)/\mathrm{Aut}(T) &\longrightarrow V(T)/\mathrm{Aut}(T) \\ O &\longmapsto f(O) \end{aligned}$$

is a permutation of $V/\mathrm{Aut}(T)$.

For the first part of the second assertion, it follows from the second assertion of Lemma 35 that $v(T)$ is odd if and only if $\mathrm{Fix}(T) \neq \emptyset$. Moreover, if $\mathrm{Fix}(T) \in V(T)/\mathrm{Aut}(T)$, then $\mathrm{Fix}(T) \neq \emptyset$. It remains to prove that if $\mathrm{Fix}(T) \neq \emptyset$, then $\mathrm{Fix}(T) \in V(T)/\mathrm{Aut}(T)$. Suppose that $\mathrm{Fix}(T) \neq \emptyset$, and consider $v \in \mathrm{Fix}(T)$. Thus, there exists an isomorphism $f$ from $T$ onto $T^\star$ such that $f(v) = v$. For each $w \in O_T(v)$, there exists $\varphi \in \mathrm{Aut}(T)$ such that $\varphi(v) = w$. Since $\varphi \circ f \circ \varphi^{-1}$ is an isomorphism from $T$ onto $T^\star$ and $(\varphi \circ f \circ \varphi^{-1})(w) = w$, we have $w \in \mathrm{Fix}(T)$. It follows that $O_T(v) \subseteq \mathrm{Fix}(T)$. Now, we show that $\mathrm{Fix}(T) \subseteq O_T(v)$. Consider $w \in \mathrm{Fix}(T)$. There exists an isomorphism $g$ from $T$ onto $T^\star$ such that $g(w) = w$. By the first assertion above, $g(O_T(v))$ is the orbit of $g(v)$ under $\mathrm{Aut}(T)$. Since $f \circ g^{-1} \in \mathrm{Aut}(T)$ and $g(O_T(v)) \in V(T)/\mathrm{Aut}(T)$, we obtain $(f \circ g^{-1})(g(O_T(v))) = g(O_T(v))$. Clearly $(f \circ g^{-1})(g(O_T(v))) = f(O_T(v))$. By the first assertion above, $f(O_T(v)) = O_T(v)$ because $f(v) = v$. Therefore

$$g(O_T(v)) = O_T(v).$$

Since $T[O_T(v)]$ is vertex-transitive, $T[O_T(v)]$ is regular, and hence $|O_T(v)|$ is odd. By the second assertion of Lemma 35 applied to $g_{\restriction O_T(v)}$, there exists $u \in O_T(v)$

such that $g(u) = u$. Since $g(w) = w$, we get $w = u$, so $w \in O_T(v)$. Thus $\mathrm{Fix}(T) \subseteq O_T(v)$. Consequently, $\mathrm{Fix}(T) = O_T(v)$.

To complete the proof of the second assertion, consider an isomorphism $f$ from $T$ onto $T^\star$. First, suppose that $v(T)$ is odd. By the second assertion of Lemma 35, there exists $v \in V(T)$ such that $f(v) = v$. Hence $v \in \mathrm{Fix}(T)$. By what precedes, $\mathrm{Fix}(T) \in V(T)/\mathrm{Aut}(T)$ because $v(T)$ is odd. Thus $\mathrm{Fix}(T) = O_T(v)$. By the first assertion above, we have $f_{\mathrm{Aut}(T)}(O_T(v)) = O_T(f(v))$. Since $f(v) = v$, we obtain $f_{\mathrm{Aut}(T)}(O_T(v)) = O_T(v)$, that is, $f_{\mathrm{Aut}(T)}(\mathrm{Fix}(T)) = \mathrm{Fix}(T)$. Second, consider $O \in V(T)/\mathrm{Aut}(T)$ such that $f_{\mathrm{Aut}(T)}(O) = O$. We get $f(O) = O$. As previously observed, $|O|$ is odd because $T[O]$ is vertex-transitive. By the second assertion of Lemma 35 applied to $f_{\restriction O}$, there exists $v \in O$ such that $f(v) = v$. By the second assertion of Lemma 35 applied to $f$, $v(T)$ is odd. By what precedes, $\mathrm{Fix}(T) \in V(T)/\mathrm{Aut}(T)$. Therefore $O = \mathrm{Fix}(T)$ because $v \in O \cap \mathrm{Fix}(T)$.

For the third assertion, consider an isomorphism $f$ from $T$ onto $T^\star$. Let $O \in V(T)/\mathrm{Aut}(T)$. Since $f \circ f \in \mathrm{Aut}(T)$, we obtain $(f \circ f)(O) = O$. It follows that $(f_{\mathrm{Aut}(T)} \circ f_{\mathrm{Aut}(T)})(O) = O$ for each $O \in V(T)/\mathrm{Aut}(T)$. Hence $f_{\mathrm{Aut}(T)}$ is involutive.

For the fourth assertion, consider isomorphisms $f$ and $g$ from $T$ onto $T^\star$. Since $f^{-1} \circ g \in \mathrm{Aut}(T)$, we obtain that for every $O \in V(T)/\mathrm{Aut}(T)$, $(f^{-1} \circ g)(O) = O$, that is, $f(O) = g(O)$. It follows that $f_{\mathrm{Aut}(T)} = g_{\mathrm{Aut}(T)}$. ■

**Notation 41.** Following the last assertion of Proposition 40, we associate with each selfdual tournament $T$ the permutation $\varphi_{\mathrm{Aut}(T)}$ of $V(T)/\mathrm{Aut}(T)$ satisfying

(5) $\varphi_{\mathrm{Aut}(T)} = f_{\mathrm{Aut}(T)}$ for every isomorphism $f$ from $T$ onto $T^\star$.

This permutation plays a crucial role in the proof of Theorem 7.

## 5. Proof of Theorem 7

We use the following result to prove Theorem 7 in the non strongly connected case.

**Lemma 42** [8]**.** *Let $T$ be a non strongly connected tournament $T$ such that $v(T) \geq 5$. If $T$ is $\{-1\}$-selfdual, then $T$ is a linear order.*

To prove Theorem 7 for tournaments $T$ such that $T/\Pi(T)$ is a 3-cycle or a critical tournament, we use the next result. It is an easy consequence of the characterization of the critical tournaments (see Theorem 23), and of [6, Theorem 1], [6, Theorem 2] and [6, Proposition 7].

**Corollary 43** [6]**.** *Let $T$ be a tournament, with $v(T) \geq 7$, such that $T/\Pi(T)$ is a 3-cycle or a critical tournament. The tournament $T$ is decomposable and*

*$\{-2,-1\}$-selfdual if and only if $T$ is a circle or $T$ is decomposed into a lexicographic product $T_{2h+1} \circ U$, where $h \geq 1$, and $U$ is a monomorphic and $\{-2,0\}$-selfdual tournament, with $v(U) \geq 2$.*

By Lemma 42 and Corollary 43, it remains to prove Theorem 7 for the tournaments $T$ satisfying

$$(6) \qquad \begin{cases} T/\Pi(T) \text{ is prime} \\ \text{and} \\ \text{there exists } X_{\neg c} \in \Pi(T) \text{ such that } (T/\Pi(T)) - X_{\neg c} \text{ is prime.} \end{cases}$$

In the next facts, $T$ denotes a tournament, with $v(T) \geq 7$, such that $T$ satisfies (6), and $T$ is decomposable and $\{-2,-1\}$-selfdual. Since $T$ is decomposable, $\mu(T) \geq 2$ (see Notation 37). Furthermore, since all the tournaments of cardinality 4 are decomposable, all the prime tournaments of cardinality 5 are critical. Hence

$$(7) \qquad |\Pi(T)| \geq 6.$$

For convenience, set

$$\tau = T/\Pi(T).$$

We use the following notation. Let $W \subseteq V(T)$ such that $|W| = 1$ or 2. Since $T$ is $\{-2,-1\}$-selfdual, $T - W$ is selfdual. Thus there exists an isomorphism from $T - W$ onto $T^\star - W$ that is denoted by $f_W$.

The following lemma is only used at the end of the proof of the next fact, when $v(T) = 7$ and $|\Pi(T)| = 6$.

**Lemma 44** [5]**.** *Let $t$ be a tournament. If $t$ contains a diamond as a subtournament, then for each $v \in V(t)$, there exists $D \subseteq V(t)$ such that $v \in D$ and $t[D]$ is a diamond.*

**Fact 45.** *We have*

$$(8) \qquad \sum_{i \in \Upsilon(T) \setminus \{1\}} \nu_i(T) \geq 2 \qquad (\textit{see Notation } 37).$$

***Proof.*** Suppose, to the contrary, that

$$\sum_{i \in \Upsilon(T) \setminus \{1\}} \nu_T(i) = 1.$$

Denote by $X$ the unique element of $\Pi(T)$ such that $|X| \geq 2$. We have

$$\Pi_1(T) = \{\{w\} : w \in V(T) \setminus X\}.$$

Since $T$ satisfies (6), we have $\Pi(T) \setminus \mathscr{C}(T/\Pi(T)) \neq \emptyset$ (see Notation 22).

To begin, suppose that

$$\Pi(T) \setminus \mathscr{C}(T/\Pi(T)) = \{X\}.$$

By Corollary 30 applied to $T/\Pi(T)$, there exist $Y, Z \in \Pi(T) \setminus \{X\}$ such that $Y \neq Z$, $YZ \in E(\mathscr{P}(T/\Pi(T)))$ (see Definition 26) and $\Pi(T) \setminus \{Y, Z\}$ is a module of $(T/\Pi(T)) - Y$. Since $\Pi_1(T) = \{\{w\} : w \in V(T) \setminus X\}$, there exist $y, z \in V(T) \setminus X$ such that $Y = \{y\}$ and $Z = \{z\}$. Since $YZ \in E(\mathscr{P}(T/\Pi(T)))$, $(T/\Pi(T))[\Pi(T) \setminus \{Y, Z\}]$ is prime. By Remark 15, $T[\cup(\Pi(T) \setminus \{Y, Z\})]$, that is, $T - \{y, z\}$ is strongly connected. Moreover, $V(T) \setminus \{y, z\}$ is a module of $T - y$ because $\Pi(T) \setminus \{Y, Z\}$ is a module of $(T/\Pi(T)) - Y$. It follows that $T - \{y, z\}$ and $T[\{z\}]$ are the only strongly connected components of $T - y$. By Remark 36, $T - y$ is not selfdual, which contradicts the $\{-1\}$-selfduality of $T$.

Now, suppose that $\Pi(T) \setminus \mathscr{C}(T/\Pi(T)) \neq \{X\}$. Since $\Pi(T) \setminus \mathscr{C}(T/\Pi(T)) \neq \emptyset$ and $\Pi_1(T) = \{\{w\} : w \in V(T) \setminus X\}$, there exists $v \in V(T) \setminus X$ such that

$$\{v\} \in \Pi(T) \setminus \mathscr{C}(T/\Pi(T)).$$

We have $(T/\Pi(T))[\Pi(T) \setminus \{\{v\}\}]$ is prime. By Remark 15, $T[\cup(\Pi(T) \setminus \{\{v\}\})]$, that is, $T - v$ is strongly connected, and $\Pi(T - v) = \Pi(T) \setminus \{\{v\}\}$. Thus $\Pi_1(T - v) = \{\{w\} : w \in V(T) \setminus (X \cup \{v\})\}$ and $\Pi(T - v) \setminus \Pi_1(T - v) = \{X\}$. It follows that $(f_{\{v\}}/\Pi(T-v))(X) = X$. By the second assertion of Lemma 35, $|\Pi(T-v)|$ is odd, so $|\Pi(T)|$ is even. We verify that $|X| = 2$. Otherwise, suppose that $|X| \geq 3$ and consider $x \in X$. By Remark 14, $\Pi(T - x) = (\Pi(T) \setminus \{X\}) \cup \{X \setminus \{x\}\}$. Thus $\Pi_1(T - x) = \Pi_1(T)$ and $\Pi(T - x) \setminus \Pi_1(T - x) = \{X \setminus \{x\}\}$. Therefore $(f_{\{x\}}/\Pi(T - x))(X \setminus \{x\}) = X \setminus \{x\}$, which contradicts the second assertion of Lemma 35 because $|\Pi(T - x)|$ is even. We verify that $|\Pi(T)| = 6$. Otherwise, suppose that $|\Pi(T)| \geq 7$. By (3), there exists $P \subseteq \Pi(T)$ such that $X \in P$, $|P| = 3$ and $(T/\Pi(T))[P]$ is prime. By Theorem 25, there exist $Y, Z \in \Pi(T) \setminus P$ such that $Y \neq Z$ and $(T/\Pi(T)) - \{Y, Z\}$ is prime. Since $\Pi_1(T) = \{\{w\} : w \in V(T) \setminus X\}$, there exist $y, z \in V(T) \setminus X$ such that $Y = \{y\}$ and $Z = \{z\}$. It follows from Remark 15 that $\Pi(T - \{y, z\}) = \Pi(T) \setminus \{\{y\}, \{z\}\}$. Hence $|\Pi(T - \{y, z\})|$ is even. Moreover, we obtain $(f_{\{y,z\}}/\Pi(T - \{y, z\}))(X) = X$, which contradicts the second assertion of Lemma 35 because $|\Pi(T - \{y, z\})|$ is even. Lastly, suppose that $|\Pi(T)| = 6$ and $|X| = 2$. Since $T/\Pi(T)$ is prime and $|\Pi(T)|$ is even, it follows from Theorem 3 that $T/\Pi(T)$ contains a diamond as a subtournament. By Lemma 44, there exists $D \subseteq \Pi(T)$ such that $X \in D$ and $(T/\Pi(T))[D]$ is a diamond. We obtain that $T[\cup D]$ has only two strongly connected components that are of sizes 1 and 4 or of sizes 2 and 3. By Remark 36, $T[\cup D]$ is not selfdual, which contradicts the $\{-2\}$-selfduality of $T$ because $|V(T) \setminus (\cup D)| = 2$.

