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TOTAL COLORINGS OF EMBEDDED GRAPHS WITH NO 3-CYCLES ADJACENT TO 4-CYCLES

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Abstract

A *total-k-coloring* of a graph G is a coloring of $V \cup E$ using k colors such that no two adjacent or incident elements receive the same color. The *total chromatic number* $\chi''(G)$ of G is the smallest integer k such that G has a total-k-coloring. Let G be a graph embedded in a surface of Euler characteristic $\varepsilon \geq 0$. If G contains no 3-cycles adjacent to 4-cycles, that is, no 3-cycle has a common edge with a 4-cycle, then $\chi''(G) \leq \max\{8, \Delta + 1\}$.

Keywords: total coloring, embedded graph, cycle.

2010 Mathematics Subject Classification: 05C15.

1. Introduction

All graphs considered in this paper are simple, finite and undirected, and we follow [2] for the terminologies and notations not defined here. Let G be a graph.

We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ (or simply V, E, Δ and δ) to denote the vertex set, the edge set, the maximum degree and the minimum degree of G, respectively. A *total-k-coloring* of a graph G is a coloring of $V \cup E$ using k colors such that no two adjacent or incident elements receive the same color. The *total chromatic number* $\chi''(G)$ of G is the smallest integer k such that G has a total-k-coloring. Clearly, $\chi''(G) \geq \Delta+1$. Behzad [1] and Vizing [18] posed independently the following famous conjecture, which is known as the Total Coloring Conjecture (TCC).

Conjecture. *For any graph G, $\chi''(G) \leq \Delta + 2$.*

This conjecture was confirmed for all graphs with $\Delta \leq 3$ independently by Vijayaditya and Rosenfeld in 1971, and in [13, 14], Kostochka proved that if $4 \leq \Delta \leq 5$, then $\chi''(G) \leq \Delta + 2$. Later, Kostochka [15] renewed the proof for $\Delta = 5$. We summary these result to the following lemma.

Lemma 1. *Let G be a graph with $\Delta(G) \leq 5$. Then $\chi''(G) \leq 7$.*

But for planar graphs, the famous conjecture was first proved by Borodin [4] for $\Delta \geq 11$ and then for $\Delta \geq 9$ [3], which was extended to $\Delta \geq 8$ by Jensen and Toft [9] and to $\Delta \geq 7$ by Sanders and Zhao [17]. So the only open case is $\Delta = 6$.

Interestingly, planar graphs with high maximum degree allow a stronger assertion, that is, every planar graph with high maximum degree Δ has a total-$(\Delta + 1)$-coloring. This result was first established in [4] for $\Delta \geq 16$, which was extended to $\Delta \geq 14$ [3], $\Delta \geq 12$ [5], $\Delta \geq 11$ [6], $\Delta \geq 10$ [25] and finally $\Delta \geq 9$ [10]. However, for $\Delta \in \{4, 5, 6, 7, 8\}$, it is not known if the assertion still holds true. Such a study has attracted a considerable amount of attention. Recently, Shen *et al.* [11] proved that if G is a planar graph with $\Delta = 8$ and G contains no chordal 5-cycles or no chordal 6-cycles, then $\chi''(G) = \Delta + 1$. Wang and Wu [19] proved that if G is a planar graph with $\Delta \geq 7$ and every vertex is incident with at most one triangle, then $\chi''(G) = \Delta + 1$. Wang and Wu [20] proved that if G is a planar graph with $\Delta \geq 7$ with no 4-cycles, then $\chi''(G) = \Delta + 1$ (later, it is extended to $\Delta \geq 6$ by Shen and Wang [12]). Chang *et al.* [7] proved that if G is a planar graph with $\Delta \geq 7$ and every vertex v has an integer $k_v \in \{3, 4, 5, 6\}$, such that v is not in any k_v-cycle, then $\chi''(G) = \Delta + 1$.

Let G be a graph embedded in a surface of Euler characteristic ε, where *surfaces* in this paper are compact, connected 2-dimensional manifolds without boundary. All embeddings considered in this paper are 2-*cell embeddings*. Wu and Wang [24] proved that if $\varepsilon < 0$ and $\Delta(G) \geq \sqrt{25 - 24\varepsilon} + 10$, then $\chi'_{list}(G) = \Delta(G)$ and $\chi''_{list}(G) = \Delta(G)+1$, which extends a result of Borodin, Kostochka and Woodall in [5]. They also proved that $\chi''(G) = \Delta(G) + 1$ if $\varepsilon \geq 0$, $\Delta(G) \geq 9$ and no two triangles have a common edge, or if $\varepsilon \geq 0$, $\Delta(G) \geq 8$ and no two triangles have a common vertex. Wang *et al.* [22] proved that if $\varepsilon \geq 0$ and $\Delta(G) \geq 7$, then $\chi''(G) \leq \Delta + 2$. Wang *et al.* [23] proved that if $\varepsilon \geq 0$ and $\Delta \geq 9$, then $\chi''(G) = \Delta + 1$. In this paper, we shall prove the following result.

Theorem 2. *Let G be a graph embedded in a surface of Euler characteristic $\varepsilon \geq 0$. If G contains no 3-cycles adjacent to 4-cycles, then $\chi''(G) \leq \max\{8, \Delta(G)+1\}$.*

The theorem shows that if a graph G can be embedded in a surface of Euler characteristic $\varepsilon \geq 0$, and contains no 3-cycles adjacent to 4-cycles, and $\Delta \geq 7$, then $\chi''(G) = \Delta + 1$.

2. PROOF OF THEOREM 2

We will introduce some more notations and definitions here for convenience. Let $G = (V, E, F)$ be an embedded graph, where F is the face set of G. For a vertex $v \in V$, let $N(v)$ denote the set of vertices adjacent to v, and let $d(v) = |N(v)|$ denote the degree of v, and for a face f, the degree of a face f, denoted by $d(f)$, is the number of edges incident with it, where each cut-edge is counted twice. A *k-vertex*, a *k^+-vertex* or a *k^--vertex* is a vertex of degree k, at least k or at most k, respectively. Similarly, A *k-face*, a *k^+-face* is a face of degree k or at least k, respectively. Let $n_t(v)$ be the number of t-vertices adjacent to a vertex v, and $f_k(v)$ the number of k-faces incident with v. Especially, let $f_3(v) = t$. Let $v_1, v_2, \ldots, v_d$ be neighbors of v in an anticlockwise order. Let f_i be face incident with v, v_i and v_{i+1}, for all i such that $i \in \{1, 2, \ldots, d\}$. Note that all the subscripts in the paper are taken modulo