It follows that (8) holds. ■

**Fact 46.** *We have*

$$\left( \bigcup_{i \in \Upsilon(T) \setminus \{1\}} \Pi_i(T) \right) \subseteq \mathrm{Fix}(\tau) \qquad (\textit{see Notation } 39). \tag{9}$$

***Proof.*** Suppose, to the contrary, that there exists $X \in \Pi_i(T) \setminus \mathrm{Fix}(\tau)$, where $i \in \Upsilon(T) \setminus \{1\}$.

Consider any $Y \in \Pi(T)$ such that $|Y| \geq 2$. Let $y \in Y$. We obtain that $f_{\{y\}}/\Pi(T-y)$ is an isomorphism from $(T-y)/\Pi(T-y)$ onto $((T-y)/\Pi(T-y))^\star$ (see (4)). Set

$$\Pi(T) - \{y\} = \{Z \setminus \{y\} : Z \in \Pi(T)\},$$

and consider the bijection

$$\begin{aligned} \pi_{\{y\}} : \Pi(T) &\longrightarrow \Pi(T) - \{y\} \\ Z \quad &\longmapsto Z \setminus \{y\} \qquad \text{(see Remark 14).} \end{aligned}$$

By Remark 14, $\Pi(T-y) = \Pi(T) - \{y\}$, and $\pi_{\{y\}}$ is an isomorphism from $\tau$ onto $(T-y)/\Pi(T-y)$. Therefore, $(\pi_{\{y\}})^{-1} \circ (f_{\{y\}}/\Pi(T-y)) \circ \pi_{\{y\}}$ is an isomorphism from $\tau$ onto $\tau^\star$. Set

$$g_{\{y\}} = (\pi_{\{y\}})^{-1} \circ (f_{\{y\}}/\Pi(T-y)) \circ \pi_{\{y\}}.$$

The next assertions follow from Proposition 40.

- $g_{\{y\}}$ induces a permutation $(g_{\{y\}})_{\mathrm{Aut}(\tau)}$ of $\Pi(T)/\mathrm{Aut}(\tau)$. Precisely, for every $Z \in \Pi(T)$,

$$(g_{\{y\}})_{\mathrm{Aut}(\tau)}(O_\tau(Z)) = g_{\{y\}}(O_\tau(Z))) = O_\tau(g_{\{y\}}(Z)).$$

- $(g_{\{y\}})_{\mathrm{Aut}(\tau)} = \varphi_{\mathrm{Aut}(\tau)}$ (see Notation 41).
- Since $X \notin \mathrm{Fix}(\tau)$, we have $O_\tau(X) \neq \mathrm{Fix}(\tau)$. Thus

$$\varphi_{\mathrm{Aut}(\tau)}(O_\tau(X)) \neq O_\tau(X).$$

  Therefore $O_\tau(X) \cap \varphi_{\mathrm{Aut}(\tau)}(O_\tau(X)) = \emptyset$, and

$$\varphi_{\mathrm{Aut}(\tau)} \text{ exchanges } O_\tau(X) \text{ and } \varphi_{\mathrm{Aut}(\tau)}(O_\tau(X)).$$

We have

$$\begin{aligned} (f_{\{y\}}/\Pi(T-y))(\pi_{\{y\}}(O_\tau(X))) &= ((f_{\{y\}}/\Pi(T-y)) \circ \pi_{\{y\}})(O_\tau(X)) \\ &= (\pi_{\{y\}} \circ g_{\{y\}})(O_\tau(X)) \\ &= (\pi_{\{y\}} \circ \varphi_{\mathrm{Aut}(\tau)})(O_\tau(X)) \\ &= \pi_{\{y\}}(\varphi_{\mathrm{Aut}(\tau)}(O_\tau(X))). \end{aligned}$$

Moreover, consider $j > 0$. Since $f_{\{y\}}$ is an isomorphism from $T-y$ onto $(T-y)^\star$, we obtain

$$\left(f_{\{y\}}/\Pi(T-y)\right)(\Pi_j(T-y)) = \Pi_j(T-y).$$

It follows that

$$\begin{aligned}&\left(f_{\{y\}}/\Pi(T-y)\right)(\pi_{\{y\}}(O_\tau(X)) \cap \Pi_j(T-y))\\ &= \pi_{\{y\}}(\varphi_{\mathrm{Aut}(\tau)}(O_\tau(X))) \cap \Pi_j(T-y).\end{aligned}$$

Since $f_{\{y\}}/\Pi(T-y)$ is an isomorphism from $(T-y)/\Pi(T-y)$ onto $((T-y)/\Pi(T-y))^\star$, we obtain

$$\left|\pi_{\{y\}}(O_\tau(X)) \cap \Pi_j(T-y)\right| = \left|\pi_{\{y\}}(\varphi_{\mathrm{Aut}(\tau)}(O_\tau(X))) \cap \Pi_j(T-y)\right|. \tag{10}$$

Choose $X$ for $Y$, and $i$ for $j$. Hence $y \in X$. We get

$$\begin{cases}\pi_{\{y\}}(O_\tau(X)) \cap \Pi_i(T-y) = (O_\tau(X) \cap \Pi_i(T)) \setminus \{X\}\\ \text{and}\\ \pi_{\{y\}}(\varphi_{\mathrm{Aut}(\tau)}(O_\tau(X))) \cap \Pi_i(T-y) = \varphi_{\mathrm{Aut}(\tau)}(O_\tau(X)) \cap \Pi_i(T).\end{cases}$$

It follows from (10) that

$$|O_\tau(X) \cap \Pi_i(T)| - 1 = |\varphi_{\mathrm{Aut}(\tau)}(O_\tau(X)) \cap \Pi_i(T)|. \tag{11}$$

By using (10) with suitable choices for $Y$ and $j$, we obtain $\Upsilon(T) \setminus \{1\} = \{i\}$ and $\nu_i(T) = 1$, which contradicts Fact 45.

To begin, suppose that $\varphi_{\mathrm{Aut}(\tau)}(O_\tau(X)) \cap \Pi_i(T) \neq \emptyset$. Choose for $Y$ an element of $\varphi_{\mathrm{Aut}(\tau)}(O_\tau(X)) \cap \Pi_i(T)$. Furthermore, choose $i$ for $j$. We get

$$\begin{cases}\pi_{\{y\}}(O_\tau(X)) \cap \Pi_i(T-y) = O_\tau(X) \cap \Pi_i(T)\\ \text{and}\\ \pi_{\{y\}}(\varphi_{\mathrm{Aut}(\tau)}(O_\tau(X))) \cap \Pi_i(T-y) = (\varphi_{\mathrm{Aut}(\tau)}(O_\tau(X)) \cap \Pi_i(T)) \setminus \{Y\}.\end{cases}$$

It follows from (10) that

$$|O_\tau(X) \cap \Pi_i(T)| = |\varphi_{\mathrm{Aut}(\tau)}(O_\tau(X)) \cap \Pi_i(T)| - 1,$$

which contradicts (11). Therefore

$$\begin{cases}|\varphi_{\mathrm{Aut}(\tau)}(O_\tau(X)) \cap \Pi_i(T)| = 0\\ \text{and}\\ |O_\tau(X) \cap \Pi_i(T)| = 1.\end{cases} \tag{12}$$

Now, suppose that there exists

$$Y \in \Pi_j(T) \setminus (O_\tau(X) \cup \varphi_{\mathrm{Aut}(\tau)}(O_\tau(X))),$$

where $j \in \Upsilon(T) \setminus \{1\}$. We get

$$\begin{cases} \pi_{\{y\}}(O_\tau(X)) \cap \Pi_i(T-y) = O_\tau(X) \cap \Pi_i(T) \\ \text{and} \\ \pi_{\{y\}}(\varphi_{\mathrm{Aut}(\tau)}(O_\tau(X))) \cap \Pi_i(T-y) = \varphi_{\mathrm{Aut}(\tau)}(O_\tau(X)) \cap \Pi_i(T). \end{cases}$$

It follows from (10) applied with $j = i$ that

$$|O_\tau(X) \cap \Pi_i(T)| = |\varphi_{\mathrm{Aut}(\tau)}(O_\tau(X)) \cap \Pi_i(T)|, \tag{13}$$

which contradicts (12). Therefore

$$\left( \bigcup_{j \in \Upsilon(T)\setminus\{1\}} \Pi_j(T) \right) \subseteq (O_\tau(X) \cup \varphi_{\mathrm{Aut}(\tau)}(O_\tau(X))). \tag{14}$$

In the same manner, we obtain (13) from (10) if there exists $Y \in \Pi_j(T)$, where $j \in \Upsilon(T) \setminus \{1\}$, and

$$\begin{cases} j \geq i+2 \\ \text{or} \\ 2 \leq j \leq i-1 \text{ (when } i \geq 3). \end{cases}$$

Thus $\Upsilon(T) \setminus \{1\} \subseteq \{i, i+1\}$, and it follows from (14) that

$$\Pi_i(T) \cup \Pi_{i+1}(T) \subseteq (O_\tau(X) \cup \varphi_{\mathrm{Aut}(\tau)}(O_\tau(X))). \tag{15}$$

To continue, suppose that there exists $Y \in \Pi_{i+1}(T) \cap O_\tau(X)$. We get

$$\begin{cases} \pi_{\{y\}}(O_\tau(X)) \cap \Pi_i(T-y) = O_\tau(X) \cap \Pi_i(T)) \cup \{Y \setminus \{y\}\} \\ \text{and} \\ \pi_{\{y\}}(\varphi_{\mathrm{Aut}(\tau)}(O_\tau(X))) \cap \Pi_i(T-y) = \varphi_{\mathrm{Aut}(\tau)}(O_\tau(X)) \cap \Pi_i(T). \end{cases}$$

It follows from (10) applied with $j = i$ that

$$|O_\tau(X) \cap \Pi_i(T)| + 1 = |\varphi_{\mathrm{Aut}(\tau)}(O_\tau(X)) \cap \Pi_i(T)|,$$

which contradicts (12). Hence

$$\Pi_{i+1}(T) \cap O_\tau(X) = \emptyset. \tag{16}$$

Lastly, suppose that $\Pi_{i+1}(T) \cap \varphi_{\mathrm{Aut}(\tau)}(O_\tau(X)) \neq \emptyset$. Choose $X$ for $Y$, and $i+1$ for $j$. We get

$$\begin{cases} \pi_{\{y\}}(O_\tau(X)) \cap \Pi_{i+1}(T-y) = O_\tau(X) \cap \Pi_{i+1}(T)) \\ \text{and} \\ \pi_{\{y\}}(\varphi_{\mathrm{Aut}(\tau)}(O_\tau(X))) \cap \Pi_{i+1}(T-y) = \varphi_{\mathrm{Aut}(\tau)}(O_\tau(X)) \cap \Pi_{i+1}(T). \end{cases}$$

Since $\Pi_{i+1}(T) \cap \varphi_{\mathrm{Aut}(\tau)}(O_\tau(X)) \neq \emptyset$, it follows from (10) that

$$\Pi_{i+1}(T) \cap O_\tau(X) \neq \emptyset,$$

which contradicts (16). Thus $\Pi_{i+1}(T) \cap \varphi_{\mathrm{Aut}(\tau)}(O_\tau(X)) = \emptyset$. It follows from (15) and (16) that $\nu_{i+1}(T) = 0$. Since $\Upsilon(T) \setminus \{1\} \subseteq \{i, i+1\}$, we obtain $\Upsilon(T) \setminus \{1\} = \{i\}$. Furthermore, it follows from (14) and (12) that $\Pi_i(T) = \{X\}$. Therefore, $\Upsilon(T) \setminus \{1\} = \{i\}$ and $\nu_i(T) = 1$, which contradicts Fact 45. In consequence, (9) holds. ■

**Fact 47.** *We have*

$$\begin{cases} \left(\bigcup_{i \in \Upsilon(T)\setminus\{1\}} \Pi_i(T)\right) = \mathrm{Fix}(\tau) \\ \textit{and} \\ \mathrm{Fix}(\tau) = \Pi(T) \setminus \mathscr{C}(\tau) \qquad (\textit{see Notation } 22). \end{cases} \tag{17}$$

***Proof.*** By Fact 46, $\mathrm{Fix}(\tau) \neq \emptyset$. By the second assertion of Proposition 40, $|\Pi(T)|$ is odd. We show that

$$\Pi_1(T) \subseteq \mathscr{C}(\tau). \tag{18}$$

Suppose, to the contrary, that there exists $v \in V(T)$ such that $\{v\} \in \Pi(T) \setminus \mathscr{C}(\tau)$. By Remark 15, $\Pi(T-v) = \Pi(T) \setminus \{\{v\}\}$. Since $|\Pi(T-v)|$ is even, it follows from the second assertion of Lemma 35 that $f_{\{v\}}/\Pi(T-v)$ does not admit a fixed point. By Remark 38, $\nu_i(T-v)$ is even for each $i \in \Upsilon(T-v)$ (see Notation 37). Since $\Pi(T-v) = \Pi(T) \setminus \{\{v\}\}$, we obtain that $\nu_1(T)$ is odd, and $\nu_i(T)$ is even for every $i \in \Upsilon(T) \setminus \{1\}$.

Now, suppose that there exists $i \in \Upsilon(T)$ such that $i \geq 3$. Let $X \in \Pi_i(T)$ and $v \in X$. By Remark 14, $\Pi(T-v) = (\Pi(T) \setminus \{X\}) \cup \{X \setminus \{v\}\}$. Since $\nu_{i-1}(T)$ and $\nu_i(T)$ are even, we obtain that $\nu_{i-1}(T-v)$ and $\nu_i(T-v)$ are odd, which contradicts Remark 38. Consequently, $\mu(T) = 2$. Consider again $v \in V(T)$ such that $\{v\} \in \Pi(T) \setminus \mathscr{C}(\tau)$. Let $X \in \Pi_2(T)$ and $w \in X$. By Remark 15, $\Pi(T-v) = \Pi(T) \setminus \{\{v\}\}$. Furthermore, by Remark 14 applied to $T-v$, $\Pi(T-\{v,w\}) = (\Pi(T) \setminus \{\{v\}, X\}) \cup \{X \setminus \{v\}\}$. Hence $\nu_2(T-\{v,w\}) = \nu_2(T)-1$, so $\nu_2(T-\{v,w\})$ is odd. Since $X \setminus \{w\} \in \Pi_1(T-\{v,w\})$, we have $\nu_1(T-\{v,w\}) = \nu_1(T)$, so

$\nu_1(T-\{v,w\})$ is odd. Therefore $\nu_2(T-\{v,w\})$ and $\nu_1(T-\{v,w\})$ are odd, which contradicts Remark 38. It follows that (18) holds.

Since $T$ satisfies (6), there exists $X \in \Pi(T) \setminus \mathscr{C}(\tau)$. Since $\Pi_1(T) \subseteq \mathscr{C}(\tau)$, $X \in \Pi_i(T)$, where $i \in \Upsilon(T) \setminus \{1\}$. By Fact 46, $X \in \mathrm{Fix}(\tau)$. Since $\mathrm{Fix}(\tau) \in \Pi(T)/\mathrm{Aut}(\tau)$ by the second assertion of Proposition 40, we obtain $\mathrm{Fix}(\tau) \subseteq \Pi(T) \setminus \mathscr{C}(\tau)$. It follows from Fact 46 that

$$\left(\bigcup_{i\in\Upsilon(T)\setminus\{1\}} \Pi_i(T)\right) \subseteq \mathrm{Fix}(\tau) \subseteq \Pi(T) \setminus \mathscr{C}(\tau).$$

Consequently, (17) holds. ∎

**Fact 48.** *We have* $\mathrm{Fix}(\tau) = \Pi_{\mu(T)}(T)$ (*see Notation* 37).

***Proof.*** Set

$$\alpha = \min(\Upsilon(T) \setminus \{1\}).$$

Let $i \in \Upsilon(T) \setminus \{1\}$, $X \in \Pi_i(T)$, and $x \in X$. We obtain that $f_{\{x\}}/\Pi(T-x)$ is an isomorphism from $(T-x)/\Pi(T-x)$ onto $((T-x)/\Pi(T-x))^\star$. Set

$$\Pi(T) - \{x\} = \{Y \setminus \{x\} : Y \in \Pi(T)\},$$

and consider the bijection

$$\begin{array}{rcll} \pi_{\{x\}} : \Pi(T) & \longrightarrow & \Pi(T) - \{x\} & \\ Y & \longmapsto & Y \setminus \{x\} & \quad \text{(see Remark 14).} \end{array}$$

By Remark 14, $\Pi(T-x) = \Pi(T) - \{x\}$ and $\pi_{\{x\}}$ is an isomorphism from $T/\Pi(T)$ onto $(T-x)/\Pi(T-x)$. By Fact 47, $\Pi(T) \setminus \Pi_1(T) = \Pi(T) \setminus \mathscr{C}(\tau)$. Since $\pi_{\{x\}}$ is an isomorphism from $\tau$ onto $(T-x)/\Pi(T-x)$, we have

$$\Pi(T-x) \setminus \mathscr{C}((T-x)/\Pi(T-x)) = \pi_{\{x\}}(\Pi(T) \setminus \Pi_1(T)).$$

Since $\pi_{\{x\}}(\Pi(T) \setminus \Pi_1(T)) = (\Pi(T) \setminus (\Pi_1(T) \cup \{X\})) \cup \{X \setminus \{x\}\}$, we obtain

$$\Pi(T-x) \setminus \mathscr{C}((T-x)/\Pi(T-x)) = (\Pi(T) \setminus (\Pi_1(T) \cup \{X\})) \cup \{X \setminus \{x\}\}.$$

Since $f_{\{x\}}/\Pi(T-x)$ is an isomorphism from $(T-x)/\Pi(T-x)$ onto $((T-x)/\Pi(T-x))^\star$ and $(f_{\{x\}}/\Pi(T-x))(Y) = f_{\{x\}}(Y)$ for every $Y \in (T-x)/\Pi(T-x)$, we obtain that

$$f_{\{x\}}(Y) \in (\Pi(T) \setminus (\Pi_1(T) \cup \{X\})) \cup \{X \setminus \{x\}\} \tag{19}$$

for every $Y \in (\Pi(T) \setminus (\Pi_1(T) \cup \{X\})) \cup \{X \setminus \{x\}\}$. Consider $i = \alpha$, $X \in \Pi_\alpha(T)$ and $x \in X$. We obtain

$$\begin{cases} \{Y \in (\Pi(T) \setminus (\Pi_1(T) \cup \{X\})) \cup \{X \setminus \{x\}\} : |Y| = \alpha - 1\} = \{X \setminus \{x\}\}, \\ \{Y \in (\Pi(T) \setminus (\Pi_1(T) \cup \{X\})) \cup \{X \setminus \{x\}\} : |Y| = \alpha\} = \Pi_\alpha(T) \setminus \{X\}, \\ \text{and} \\ \text{for } j > \alpha,\ \{Y \in (\Pi(T) \setminus (\Pi_1(T) \cup \{X\})) \cup \{X \setminus \{x\}\} : |Y| = j\} = \Pi_j(T). \end{cases}$$

Hence

$$\begin{cases} \nu_{\alpha-1,\neg c}(T - x) = 1, \\ \nu_{\alpha,\neg c}(T - x) = \nu_\alpha(T) - 1, \\ \text{and} \\ \text{for } j > \alpha,\ \nu_{j,\neg c}(T - x) = \nu_j(T). \end{cases}$$

It follows from Remark 38 that

$$\begin{cases} \nu_\alpha(T) \text{ is odd} \\ \text{and} \\ \text{for } j > \alpha,\ \nu_j(T) \text{ is even.} \end{cases} \tag{20}$$

Now, suppose that there exists $i \in \Upsilon(T)$ such that $i > \alpha+1$. Consider $X \in \Pi_i(T)$ and $x \in X$. We obtain

$$\begin{cases} \{Y \in (\Pi(T) \setminus (\Pi_1(T) \cup \{X\})) \cup \{X \setminus \{x\}\} : |Y| = \alpha\} = \Pi_\alpha(T), \\ \text{and} \\ \{Y \in (\Pi(T) \setminus (\Pi_1(T) \cup \{X\})) \cup \{X \setminus \{x\}\} : |Y| = i-1\} = \Pi_{i-1}(T) \cup \{X \setminus \{x\}\}. \end{cases}$$

It follows from (20) that $\nu_{\alpha,\neg c}(T-x)$ and $\nu_{i-1,\neg c}(T-x)$ are odd, which contradicts Remark 38. It follows that $\mu(T) \leq \alpha + 1$. Lastly, suppose that $\mu(T) = \alpha + 1$. Consider $X \in \Pi_\alpha(T)$, $Y \in \Pi_{\alpha+1}(T)$, $x \in X$ and $y \in Y$. We have

$$\begin{aligned} &\Pi(T - \{x, y\}) \setminus \mathscr{C}((T - \{x, y\})/\Pi(T - \{x, y\})) \\ &= (\Pi(T) \setminus (\Pi_1(T) \cup \{X, Y\})) \cup \{X \setminus \{x\}, Y \setminus \{y\}\}. \end{aligned}$$

Therefore, (19) becomes

$$f_{\{x,y\}}(Z) \in (\Pi(T) \setminus (\Pi_1(T) \cup \{X, Y\})) \cup \{X \setminus \{x\}, Y \setminus \{y\}\}$$

for every $Z \in (\Pi(T) \setminus (\Pi_1(T) \cup \{X, Y\})) \cup \{X \setminus \{x\}, Y \setminus \{y\}\}$. We obtain

$$\begin{aligned} \{Z \in (\Pi(T) \setminus (\Pi_1(T) \cup \{X, Y\}))\ &\cup \{X \setminus \{x\}, Y \setminus \{y\}\} : |Z| = \alpha\} \\ &= (\Pi_\alpha(T) \setminus \{X\}) \cup \{Y \setminus \{y\}\} \end{aligned}$$

and

$$\{Z \in (\Pi(T) \setminus (\Pi_1(T) \cup \{X,Y\})) \cup \{X \setminus \{x\}, Y \setminus \{y\}\} : |Z| = \alpha + 1\} = \Pi_{\alpha+1}(T) \setminus \{Y\}.$$

It follows from (20) that $\nu_{\alpha,\neg c}(T - \{x,y\})$ and $\nu_{\alpha+1,\neg c}(T - \{x,y\})$ are odd, which contradicts Remark 38. Consequently $\mu(T) = \alpha$. By Fact 47, $\text{Fix}(\tau) = \Pi_{\mu(T)}(T)$. ∎

**Fact 49.** *We have* $\text{Fix}(\tau) = \Pi(T)$ (*see Notation* 39).

***Proof.*** Suppose, to the contrary, that

$$\text{Fix}(\tau) \subsetneq \Pi(T). \tag{21}$$

Since $\text{Fix}(\tau) \neq \emptyset$, it follows from the second assertion of Proposition 40 that $|\Pi(T)|$ is odd and $\text{Fix}(\tau) \in \Pi(T)/\text{Aut}(\tau)$. Since $\text{Fix}(\tau) \in \Pi(T)/\text{Aut}(\tau)$, $\tau[\text{Fix}(\tau)]$ is vertex-transitive. Thus $|\text{Fix}(\tau)|$ is odd and $\tau[\text{Fix}(\tau)]$ is strongly connected. Furthermore, it follows from Fact 45 that $|\text{Fix}(\tau)| \geq 3$.

We show that for each $v \in V(T)$,

$$\text{if } \{v\} \in \Pi_1(T), \text{ then } d_{\mathscr{P}(\tau)}(\{v\}) = 0 \qquad \text{(see Definition 26)}. \tag{22}$$

Suppose, to the contrary, that there exist $v \in V(T)$, with $\{v\} \in \Pi_1(T)$, and $X \in \Pi(T)$ such that $X \in N_{\mathscr{P}(\tau)}(\{v\})$. Denote by $\Gamma$ the connected component of $\mathscr{P}(\tau)$ such that $\{v\}, X \in V(\Gamma)$. By Fact 47, $\{v\} \in V(\Gamma) \cap \mathscr{C}(\tau)$ (see Notation 22). Moreover, $|\Pi(T)| \geq 6$ by (7). Recall that all the tournaments of cardinality 4 are decomposable. Hence, since $\mathscr{P}(\tau)$ admits an edge, we have $|\Pi(T)| \geq 7$. By Theorem 28, $V(\Gamma) \cap (\Pi(T) \setminus \mathscr{C}(\tau)) \neq \emptyset$. Thus, $V(\Gamma) \cap \mathscr{C}(\tau) \neq \emptyset$ and $V(\Gamma) \cap (\Pi(T) \setminus \mathscr{C}(\tau)) \neq \emptyset$. Consequently, there exist $Y \in V(\Gamma) \cap \mathscr{C}(\tau)$ and $Z \in V(\Gamma) \cap (\Pi(T) \setminus \mathscr{C}(\tau))$ such that $YZ \in E(\mathscr{P}(\tau))$. It follows from Facts 47 and 48 that $Z \in \Pi_{\mu(T)}(T)$ (see Notation 37) and there exists $w \in V(T)$ such that $Y = \{w\}$. Since $\{w\} \in \mathscr{C}(\tau)$ and $Z \in N_{\mathscr{P}(\tau)}(\{w\})$, it follows from Lemma 27 that $d_{\mathscr{P}(\tau)}(\{w\}) = 1$ or 2. We distinguish the following two cases, obtaining a contradiction in each case.

- Suppose that $N_{\mathscr{P}(\tau)}(\{w\}) = \{Z\}$. Set $P = \Pi(T) \setminus \{\{w\}, Z\}$. Since $\{w\}Z \in E(\mathscr{P}(\tau))$, we have $\tau[P]$ is prime. By Remark 15, $T[\cup P]$ is strongly connected and $\Pi(T[\cup P]) = P$. By the first assertion of Lemma 27, $P$ is a module of $\tau - \{w\}$. Thus, $Z$ and $\cup P$ are modules of $T - w$ and hence $T - w$ is not strongly connected. Since $T[\cup P]$ is strongly connected, it is a strongly connected component of $T - w$. Moreover, since $|\text{Fix}(\tau)| \geq 3$, that is, $\nu_{\mu(T)}(T) \geq 3$, we have $|\cup P| \geq 2\mu(T)$, so $|\cup P| > |Z|$. Therefore, $T[\cup P]$ is not isomorphic to any of the strongly connected components of $T[Z]$. It follows from Remark 36 that $T - w$ is not selfdual, contradicting the $\{-1\}$-selfduality of $T$.

• Suppose that there exists $Z' \in \Pi(T) \setminus \{\{w\}, Z\}$ such that $N_{\mathscr{P}(\tau)}(\{w\}) = \{Z, Z'\}$. By the second assertion of Lemma 27, $\{Z, Z'\}$ is a module of $\tau - \{w\}$. Since $\{w\}Z \in E(\mathscr{P}(\tau))$, we have $\tau - \{\{w\}, Z\}$ is prime. Thus $\{\{Z, Z'\}\} \cup \{\{Z''\} : Z'' \in \Pi(T) \setminus \{\{w\}, Z, Z'\}\}$ is a modular partition of $\tau - \{w\}$. Moreover, the function

$$\begin{aligned} \Pi(T) \setminus \{\{w\}, Z\} &\longrightarrow \{\{Z, Z'\}\} \cup \\ &\qquad \{\{Z''\} : Z'' \in \Pi(T) \setminus \{\{w\}, Z, Z'\}\} \\ Z' &\longmapsto \{Z, Z'\} \\ Z'' \in \Pi(T) \setminus \{\{w\}, Z, Z'\} &\longmapsto \{Z''\}, \end{aligned}$$

is an isomorphism from $\tau - \{\{w\}, Z\}$ onto $(\tau - \{w\})/(\{\{Z, Z'\}\} \cup \{\{Z''\} : Z'' \in \Pi(T) \setminus \{\{w\}, Z, Z'\}\})$. Hence, $(\tau - \{w\})/(\{\{Z, Z'\}\} \cup \{\{Z''\} : Z'' \in \Pi(T) \setminus \{\{w\}, Z, Z'\}\})$ is prime. It follows from the third assertion of Remark 13 that $\Pi(\tau - \{w\}) = \{\{Z, Z'\}\} \cup \{\{Z''\} : Z'' \in \Pi(T) \setminus \{\{w\}, Z, Z'\}\}$. By the first assertion of Remark 13, $\tau - \{w\}$ is strongly connected. By Remark 15, $T - w$ is strongly connected and

$$\Pi(T - w) = \{Z \cup Z'\} \cup (\Pi(T) \setminus \{\{w\}, Z, Z'\}).$$

Thus $\nu_{\mu(T)+|Z'|}(T - w) = 1$. Recall that $|\Pi(T)|$ and $\nu_{\mu(T)}(T)$ are odd. It follows that $\nu_1(T)$ is even. Suppose that $Z' \in \Pi_{\mu(T)}(T)$. We obtain $\nu_{\mu(T)}(T - w) = \nu_{\mu(T)}(T) - 2$, so $\nu_{\mu(T)}(T - w)$ is odd. Since $\nu_{2\mu(T)}(T - w) = 1$, it follows from Remark 38 that $T - w$ is not selfdual, contradicting the $\{-1\}$-selfduality of $T$. Suppose that $Z' \in \Pi_1(T)$. Consider $x \in X$, where $X \in \Pi_{\mu(T)}(T) \setminus \{Z\}$. We obtain that $\Pi(T - \{x, w\}) = \{X \setminus \{x\}, Z \cup Z'\} \cup (\Pi(T) \setminus \{\{w\}, X, Z, Z'\})$. Therefore, $\nu_{\mu(T)+1}(T - \{x, w\}) = 1$ and $\nu_{\mu(T)}(T - \{x, w\}) = \nu_{\mu(T)}(T) - 2$. Hence $\nu_{\mu(T)+1}(T - \{x, w\})$ and $\nu_{\mu(T)}(T - \{x, w\})$ are odd, contradicting the $\{-2\}$-selfduality of $T$.

It follows that (22) holds. By Fact 47, $\Pi_1(T) = \mathscr{C}(\tau)$. Consequently, it follows from (22) that

(23) for any $v, w \in V(T)$ such that $\{v\}, \{w\} \in \Pi_1(T)$, $\tau - \{\{v\}, \{w\}\}$ is decomposable.

Let $P$ be a subset of $\Pi(T)$ such that $\mathrm{Fix}(\tau) \subseteq P \subsetneq \Pi(T)$. Suppose that $\tau[P]$ is prime. Using Corollary 18 several times from $\tau[P]$, we obtain $Q \subseteq \Pi(T)$ such that $\tau[Q]$ is prime and $|\Pi(T) \setminus Q| = 1$ or 2, which contradicts (23) because $\Pi(T) \setminus Q \subseteq \Pi(T) \setminus P \subseteq \Pi(T) \setminus \mathrm{Fix}(\tau)$, and $\Pi(T) \setminus \mathrm{Fix}(\tau) = \Pi_1(T)$ by Fact 47.

It follows that for every subset $P$ of $\Pi(T)$,

(24) if $\mathrm{Fix}(\tau) \subseteq P \subsetneq \Pi(T)$, then $\tau[P]$ is decomposable.

Consider the set $\mathbb{P}$ of $P \subseteq \Pi(T)$ satisfying

$$\begin{cases} \text{Fix}(\tau) \subseteq P \subsetneq \Pi(T) \\ \tau[P] \text{ is strongly connected,} \\ \text{and} \\ \text{for every } v \in V(T) \text{ such that } \{v\} \in P \setminus \text{Fix}(\tau),\ \{\{v\}\} \in \Pi_1(\tau[P]). \end{cases}$$

Since $\text{Fix}(\tau) \subsetneq \Pi(T)$ by (21), we have $\text{Fix}(\tau) \in \mathbb{P}$. Hence $\mathbb{P} \neq \emptyset$, and $\mathbb{P}$ admits a maximal element $Q$ under inclusion. Suppose that $|\Pi(T) \setminus Q| \geq 3$. By Theorem 33 applied to $\tau$ and $\tau[Q]$, there exist $\{v\}, \{w\} \in \Pi(T) \setminus Q$ such that $\tau[Q \cup \{\{v\},\{w\}\}]$ is strongly connected and $\{\{v\}\}, \{\{w\}\} \in \Pi_1(\tau[Q \cup \{\{v\},\{w\}\}])$. Since $Q \in \mathbb{P}$, we have $\{\{\{u\}\} : \{u\} \in Q \setminus \text{Fix}(\tau)\} \subseteq \Pi_1(\tau[Q])$. By Remark 34, $\{\{\{u\}\} : \{u\} \in Q \setminus \text{Fix}(\tau)\} \subseteq \Pi_1(\tau[Q \cup \{\{v\},\{w\}\}])$. Since $\{\{v\}\}, \{\{w\}\} \in \Pi_1(\tau[Q \cup \{\{v\},\{w\}\}])$, we obtain

$$\{\{\{u\}\} : \{u\} \in (Q \cup \{\{v\},\{w\}\}) \setminus \text{Fix}(\tau)\} \subseteq \Pi_1(\tau[Q \cup \{\{v\},\{w\}\}]).$$

Therefore $Q \cup \{\{v\},\{w\}\} \in \mathbb{P}$, which contradicts the maximality of $Q$. It follows that $|\Pi(T) \setminus Q| = 1$ or 2. For convenience, set

$$\Pi_{\geq 2}(\tau[Q]) = \bigcup_{i \geq 2} \Pi_i(\tau[Q]).$$

Since $\{\{\{v\}\} : \{v\} \in Q \setminus \text{Fix}(\tau)\} \subseteq \Pi_1(\tau[Q])$, we have

$$\text{for every } M \in \Pi_{\geq 2}(\tau[Q]),\ M \subseteq \text{Fix}(\tau). \tag{25}$$

By (24), $\tau[Q]$ is decomposable. Thus $\Pi_{\geq 2}(\tau[Q]) \neq \emptyset$. Finally, we distinguish the following two cases.

1. Suppose that $|\Pi(T) \setminus Q| = 2$. We verify that $\text{Ext}_\tau(Q) = \emptyset$. Otherwise, there exists $x \in V(T)$ such that $\{x\} \in \text{Ext}_\tau(Q)$. By definition of $\text{Ext}_\tau(Q)$, $\tau[Q \cup \{\{x\}\}]$ is strongly connected and $\{\{x\}\} \in \Pi_1(\tau[Q \cup \{\{x\}\}])$. Since $Q \in \mathbb{P}$, we have $\{\{\{u\}\} : \{u\} \in Q \setminus \text{Fix}(\tau)\} \subseteq \Pi_1(\tau[Q])$. It follows from Remark 34 that $Q \cup \{\{x\}\} \in \mathbb{P}$, which contradicts the maximality of $Q$. Consequently

$$\text{Ext}_\tau(Q) = \emptyset. \tag{26}$$

Since $\text{Ext}_\tau(Q) = \emptyset$, it follows from Theorem 33 that

$$\text{for every } M \in \Pi_{\geq 2}(\tau[Q]),\ Q_\tau(M) \neq \emptyset. \tag{27}$$

Since $q_{(\tau,Q)}$ is a partition of $\Pi(T) \setminus Q$ by Proposition 32, it follows from (27) that $|\Pi_{\geq 2}(\tau[Q])| \leq |\Pi(T) \setminus Q|$. Since $|\Pi_{\geq 2}(\tau[Q])| \neq \emptyset$ and $|\Pi(T) \setminus Q| = 2$, we obtain

$$|\Pi_{\geq 2}(\tau[Q])| = 1 \text{ or } 2. \tag{28}$$

Now, consider $M \in \Pi_{\geq 2}(\tau[Q]) \neq \emptyset$. By (27), there exists $v \in V(T)$ such that $\{v\} \in Q_\tau(M)$. Since $|\Pi(T) \setminus Q| = 2$, set $\Pi(T) \setminus Q = \{\{v\}, \{w\}\}$. If $\{w\} \in Q_\tau(M)$, then $M \cup Q_\tau(M)$ is a module of $\tau$, which contradicts the fact that $\tau$ is prime. By (26), $\mathrm{Ext}_\tau(Q) = \emptyset$. Since $q_{(\tau,Q)}$ is a partition of $\Pi(T) \setminus Q$ by Proposition 32, we get $\{w\} \in \langle Q \rangle_\tau$ or there exists $N \in \Pi(\tau[Q]) \setminus \{M\}$ such that $\{w\} \in Q_\tau(N)$. Furthermore, suppose that $|\Pi_{\geq 2}(\tau[Q])| = 2$. Since $q_{(\tau,Q)}$ is a partition of $\Pi(T) \setminus Q$ by Proposition 32, it follows from (27) that $\{w\} \in Q_\tau(N)$, where $N$ is the unique element of $\Pi_{\geq 2}(\tau[Q]) \setminus \{M\}$. Hence,

$$\text{if } |\Pi_{\geq 2}(\tau[Q])| = 2, \text{ then } \{w\} \in Q_\tau(N), \text{ where } \Pi_{\geq 2}(\tau[Q]) = \{M, N\}. \tag{29}$$

We distinguish the following two subcases. In each of them, we obtain a contradiction.

(a) Suppose that $\{w\} \in Q_\tau(N)$, where $N \in \Pi(\tau[Q]) \setminus \{M\}$. We can have

$$\begin{aligned} &N \in \Pi_{\geq 2}(\tau[Q]) \text{ (and hence } N \subseteq \mathrm{Fix}(\tau) \text{ by (25))} \\ &\text{or } N \in \Pi_1(\tau[Q]) \text{ and } N \subseteq \mathrm{Fix}(\tau) \\ &\text{or } N \in \Pi_1(\tau[Q]) \text{ and } N \subseteq Q \setminus \mathrm{Fix}(\tau). \end{aligned} \tag{30}$$

By Remark 31, $\Pi(\tau[Q \cup \{\{w\}\}]) = (\Pi(\tau[Q]) \setminus \{N\}) \cup \{N \cup \{\{w\}\}\}$. It follows from (28) and (29) that

$$\Pi(\tau[Q \cup \{\{w\}\}]) = (\Pi_1(\tau[Q]) \setminus \{N\}) \cup \{N \cup \{\{w\}\}, M\}.$$

Since $\{\{\{u\}\} : \{u\} \in Q \setminus \mathrm{Fix}(\tau)\} \subseteq \Pi_1(\tau[Q])$, it follows from (30) that

$$\begin{aligned} \Pi_1(\tau[Q] \cup \{\{w\}\}) = \{&\{\{u\}\} : \{u\} \in (Q \setminus \mathrm{Fix}(\tau)) \setminus N\} \\ &\cup \{\{X\} : X \in \mathrm{Fix}(\tau) \setminus (M \cup N)\}. \end{aligned}$$

Therefore

$$\begin{aligned} \Pi(\tau[Q \cup \{\{w\}\}]) = \{&\{\{u\}\} : \{u\} \in (Q \setminus \mathrm{Fix}(\tau)) \setminus N\} \\ &\cup \{\{X\} : X \in \mathrm{Fix}(\tau) \setminus (M \cup N)\} \cup \{N \cup \{\{w\}\}, M\}. \end{aligned} \tag{31}$$

Since $\{w\} \in Q_\tau(N)$, $\tau[Q \cup \{\{w\}\}]$ is strongly connected. By Remark 15, $T[\cup(Q \cup \{\{w\}\})]$, that is, $T - v$ is strongly connected, and $\Pi(T - v) = \{\cup\xi : \xi \in \Pi(\tau[Q \cup \{\{w\}\}])\}$. It follows from (31) that

$$\begin{aligned} \Pi(T - v) = ((&Q \setminus \mathrm{Fix}(\tau)) \setminus \{\cup N\}) \\ &\cup (\Pi_{\mu(T)}(T) \setminus (M \cup N)) \cup \{(\cup N) \cup \{w\}, \cup M\}. \end{aligned} \tag{32}$$

Therefore

$$\Upsilon(T - v) \subseteq \{1, \mu(T), |\cup M|, |(\cup N) \cup \{w\}|\}. \tag{33}$$

Recall that $\text{Fix}(\tau) = \Pi_{\mu(T)}(T)$ by Fact 48, and $M \subseteq \text{Fix}(\tau)$ by (25). Hence $|\cup M| = \mu(T)|M|$. Furthermore, it follows from (30) that

$$|(\cup N) \cup \{w\}| = \begin{cases} \mu(T)|N| + 1 \text{ if } N \in \Pi_{\geq 2}(\tau[Q]), \\ \mu(T) + 1 \text{ if } N \in \Pi_1(\tau[Q]) \text{ and } N \subseteq \text{Fix}(\tau), \\ 2 \text{ if } N \in \Pi_1(\tau[Q]) \text{ and } N \subseteq Q \setminus \text{Fix}(\tau). \end{cases} \tag{34}$$

Therefore

$$|\cup M| \neq |(\cup N) \cup \{w\}|. \tag{35}$$

It follows from (33) that

$$\nu_{|\cup M|}(T - v) = 1. \tag{36}$$

We conclude as follows.

- Suppose that $|(\cup N) \cup \{w\}| \neq \mu(T)$. It follows from (33), (34) and (35) that $\nu_{|(\cup N)\cup\{w\}|}(T - v) = 1$. By (36), $\nu_{|\cup M|}(T - v) = 1$. It follows from Remark 38 that $T - v$ is not selfdual, contradicting the $\{-1\}$-selfduality of $T$.

- Suppose that $|(\cup N) \cup \{w\}| = \mu(T)$. It follows from (34) that $N \in \Pi_1(\tau[Q])$ and $N \subseteq Q \setminus \text{Fix}(\tau)$. Hence $\mu(T) = 2$. Moreover, it follows from (32) that $\Pi_1(T - v) = (Q \setminus \text{Fix}(\tau)) \setminus \{\cup N\}$. Recall that $|\Pi(T)|$ and $|\text{Fix}(\tau)|$ are odd. Since $|\Pi(T) \setminus Q| = 2$, $|Q|$ is odd. Hence $|Q \setminus \text{Fix}(\tau)|$ is even, so $|\Pi_1(T - v)|$ is odd. By (36), $\nu_{|\cup M|}(T - v) = 1$. By Remark 38, $T - v$ is not selfdual, contradicting the $\{-1\}$-selfduality of $T$.

(b) Suppose that $\{w\} \in \langle Q\rangle_\tau$. Since $Q \in \mathbb{P}$, $\tau[Q]$ is strongly connected. By Remark 15, $T[\cup Q]$, that is, $T - \{v, w\}$ is strongly connected. Furthermore, $Q$ is a module of $\tau - \{v\}$ because $\{w\} \in \langle Q\rangle_\tau$. It follows that $\cup Q$ is a module of $T - v$. Therefore, $T - v$ is not strongly connected, and its only strongly connected components are $T[\{w\}]$ and $T - \{v, w\}$. By Remark 36, $T - v$ is not selfdual, contradicting the $\{-1\}$-selfduality of $T$.

2. Suppose that $|\Pi(T) \setminus Q| = 1$. Set $\Pi(T) \setminus Q = \{\{v\}\}$. As previously seen, $T[\cup Q]$, that is, $T - v$ is strongly connected. Since $\{\{\{u\}\} : \{u\} \in Q \setminus \text{Fix}(\tau)\} \subseteq \Pi_1(\tau[Q])$, we have

$$\begin{aligned} \Pi(\tau[Q]) = {} & \{\{\{u\}\} : \{u\} \in Q \setminus \text{Fix}(\tau)\} \\ & \cup \left\{ \{X\} : X \in \text{Fix}(\tau) \setminus \bigcup_{M \in \Pi_{\geq 2}(\tau[Q])} M \right\} \cup \Pi_{\geq 2}(\tau[Q]). \end{aligned} \tag{37}$$

By Remark 15, $\Pi(T - v) = \{\cup \xi : \xi \in \Pi(\tau[Q])\}$. It follows from (37) that

$$\Upsilon(T - v) \subseteq \{1, \mu(T)\} \cup \{i\mu(T) : i \in \Upsilon(\tau[Q]) \setminus \{1\}\} \tag{38}$$

and

$$\begin{cases} \nu_1(T-v) = |Q \setminus \mathrm{Fix}(\tau)|, \\ \nu_{\mu(T)}(T-v) = |\mathrm{Fix}(\tau) \setminus \bigcup_{M\in\Pi_{\geq 2}(\tau[Q])} M|, \\ \text{and} \\ \nu_{i\mu(T)}(T-v) = \nu_i(\tau[Q]) \text{ for every } i \in \Upsilon(\tau[Q]) \setminus \{1\}. \end{cases} \tag{39}$$

Recall that $|\Pi(T)|$ and $|\mathrm{Fix}(\tau)|$ are odd. Thus $|\Pi(T) \setminus \mathrm{Fix}(\tau)|$ is even. Since $|\Pi(T) \setminus Q| = 1$, we obtain that $|Q \setminus \mathrm{Fix}(\tau)|$ is odd. By (39), $\nu_1(T-v)$ is odd. Lastly, consider $M \in \Pi_{\geq 2}(\tau[Q])$. We have $\cup M \in \Pi(T-v)$. Let $Y \in M$ and $y \in Y$. By Remark 14, $\Pi(T-\{y,v\}) = (\Pi(T-v) \setminus \{\cup M\}) \cup \{(\cup M) \setminus \{y\}\}$. Therefore, $\nu_1(T-\{y,v\}) = \nu_1(T-v)$ and hence $\nu_1(T-\{y,v\})$ is odd. Moreover, it follows from (38) and (39) that $\nu_{\mu(T)|M|-1}(T-\{y,v\}) = 1$. By Remark 38, $T-\{y,v\}$ is not selfdual, contradicting the $\{-2\}$-selfduality of $T$. ∎

**Fact 50.** *We have* $\Upsilon(T) = \{\mu(T)\}$ (*see Notation* 37).

***Proof.*** By Fact 48, $\mathrm{Fix}(\tau) = \Pi_{\mu(T)}(T)$. Furthermore, $\mathrm{Fix}(\tau) = \Pi(T)$ by Fact 49. Therefore $\Upsilon(T) = \{\mu(T)\}$. ∎

Using the facts above, we prove Theorem 7 as follows.

***Proof of Theorem 7.*** Let $T$ be a tournament such that $v(T) \geq 7$. If $T$ is a linear order or a circle, then $T$ is clearly decomposable and $\{-2,-1\}$-selfdual. Now, suppose that $T$ is decomposed into a lexicographic product $\mathbb{Q} \circ U$, where $\mathbb{Q}$ is a prime and vertex-selfdual tournament, and $U$ is a monomorphic and $\{-2,0\}$-selfdual tournament, with $v(U) \geq 2$. For every $q \in V(\mathbb{Q})$, $\{q\} \times V(U)$ is a module of $T$. Thus $T$ is decomposable. We verify that $T$ is $\{-2,-1\}$-selfdual. Let $q, q' \in V(\mathbb{Q})$ and $u, u' \in V(U)$. Since $\mathbb{Q}$ is vertex-selfdual, there exists an isomorphism $f$ from $\mathbb{Q}$ onto $\mathbb{Q}^\star$ such that $f$ exchanges $q$ and $q'$. Since $U$ is selfdual, there exists an isomorphism $g$ from $U$ onto $U^\star$. Furthermore, there exists an isomorphism $h_u$ from $U-u$ onto $U-g(u')$ because $U$ is monomorphic. Similarly, there exists an isomorphism $h_{u'}$ from $U-u'$ onto $U-g(u)$. If $q \neq q'$, then the function

$$\begin{aligned} (V(\mathbb{Q}) \times V(U)) \setminus \{(q,u),(q',u')\} &\longrightarrow (V(\mathbb{Q} \times V(U)) \setminus \{(q,u),(q',u')\} \\ \text{for } r \notin \{q,q'\}, (r,v) &\longmapsto (f(r), g(v)) \\ \text{for } v \neq u, (q,v) &\longmapsto (q', ((h_{u'})^{-1} \circ g)(v)) \\ \text{for } v \neq u', (q',v), &\longmapsto (q, ((h_u)^{-1} \circ g)(v)), \end{aligned}$$

is an isomorphism from $(\mathbb{Q} \circ U) - \{(q,u),(q',u')\}$ onto $((\mathbb{Q} \circ U) - \{(q,u),(q',u')\})^\star$. Suppose that $q = q'$. Since $U$ is monomorphic and selfdual, $U$ is $\{-1\}$-selfdual.

Hence $U$ is $\{-2,-1\}$-selfdual. Thus, there exists an isomorphism $h$ from $U - \{u,u'\}$ onto $(U - \{u,u'\})^\star$. The function

$$\begin{aligned}(V(\mathbb{Q}) \times V(U)) \setminus \{(q,u),(q,u')\} &\longrightarrow (V(\mathbb{Q} \times V(U)) \setminus \{(q,u),(q,u')\}\\ \text{for } r \neq q, (r,v) &\longmapsto (f(r), g(v))\\ \text{for } v \notin \{u,u'\}, (q,v) &\longmapsto (q, h(v)),\end{aligned}$$

is an isomorphism from $(\mathbb{Q} \circ U) - \{(q,u),(q,u')\}$ onto $((\mathbb{Q} \circ U) - \{(q,u),(q,u')\})^\star$.

Conversely, suppose that $T$ is decomposable and $\{-2,-1\}$-selfdual. If $T$ is not strongly connected, then $T$ is a linear order by Lemma 42.

Now, suppose that $T$ is strongly connected. By the first assertion of Remark 13, $T/\Pi(T)$ is prime. If $T/\Pi(T)$ is a 3-cycle or a critical tournament, then it follows from Corollary 43 that $T$ is a circle or $T$ is decomposed into a lexicographic product $T_{2h+1} \circ U$, where $h \geq 1$, and $U$ is a monomorphic and $\{-2,0\}$-selfdual tournament, with $v(U) \geq 2$. As noted before the statement of Theorem 7, $T_{2n+1}$ is vertex-selfdual. Furthermore, $T_{2n+1}$ is prime by Theorem 23.

Lastly, suppose that $T/\Pi(T)$ is prime and non-critical, with $|\Pi(T)| \geq 4$. We obtain that $T$ satisfies (6). By (7), $|\Pi(T)| \geq 6$. By Fact 49, $\mathrm{Fix}(\tau) = \Pi(T)$ (see Notation 39). Thus $|\Pi(T)|$ is odd by the second assertion of Proposition 40. Furthermore, it follows from Fact 50 that for any $X, Y \in \Pi(T)$, we have $|X| = |Y|$. We show that

(40) that for any $X, Y \in \Pi(T)$, $T[X]$ and $T[Y]$ are isomorphic.

Suppose, to the contrary, that (40) does not hold. For each $X \in \Pi(T)$, consider the set $\Pi_X(T)$ of $Y \in \Pi(T)$ such that $T[Y]$ is isomorphic to $T[X]$. Since (40) does not hold, we have $|\Pi_X(T)| < |\Pi(T)|$ for every $X \in \Pi(T)$. Since $|\Pi(T)|$ is odd, there exists $X \in \Pi(T)$ such that $|\Pi_X(T)|$ is odd. Consider $Y \in \Pi(T) \setminus \Pi_X(T)$ and $y \in Y$. By Remark 14, $\Pi(T-y) = (\Pi(T) \setminus \{Y\}) \cup \{Y \setminus \{y\}\}$. Since $|Z| = |Y|$ for every $Z \in \Pi(T)$, we have $\Pi_{|Y|-1}(T-y) = \{Y \setminus \{y\}\}$ (see Notation 37) and hence $f_{\{y\}}(Y \setminus \{y\}) = Y \setminus \{y\}$. Moreover, we have

$$\{Z \in \Pi(T-y) : T[Z] \text{ is isomorphic to } T[X]\} = \Pi_X(T).$$

Thus $f_{\{y\}}(\Pi_X(T)) = \Pi_X(T)$. Since $|\Pi_X(T)|$ is odd, it follows from the second assertion of Lemma 35 that there exists $X' \in \Pi_X(T)$ such that $f_{\{y\}}(X') = X'$, which is impossible because $f_{\{y\}}(Y \setminus \{y\}) = Y \setminus \{y\}$. Consequently, (40) holds. It follows that

$$T \text{ is isomorphic to } (T/\Pi(T)) \circ T[X],$$

where $X \in \Pi(T)$. We verify that $T/\Pi(T)$ is vertex-selfdual. Let $Y, Z \in \Pi(T)$. Consider $y \in Y$ and $z \in Z$. If $Y = Z$, then we require that $y = z$. By Remark 14,

$\Pi(T-\{y,z\}) = (\Pi(T)\setminus\{Y,Z\})\cup\{Y\setminus\{y\}, Z\setminus\{z\}\}$. Therefore,

$$\begin{cases} f_{\{y,z\}}(Y\setminus\{y\}) = Y\setminus\{y\} \text{ when } Y = Z, \\ \text{and} \\ f_{\{y,z\}} \text{ exchanges } Y\setminus\{y\} \text{ and } Z\setminus\{z\} \text{ when } Y\neq Z. \end{cases} \tag{41}$$

Recall that the permutation $f_{\{y,z\}}/\Pi(T-\{y,z\})$ of $\Pi(T-\{y,z\})$ defined by

$$\begin{array}{ccc} \Pi(T-\{y,z\}) & \longrightarrow & \Pi(T-\{y,z\}) \\ X' & \longmapsto & f_{\{y,z\}}(X'), \end{array}$$

is an isomorphism from $(T-\{y,z\})/\Pi(T-\{y,z\})$ onto $((T-\{y,z\})/\Pi(T-\{y,z\}))^\star$. Moreover, it follows from Remark 14 that

$$\begin{array}{rccc} \pi_{\{y,z\}}: & \Pi(T) & \longrightarrow & \Pi(T-\{y,z\}) \\ & X' & \longmapsto & X'\setminus\{y,z\} \end{array}$$

is an isomorphism from $T/\Pi(T)$ onto $(T-\{y,z\})/\Pi(T-\{y,z\})$. We obtain that

$$g_{\{y,z\}} = (\pi_{\{y,z\}})^{-1}\circ(f_{\{y,z\}}/\Pi(T-\{y,z\}))\circ\pi_{\{y,z\}}$$

is an isomorphism from $T/\Pi(T)$ onto $(T/\Pi(T))^\star$. Furthermore, it follows from (41) that $g_{\{y,z\}}(Y) = Y$ when $Y = Z$, and $g_{\{y,z\}}$ exchanges $Y$ and $Z$ when $Y\neq Z$. Thus, $T/\Pi(T)$ is vertex-selfdual. We complete the proof as follows.

- We verify that $T[X]$ is monomorphic. Consider again $Y, Z\in\Pi(T)$, with $Y\neq Z$. Let $y, y'\in Y$ and $z\in Z$. As previously seen, $f_{\{y,z\}}$ exchanges $Y\setminus\{y\}$ and $Z\setminus\{z\}$. Hence $T[Y\setminus\{y\}]$ is isomorphic to $(T[Z\setminus\{z\}])^\star$. Similarly, $T[Y\setminus\{y'\}]$ is isomorphic to $(T[Z\setminus\{z\}])^\star$. Thus $T[Y\setminus\{y\}]$ and $T[Y\setminus\{y'\}]$ are isomorphic. It follows that $T[Y]$ and hence $T[X]$ are monomorphic.

- We verify that $T[X]$ is selfdual. Let $Y, Z\in\Pi(T)$, with $Y\neq Z$. Consider $y\in Y$ and $z\in Z$. By Remark 14, $\Pi(T-\{y,z\}) = (\Pi(T)\setminus\{Y,Z\})\cup\{Y\setminus\{y\}, Z\setminus\{z\}\}$. It follows from (41) that $f_{\{y,z\}}$ exchanges $Y\setminus\{y\}$ and $Z\setminus\{z\}$. Recall that the permutation $f_{\{y,z\}}/\Pi(T-\{y,z\})$ of $\Pi(T-\{y,z\})$ defined by

$$\begin{array}{ccc} \Pi(T-\{y,z\}) & \longrightarrow & \Pi(T-\{y,z\}) \\ X' & \longmapsto & f_{\{y,z\}}(X'), \end{array}$$

is an isomorphism from $(T-\{y,z\})/\Pi(T-\{y,z\})$ onto $((T-\{y,z\})/\Pi(T-\{y,z\}))^\star$. Since $\Pi(T-\{y,z\}) = (\Pi(T)\setminus\{Y,Z\})\cup\{Y\setminus\{y\}, Z\setminus\{z\}\}$, and $|\Pi(T)|$ is odd, we get $|\Pi(T-\{y,z\})|$ is odd. By Lemma 35, there exists $X'\in\Pi(T-\{y,z\})$ such that $(f_{\{y,z\}}/\Pi(T-\{y,z\}))(X') = X'$. Hence $f_{\{y,z\}}(X') = X'$. Thus $X'\in\Pi(T)\setminus\{Y,Z\}$ because $f_{\{y,z\}}$ exchanges $Y\setminus\{y\}$ and $Z\setminus\{z\}$. Since $f_{\{y,z\}}$ is an isomorphism from $T-\{y,z\}$ onto $(T-\{y,z\})^\star$, $(f_{\{y,z\}})_{\restriction X'}$ is an isomorphism from $T[X']$ onto $T[X']^\star$. Therefore $T[X']$ and hence $T[X]$ are selfdual.

• Suppose that $|X|>2$. We verify that $T[X]$ is $\{-2\}$-seldual. Given $Y\in\Pi(T)$, consider $y,z\in Y$, with $y\neq z$. By Remark 14, $\Pi(T-\{y,z\})=(\Pi(T)\setminus\{Y,Z\})\cup\{Y\setminus\{y,z\}\}$. Therefore $f_{\{y,z\}}(Y\setminus\{y,z\})=Y\setminus\{y,z\}$. Since $f_{\{y,z\}}$ is an isomorphism from $T-\{y,z\}$ onto $(T-\{y,z\})^\star$, $T[Y\setminus\{y,z\}]$ is isomorphic to $(T[Y\setminus\{y,z\}])^\star$. Consequently, $T[Y]$ and hence $T[X]$ are $\{-2\}$-selfdual. ∎

The threshold 7 of Theorem 7 is sharp. Indeed, $T_7-0$ is decomposable and $\{-2,-1\}$-seldual. We have $\Pi(T_7-0)=\{\{1\},\{2\},\{3,4\},\{5\},\{6\}\}$ and $(T_7-0)/\Pi(T_7-0)$ is isomorphic to $T_5$. Thus $T_7-0$ does not satisfy the conclusion of Theorem 7. The next result is obtained by using Theorem 7 iteratively.

**Corollary 51.** *Given a tournament such that $v(T)\geq 7$, the following two assertions are equivalent*

1. *$T$ is decomposable and $\{-2,-1\}$-selfdual, and $T$ is neither a linear order nor a circle;*
2. *$T$ is decomposed into*

$$\mathbb{Q}_0\circ\cdots\circ\mathbb{Q}_k\circ R,$$

*where $\mathbb{Q}_0,\dots,\mathbb{Q}_k$ are prime and vertex-selfdual tournaments, and $R$ is a linear order, with $v(R)\geq 2$, or a prime, monomorphic, and $\{-2,0\}$-selfdual tournament.*

***Proof.*** To begin, suppose that $T$ is decomposed as in the second assertion. Since $v(R)\geq 2$, $T$ is decomposable. Furthermore, it is easy to verify that a lexicographic product of two vertex-selfdual tournaments is vertex-selfdual. Thus $\mathbb{Q}_0\circ\cdots\circ\mathbb{Q}_k$ is vertex-selfdual. As in the proof of Theorem 7, we verify that $T$ is $\{-2,-1\}$-selfdual by using the fact that $\mathbb{Q}_0\circ\cdots\circ\mathbb{Q}_k$ is vertex-selfdual, and $R$ is monomorphic and $\{-2,0\}$-selfdual.

Conversely, suppose that the first assertion holds. By Theorem 7, $T$ is decomposed onto $\mathbb{Q}_0\circ U_0$, where $\mathbb{Q}_0$ is a prime and vertex-selfdual tournament, and $U_0$ is a monomorphic and $\{-2,0\}$-selfdual tournament, with $v(U_0)\geq 2$. If $v(U_0)\geq 4$, then $U_0$ is not a circle, because a circle on at least 4 vertices is not monomorphic. Furthermore, if $U_0$ is a circle, with $v(U_0)=3$, then $U_0$ is isomorphic to the 3-cycle, and hence $U_0$ is prime. Therefore, if $U_0$ is a circle, then $U_0$ is prime or $U_0$ is a linear order. Moreover, if $U_0$ is a linear order or a prime tournament, then we obtain that $T=\mathbb{Q}_0\circ R$, where $R=U_0$, and we can stop here. Hence suppose that $U_0$ is decomposable, and $U_0$ is neither a linear order nor a circle. Since $U_0$ is monomorphic and $\{-2,0\}$-selfdual, it is $\{-2,-1\}$-selfdual. Suppose that $v(U_0)\leq 6$. It is easy to verify that $U_0=T_3\circ R$, where $v(R)=2$.

Thus $T = \mathbb{Q}_0 \circ \mathbb{Q}_1 \circ R$, where $\mathbb{Q}_1 = T_3$. Lastly, suppose that $v(U_0) \geq 7$. By Theorem 7 applied to $U_0$, we obtain $U_0 = \mathbb{Q}_1 \circ U_1$, where $\mathbb{Q}_1$ is a prime and vertex-selfdual tournament, and $U_1$ is a monomorphic and $\{-2, 0\}$-selfdual tournament, with $v(U_1) \geq 2$. Consequently $T = \mathbb{Q}_0 \circ \mathbb{Q}_1 \circ U_1$. To complete the proof, we continue the decomposition process above from $U_1$. ∎

We end the section with remarks on vertex-selfdual tournaments.

**Remark 52.** As previously noted, the tournament $T_{2n+1}$ (where $n \geq 1$, see Figure 1) is vertex-selfdual. Furthermore, $T_{2n+1}$ is the Cayley tournament defined on $(\mathbb{Z}_{2n+1}, +)$ by

$$N^+_{T_{2n+1}}(0) = \{2p : p \in \{1, \ldots, n\}\}.$$

It is easy to verify that a Cayley tournament defined from an odd and abelian group is vertex-selfdual. In particular, every Paley tournament is vertex-selfdual.

Clearly, a vertex-selfdual tournament is $\{-2, -1\}$-selfdual. Furthermore, a vertex-selfdual tournament is vertex-transitive. Therefore, given a tournament $T$ with $v(T) \geq 3$, $T$ is decomposable and vertex-selfdual if and only if $T$ is decomposed into

$$\mathbb{Q}_0 \circ \cdots \circ \mathbb{Q}_k,$$

where $k \geq 1$, $\mathbb{Q}_0, \ldots, \mathbb{Q}_k$ are prime and vertex-selfdual tournaments (see Corollary 51).

Now, consider a prime and vertex-selfdual tournament $T$ with $v(T) \geq 5$. Since $T$ is vertex-transitive, $T$ is critical or $\mathscr{C}(T) = \emptyset$ (see Notation 22). If $T$ is critical, then $T$ is isomorphic to $T_{2n+1}$ because $T_{2n+1}$ is the single critical tournament which is vertex-transitive (see Theorem 23).

Lastly, consider a Paley tournament $T$. By [22, Proposition 3.1], $T$ is arc-transitive. It follows that $T$ is prime. Furthermore, as previously mentioned, $T$ is vertex-selfdual. Since $T_{2n+1}$ is not arc-transitive, we obtain $\mathscr{C}(T) = \emptyset$. We do not know if there exist prime and vertex-selfdual tournaments that are not Cayley tournaments.

## 6. Proof of Theorem 8

To prove Theorem 8, we use the following consequence of Theorem 7.

**Corollary 53.** *Let $T$ be a prime tournament, with $v(T) \geq 8$, such that $\mathscr{C}(T) \neq \emptyset$ (see Notation 22). If $T$ is $\{-3, -2\}$-selfdual, then there exists $x \in \mathscr{C}(T)$ such that $T - x$ is decomposed into a lexicographic product $\mathbb{Q} \circ U$, where $\mathbb{Q}$ is a prime and vertex-selfdual tournament, and $U$ is a monomorphic and $\{-2, 0\}$-selfdual tournament, with $v(U) \geq 2$.*

***Proof.*** To begin, suppose that there exists $x \in \mathscr{C}(T)$ such that $T-x$ is a linear order. Since $T$ is prime, we obtain that $v(T)$ is odd and $T$ is isomorphic to $W_{v(T)}$ (see Figure 3). It is easy to verify that $W_{v(T)} - \{0,2\}$ is not seldual, which contradicts the $\{-2\}$-selfduality of $T$. Therefore,

$$\text{for every } x \in \mathscr{C}(T),\ T-x \text{ is not a linear order.} \tag{42}$$

Now, we prove that there exists $x \in \mathscr{C}(T)$ such that

$$T-x \text{ is neither a linear order nor a circle.} \tag{43}$$

Let $x \in \mathscr{C}(T)$. By (42), $T-x$ is not a linear order. Suppose that $T-x$ is a circle. We show that there exists $y \in \mathscr{C}(T)$ such that $T-y$ is not a circle. Since $T-x$ is a circle, there exist distinct $u,v \in V(T)\setminus\{x\}$ such that $\Pi(T-x) = \{\{u\},\{v\},V(T)\setminus\{x,u,v\}\}$, $(T-x)/\Pi(T-x)$ is a 3-cycle, and $T-\{x,u,v\}$ is a linear order. Hence we can denote the elements of $V(T)\setminus\{x,u,v\}$ by $w_0,\dots,w_{v(T)-4}$ in such a way that $w_i \longrightarrow w_j$ for any $i,j \in \{0,\dots,v(T)-4\}$ with $i<j$. For each $i \in \{0,\dots,v(T)-5\}$, $\{w_i,w_{i+1}\}$ is a module of $T-x$. Since $T$ is prime, we obtain $w_i \longrightarrow x \longrightarrow w_{i+1}$ or $w_{i+1} \longrightarrow x \longrightarrow w_i$. Therefore, for $i=0$ or 1, $T[\{x,w_i,w_{i+1}\}]$ is a 3-cycle. Furthermore, $\{w_2,w_4\}$ is a module of $T-w_3$. Hence $w_3 \in \mathscr{C}(T)$. Moreover, $T[\{x,w_i,w_{i+1}\}]$ and $T[\{u,v,w_0\}]$ are 3-cycles of $T-w_3$ such that $|\{x,w_i,w_{i+1}\} \cap \{u,v,w_0\}| \leq 1$. It follows that $T-w_3$ is not a circle. By (42), $T-w_3$ is not a linear order. Consequently, (43) holds.

By (43), there exists $x \in \mathscr{C}(T)$ such that $T-x$ is neither a linear order nor a circle. Since $x \in \mathscr{C}(T)$, $T-x$ is decomposable. Furthermore $T-x$ is $\{-2,-1\}$-selfdual because $T$ is $\{-3,-2\}$-selfdual. To conclude, it suffices to apply Theorem 7 to $T-x$. ■

We prove Theorem 8 after showing the next result.

**Lemma 54.** *Let $T$ be a prime tournament with $v(T) \geq 8$. If $T$ is $\{-3,-2\}$-selfdual, then $|\mathscr{C}(T)| \leq 1$ (see Notation 22).*

***Proof.*** Suppose, to the contrary, that $|\mathscr{C}(T)| \geq 2$. It follows from Corollary 53 that there exists $x \in \mathscr{C}(T)$ such that $(T-x)/\Pi(T-x)$ is prime, and there exists $k \geq 2$ such that $|X| = k$ for each $X \in \Pi(T-x)$. By the first assertion of Remark 13, $T-x$ is strongly connected. Since $|\mathscr{C}(T)| \geq 2$, there exists $y \in \mathscr{C}(T)\setminus\{x\}$. Denote by $X_y$ the element of $\Pi(T-x)$ containing $y$. By Remark 14,

$$\Pi(T-\{x,y\}) = (\Pi(T-x)\setminus\{X_y\}) \cup \{X_y\setminus\{y\}\}. \tag{44}$$

Since $\Pi(T-x)\setminus\{X_y\} \subseteq \Pi_k(T-x)$ (see Notation 37), there exist $W,W' \subseteq V(T)\setminus\{x,y\}$ satisfying

$$\begin{cases} \text{for every } X' \in \Pi(T - \{x, y\}),\ |X' \cap W| = |X' \cap W'| = 1 \\ \text{and} \\ \text{for every } X' \in \Pi(T - \{x, y\}) \setminus \{X_y \setminus \{y\}\},\ X' \cap W \neq X' \cap W'. \end{cases}$$

Clearly, $T[W]$ and $T[W']$ are prime, and $|W \cap W'| \leq 1$. Thus $T - y$ is neither a linear order nor a circle. As seen at the end of the proof of Corollary 53, it follows from Theorem 7 applied to $T - y$ that $(T - y)/\Pi(T - y)$ is prime, and there exists $l \geq 2$ such that $|X| = l$ for each $X \in \Pi(T - y)$. By denoting by $Y_x$ the element of $\Pi(T - y)$ containing $x$, we have again

$$\Pi(T - \{x, y\}) = (\Pi(T - y) \setminus \{Y_x\}) \cup \{Y_x \setminus \{x\}\}. \tag{45}$$

Since $|\Pi(T - \{x, y\})| = |\Pi(T - x)|$ and $|\Pi(T - x)| \geq 3$, there exists $Z \in \Pi(T - \{x, y\}) \setminus \{X_y \setminus \{y\}, Y_x \setminus \{x\}\}$. It follows from (44) and (45) that

$$Z \in \Pi(T - x) \cap \Pi(T - y).$$

Hence $Z$ is a module of $T - x$ and $T - y$. Thus $Z$ is a module of $T$, which contradicts the primality of $T$. Consequently, $|\mathscr{C}(T)| \leq 1$. ■

***Proof of Theorem* 8.** Consider a $\{-3, -2\}$-selfdual and prime tournament $T$ with $v(T) \geq 8$. Suppose, to the contrary, that $\mathscr{C}(T) \neq \emptyset$ (see Notation 22). By Lemma 54, $|\mathscr{C}(T)| = 1$. Furthermore, it follows from Corollary 53 that there exists $x \in \mathscr{C}(T)$ such that $(T - x)/\Pi(T - x)$ is prime and vertex-seldual, and there exists $k \geq 2$ such that $|X| = k$ for each $X \in \Pi(T - x)$. Since $(T - x)/\Pi(T - x)$ is vertex-seldual, $(T - x)/\Pi(T - x)$ is vertex-transitive and hence regular. Thus

$$|\Pi(T - x)| \text{ is odd.}$$

Let $X \in \Pi(T - x)$. Since $T$ is prime and $|X| \geq 2$, $X$ is not a module of $T$. Hence $N_T^+(x) \cap X \neq \emptyset$ and $N_T^-(x) \cap X \neq \emptyset$. Suppose that $N_T^+(x) \cap X$ is a singleton, and denote by $u^+$ its unique element. We obtain that $N_T^-(x) \cap X$ is a module of $T - u^+$. Since $|\mathscr{C}(T)| = 1$ and $x \in \mathscr{C}(T)$, $u^+ \notin \mathscr{C}(T)$, that is, $T - u^+$ is prime. It follows that $N_T^-(x) \cap X$ is a singleton as well. In particular, we get $k = 2$. Similarly, if $|N_T^-(x) \cap X| = 1$, then $|N_T^+(x) \cap X| = 1$. Consequently,

$$\text{either } \begin{cases} k = 2 \\ \text{and} \\ \text{for every } X \in \Pi(T - x),\ |N_T^+(x) \cap X| = |N_T^-(x) \cap X| = 1 \end{cases}$$

$$\text{or } \begin{cases} k \geq 4 \\ \text{and} \\ \text{for every } X \in \Pi(T - x),\ |N_T^+(x) \cap X| \geq 2 \text{ and } |N_T^-(x) \cap X| \geq 2. \end{cases} \tag{46}$$

This leads us to distinguish the following two cases. In each of them, we obtain a contradiction.

1. Suppose that $k = 2$. Since $v(T-x) \geq 7$ and $|\Pi(T-x)|$ is odd, we have $|\Pi(T-x)| \geq 5$. By the first assertion of Corollary 19, there exist $X, Y \in \Pi(T-x)$ such that $X \neq Y$ and $((T-x)/\Pi(T-x)) - \{X, Y\}$ is prime. Consider $u^-, v^- \in V(T)$ such that $N_T^-(x) \cap X = \{u^-\}$ and $N_T^-(x) \cap Y = \{v^-\}$. Set

$$t = T - \{u^-, v^-\}.$$

Since $T$ is $\{-2\}$-seldual, there exists an isomorphism $f_{\{u^-,v^-\}}$ from $t$ onto $t^\star$. We show that

$$f_{\{u^-,v^-\}}(x) = x.$$

We have $t[N_t^+(x)] = T[N_T^+(x)]$. Since $|N_T^+(x) \cap Z| = 1$ for each $Z \in \Pi(T-x)$, we obtain that $T[N_T^+(x)]$ is isomorphic to $(T-x)/\Pi(T-x)$. It follows that $t[N_t^+(x)]$ is prime. We have $t[N_t^-(x)] = T[N_T^-(x)] - \{u^-, v^-\}$. Since $|N_T^-(x) \cap Z| = 1$ for each $Z \in \Pi(T-x)$, we obtain that $T[N_T^-(x)] - \{u^-, v^-\}$ is isomorphic to $((T-x)/\Pi(T-x) - \{X, Y\}$. It follows that $t[N_t^-(x)]$ is prime. Now, we prove that

(47) for each $w \in (V(T) \setminus \{x, u^-, v^-\}) \cap N_T^+(x)$, $t[N_t^+(w)]$ is not prime.

We distinguish the following two subcases.

- Suppose that $|\Pi(T-x)| = 5$. Hence $(T-x)/\Pi(T-x)$ is critical. By Theorem 23, $(T-x)/\Pi(T-x)$ is isomorphic to $T_5$, $U_5$ or $W_5$. Since $(T-x)/\Pi(T-x)$ is vertex-selfdual, $(T-x)/\Pi(T-x)$ is isomorphic to $T_5$. It follows that $T[N_{T-x}^+(w)]$ is a linear order on at least 4 vertices. Hence $t[N_t^+(w)]$ is a linear order on at least 2 vertices, so $t[N_t^+(w)]$ is not prime.

- Suppose that $|\Pi(T-x)| \geq 7$. Denote by $Z$ the element of $\Pi(T-x)$ containing $w$. Since $(T-x)/\Pi(T-x)$ is regular, we have $d^+_{(T-x)/\Pi(T-x)}(Z) = (|\Pi(T-x)| - 1)/2$, and hence $d^+_{(T-x)/\Pi(T-x)}(Z) \geq 3$. Thus, there exists $Z^+ \in N^+_{(T-x)/\Pi(T-x)}(Z) \setminus \{X, Y\}$. We obtain

$$Z^+ \subseteq N_T^+(w) \setminus (X \cup Y) \subseteq N_t^+(w).$$

Since $Z^+$ is a module of $T-x$ and $x \notin N_t^+(w)$, $Z^+$ is a module of $t[N_t^+(w)]$. Therefore, $t[N_t^+(w)]$ is not prime.

It follows that (47) holds. Similarly, we obtain that for each $w \in (V(T) \setminus \{x, u^-, v^-\}) \cap N_T^-(x)$, $t[N_t^-(w)]$ is not prime. It follows that $f_{\{u^-,v^-\}}(x) = x$, which is impossible because

$$\begin{cases} d_t^+(x) = |\Pi(T-x)| \\ \text{and} \\ d_t^-(x) = |\Pi(T-x)| - 2. \end{cases}$$

2. Suppose that $k \geq 4$. Let $X$ and $Y$ be distinct elements of $\Pi(T-x)$. Consider $u^- \in N_T^-(x) \cap X$, $u^+ \in N_T^+(x) \cap X$ and $v^- \in N_T^-(x) \cap Y$. Set

$$t = T - \{u^-, u^+, v^-\}.$$

Since $T$ is $\{-3\}$-selfdual, there exists an isomorphism $f_{\{u^-,u^+,v^-\}}$ from $t$ onto $t^\star$. We prove that

$$f_{\{u^-,u^+,v^-\}}(x) = x. \tag{48}$$

To determine $\Pi(t[N_t^-(x)])$ and $t[N_t^-(x)]/\Pi(t[N_t^-(x)])$, we use Remark 14 as follows. Set $W = N_T^+(x) \cup \{u^-, v^-\}$. For each $Z \in \Pi(T-x)$, we have $Z \setminus W = (N_T^-(x) \cap Z) \setminus \{u^-, v^-\}$. Since $|N_T^-(x) \cap Z| \geq 2$ by (46), we obtain $Z \setminus W \neq \emptyset$. Set

$$\begin{aligned} Q_x^- = &\{(N_T^-(x) \cap X) \setminus \{u^-\}, (N_T^-(x) \cap Y) \setminus \{v^-\}\} \\ &\cup \{N_T^-(x) \cap Z : Z \in \Pi(T-x) \setminus \{X, Y\}\}. \end{aligned}$$

By (46),

$$|Q_x^-| = |\Pi(T-x)|. \tag{49}$$

Moreover, it follows from Remark 14 that

$$\begin{cases} Q_x^- = \Pi(t[N_t^-(x)]) \\ \text{and} \\ t[N_t^-(x)]/Q_x^- \text{ is prime.} \end{cases} \tag{50}$$

Analagously, by denoting $\{(N_T^+(x) \cap X) \setminus \{u^+\}\} \cup \{N_T^+(x) \cap Z : Z \in \Pi(T-x) \setminus \{X\}\}$ by $Q_x^+$, we obtain that $Q_x^+ = \Pi(t[N_t^+(x)])$ and $t[N_t^+(x)]/Q_x^+$ is prime. Now, suppose that (48) does not hold. For instance, suppose that $x \in N_T^-(f_{\{u^-,u^+,v^-\}}(x))$. We look for a modular partition of $t[N_t^+(f_{\{u^-,u^+,v^-\}}(x))]$. Denote by $Z$ the unique element of $\Pi(T-x)$ containing $f_{\{u^-,u^+,v^-\}}(x)$. Consider

$$Q^+_{f_{\{u^-,u^+,v^-\}}(x)} = \left\{Z' \setminus \{u^-, u^+, v^-\} : Z' \in N^+_{(T-x)/\Pi(T-x)}(Z)\right\}$$

if $N^+_{T[Z]}(f_{\{u^-,u^+,v^-\}}(x)) \subseteq \{u^-, u^+, v^-\}$, and

$$\begin{aligned} Q^+_{f_{\{u^-,u^+,v^-\}}(x)} = &\left\{Z' \setminus \{u^-, u^+, v^-\} : Z' \in N^+_{(T-x)/\Pi(T-x)}(Z)\right\} \\ &\cup \left\{N^+_{T[Z]}\left(f_{\{u^-,u^+,v^-\}}(x)\right) \setminus \{u^-, u^+, v^-\}\right\} \end{aligned}$$

if $N^+_{T[Z]}(f_{\{u^-,u^+,v^-\}}(x)) \setminus \{u^-,u^+,v^-\} \neq \emptyset$. It follows from (46) that

$$\left|Q^+_{f_{\{u^-,u^+,v^-\}}(x)}\right| = d^+_{(T-x)/\Pi(T-x)}(Z) \text{ or } d^+_{(T-x)/\Pi(T-x)}(Z)+1.$$

Since $(T-x)/\Pi(T-x)$ is regular, we obtain

$$\left|Q^+_{f_{\{u^-,u^+,v^-\}}(x)}\right| = \frac{|\Pi(T-x)|-1}{2} \quad \text{or} \quad \frac{|\Pi(T-x)|-1}{2}+1.$$

Since $\Pi(T-x)$ is a modular partition of $T-x$ and $x \notin N^+_T(f_{\{u^-,u^+,v^-\}}(x))$, $Q^+_{f_{\{u^-,u^+,v^-\}}(x)}$ is a modular partition of $t[N^+_t(f_{\{u^-,u^+,v^-\}}(x))]$. Thus,

$$\left(f_{\{u^-,u^+,v^-\}}\right)^{-1}\left(Q^+_{f_{\{u^-,u^+,v^-\}}(x)}\right)$$

is a modular partition of $t[N^-_t(x)]$. Since $t[N^-_t(x)]/Q^-_x$ is prime by (50), $t[N^-_t(x)]$ is strongly connected by the first assertion of Remark 13. It follows from the fourth assertion of Remark 13 that

$$\text{for each } X' \in \left(f_{\{u^-,u^+,v^-\}}\right)^{-1}\left(Q^+_{f_{\{u^-,u^+,v^-\}}(x)}\right),$$
$$\text{there exists } Y' \in Q^-_x \text{ such that } Y' \supseteq X'.$$

Therefore

$$\left|\left(f_{\{u^-,u^+,v^-\}}\right)^{-1}\left(Q^+_{f_{\{u^-,u^+,v^-\}}(x)}\right)\right| \geq |Q^-_x|,$$

which is impossible because

$$\begin{cases} \left|\left(f_{\{u^-,u^+,v^-\}}\right)^{-1}\left(Q^+_{f_{\{u^-,u^+,v^-\}}(x)}\right)\right| = \left|Q^+_{f_{\{u^-,u^+,v^-\}}(x)}\right| \leq \frac{|\Pi(T-x)|-1}{2}+1 \\ \text{and, by (49),} \\ |Q^-_x| = |\Pi(T-x)|. \end{cases}$$

It follows that (48) holds. Thus $f_{\{u^-,u^+,v^-\}}(x)_{\restriction V(t)\setminus\{x\}}$ is an isomorphism from $t-x$ onto $(t-x)^\star$. Therefore $(f_{\{u^-,u^+,v^-\}}(x)_{\restriction V(t)\setminus\{x\}})/\Pi(t-x)$ is an isomorphism from $(t-x)/\Pi(t-x)$ onto $((t-x)/\Pi(t-x))^\star$. Lastly, by Remark 14 applied to $T-x$, $\Pi(t-x) = (\Pi(T-x)\setminus\{X,Y\}) \cup \{X\setminus\{u^-,u^+\}, Y\setminus\{v^-\}\}$. Consequently, $\Pi_{k-2}(t-x) = \{X\setminus\{u^-,u^+\}\}$ and $\Pi_{k-1}(t-x) = \{Y\setminus\{v^-\}\}$, which contradicts the selfduality of $(t-x)/\Pi(t-x)$ by Remark 38.

Consequently, $\mathscr{C}(T) = \emptyset$. ■

The threshold 8 of Theorem 8 is sharp because $T_7$ is a $\{-3,-2\}$-selfdual and prime tournament that is critical (by Theorem 23).

## 7. Applications to Pouzet's Reconstruction

In this section, we prove Corollary 10 and Theorem 11. Corollary 10 is an easy consequence of Lemma 9, Theorem 20 and Theorem 8.

***Proof of Corollary* 10.** Let $T$ be a prime tournament such that $v(T) \geq 8$ and $\mathscr{C}(T) \neq \emptyset$ (see Notation 22). To show that $T$ is $\{-3,-2\}$-reconstructible, consider a tournament $U$ that is $\{-3,-2\}$-hypomorphic to $T$. By Lemma 9, $T$ and $U$ are $\{3\}$-hypomorphic. It follows from Theorem 20 that $U = T$ or $T^\star$. If $U = T^\star$, then $T$ is a $\{-3,-2\}$-selfdual and prime tournament such that $\mathscr{C}(T) \neq \emptyset$, which contradicts Theorem 8. It follows that $U = T$. Therefore $T$ is $\{-3,-2\}$-reconstructible. ■

It is easily verified that Corollary 10 is also satisfied by prime tournaments $T$ such that $\mathscr{C}(T) \neq \emptyset$, when $v(T) \leq 7$. We use the next two results to prove Theorem 11.

**Proposition 55** [18]**.** *Let $T$ be a tournament such that $v(T) \geq 5$. If $T$ is not strongly connected, then $T$ is $\{-1\}$-reconstructible.*

**Lemma 56** [2]**.** *Let $T$ and $U$ be strongly connected tournaments such that $\Pi(T) = \Pi(U)$. Suppose that $|\Pi(T) \setminus \Pi_1(T)| \geq 2$. If $T$ and $U$ are $\{-1\}$-hypomorphic, then for each $X \in \Pi(T)$, $T[X]$ and $U[X]$ are isomorphic.*

***Proof of Theorem* 11.** Consider a decomposable tournament $T$ such that $v(T) \geq 7$. By Proposition 55, if $T$ is not strongly connected, then

$$T \text{ is } \{-1\}\text{-reconstructible},$$

so $T$ is $\{-2,-1,3\}$-reconstructible. Thus suppose that $T$ is strongly connected. By the first assertion of Remark 13,

$$T/\Pi(T) \text{ is prime.} \tag{51}$$

Consider a tournament $U$ such that $T$ and $U$ are $\{-2,-1,3\}$-hypomorphic. We have to prove that $T$ and $U$ are isomorphic. Since $T$ and $U$ are $\{3\}$-hypomorphic, it follows from Corollary 21 that $U$ is strongly connected,

$$\Pi(T) = \Pi(U), \tag{52}$$

and

$$T/\Pi(T) = U/\Pi(U) \text{ or } (U/\Pi(U))^\star. \tag{53}$$

We prove that for each $X \in \Pi(T)$,

$$\text{there exists an isomorphism } \varphi_X \text{ from } T[X] \text{ onto } U[X]. \tag{54}$$

By Lemma 56, (54) holds when $|\Pi(T) \setminus \Pi_1(T)| \geq 2$ (see Notation 37). Hence suppose that $\Pi(T)$ admits a unique element $X$ such that $|X| \geq 2$. If $|\Pi(T)| = 3$, then $T[X]$ and $U[X]$ are isomorphic because $T$ and $U$ are $\{-2\}$-hypomorphic, and $|V(T) \setminus X| = 2$. Thus suppose that $|\Pi(T)| \geq 5$. It follows from Corollary 19 that there exist $Y, Z \in \Pi(T) \setminus \{X\}$ such that $(T/\Pi(T)) - \{Y, Z\}$ is prime. Since $\Pi(T) \setminus \Pi_1(T) = \{X\}$, there exist $u, v \in V(T) \setminus X$ such that $Y = \{u\}$ and $Z = \{v\}$. By Remark 15, $\Pi(T - \{u, v\}) = \Pi(T) \setminus \{\{u\}, \{v\}\}$. It follows from (52) and (53) that $\Pi(U - \{u, v\}) = \Pi(T - \{u, v\})$. Therefore

$$\Pi(T - \{u,v\}) \setminus \Pi_1(T - \{u,v\}) = \Pi(U - \{u,v\}) \setminus \Pi_1(U - \{u,v\}) = \{X\}. \tag{55}$$

Since $T$ and $U$ are $\{-2,-1\}$-hypomorphic, there exists an isomorphism $g_{\{u,v\}}$ from $T - \{u, v\}$ onto $U - \{u, v\}$. It follows from (55) that $g_{\{u,v\}}(X) = X$, so $T[X]$ and $U[X]$ are isomorphic. Consequently, (54) holds.

If $T/\Pi(T) = U/\Pi(U)$ (see (53)), then the common extension of the $\varphi_X$'s

$$\begin{aligned} V(T) &\longrightarrow V(U) \\ v &\longmapsto \varphi_X(v), \text{ where } X \in \Pi(T) \text{ and } v \in X, \end{aligned}$$

is an isomorphism from $T$ onto $U$. Now, by (53), we can suppose that

$$T/\Pi(T) = (U/\Pi(U))^\star.$$

We show that

$$\text{there exists } i \in \Upsilon(T) \text{ such that } i \geq 2 \text{ and } i - 1 \notin \Upsilon(T). \tag{56}$$

Seeking a contradiction, suppose that (56) does not hold. We obtain

$$\Upsilon(T) = \{1, \ldots, \mu(T)\}.$$

We distinguish the following two cases. In both cases, we obtain a contradiction.

1. Suppose that $\mu(T) \leq 3$. We have $\mu(T) = 2$ or $3$ because $T$ is decomposable. Since $T$ and $U$ are $\{3\}$-hypomorphic, we obtain that $T^\star[W]$ and $U[W]$ are isomorphic for each $W \subseteq V(T)$. It follows that $T$ and $T^\star$ are $\{-2,-1\}$-hypomorphic, that is, $T$ is $\{-2,-1\}$-selfdual, which contradicts Theorem 7 because $T$ is neither a linear order nor a circle nor a lexicographic product. Indeed, $T$ is not a linear order because $T$ is strongly connected. Furthermore, $T$ is not a circle because $\Upsilon(T) = \{1, \ldots, \mu(T)\}$ and $v(T) \geq 7$. Lastly, since $1, 2 \in \Upsilon(T)$, $T$ is not a lexicographic product.

2. Suppose that $\mu(T) \geq 4$. To begin, suppose that $\nu_2(T)$ is even and $\nu_3(T)$ is odd. Consider $X \in \Pi_2(T)$ and $v \in X$. Since $T$ and $U$ are $\{-1\}$-hypomorphic,

there exists an isomorphism $g_{\{v\}}$ from $T-v$ onto $U-v$. By (52), $\Pi(T)=\Pi(U)$. It follows from Remark 14 that

$$\Pi(T-v)=\Pi(U-v)=(\Pi(T)\setminus\{X\})\cup\{X\setminus\{v\}\}. \tag{57}$$

Since $g_{\{v\}}$ is an isomorphism from $T-v$ onto $U-v$, $g_{\{v\}}$ induces an isomorphism

$$\begin{array}{rcl} g_{\{v\}}/\Pi(T-v): \Pi(T-v) & \longrightarrow & \Pi(U-v) \\ X' & \longmapsto & g_{\{v\}}(X'), \end{array} \tag{58}$$

from $(T-v)/\Pi(T-v)$ onto $(U-v)/\Pi(U-v)$, that is, $((T-v)/\Pi(T-v))^\star$, which is impossible because of Lemma 35. Indeed, it follows from (57) that $\Pi_2(T-v)=\Pi_2(T)\setminus\{X\}$. Since $\nu_2(T)$ is even, we obtain that $|\Pi_2(T-v)|$ is odd. Moreover, $g_{\{v\}}(\Pi_2(T-v))=\Pi_2(T-v)$ by definition of $g_{\{v\}}/\Pi(T-v)$. By the second assertion of Lemma 35, there exists $Y\in\Pi_2(T-v)$ such that $(g_{\{v\}}/\Pi(T-v))(Y)=Y$. Similarly, it follows from (57) that $\Pi_3(T-v)=\Pi_3(T)$. Since $\nu_3(T)$ is odd, $|\Pi_3(T-v)|$ is odd. Thus, there also exists $Z\in\Pi_3(T-v)$ such that $(g_{\{v\}}/\Pi(T-v))(Z)=Z$, which contradicts the fact that $g_{\{v\}}/\Pi(T-v)$ is an isomorphism from $(T-v)/\Pi(T-v)$ onto $((T-v)/\Pi(T-v))^\star$. We get an analogous contradiction when $\nu_2(T)$ and $\nu_3(T)$ are even, by considering $X\in\Pi_3(T)$ and $v\in X$. Lastly, suppose that $\nu_2(T)$ is odd. The contradiction is obtained in the following manner. If $\nu_3(T)$ is even or $\nu_4(T)$ is even, then it suffices to consider $X\in\Pi_4(T)$ and $v\in X$. If $\nu_3(T)$ and $\nu_4(T)$ are odd, then it suffices to consider $X\in\Pi_2(T)$ and $v\in X$.

It follows that (56) holds. Hence, there exists $i\in\Upsilon(T)$ such that $i\geq 2$ and $i-1\notin\Upsilon(T)$. Consider $X\in\Pi_i(T)$ and $v\in X$. Since $T$ and $U$ are $\{-1\}$-hypomorphic, there exists an isomorphism $g_{\{v\}}$ from $T-v$ onto $U-v$. By Remark 14,

$$\Pi(T-v)=\Pi(U-v)=(\Pi(T)\setminus\{X\})\cup\{X\setminus\{v\}\}.$$

As previously (see (58)), $g_{\{v\}}$ induces an isomorphism

$$\begin{array}{rcl} g_{\{v\}}/\Pi(T-v): \Pi(T-v) & \longrightarrow & \Pi(U-v) \\ X' & \longmapsto & g_{\{v\}}(X'), \end{array} \tag{59}$$

from $(T-v)/\Pi(T-v)$ onto $((T-v)/\Pi(T-v))^\star$. Since $\Pi_{i-1}(T-v)=\{X\setminus\{v\}\}$, we obtain $(g_{\{v\}}/\Pi(T-v))(X\setminus\{v\})=X\setminus\{v\}$, that is, $g_{\{v\}}(X\setminus\{v\})=X\setminus\{v\}$. It follows that

$$\begin{array}{rcl} V(T) & \longrightarrow & V(U) \\ x & \longmapsto & \begin{cases} g_{\{v\}}(x) \text{ if } x\in V(T)\setminus X \\ \text{or} \\ \varphi_X(x) \text{ if } x\in X \text{ (see (54))}, \end{cases} \end{array}$$

is an isomorphism from $T$ onto $U$. ∎

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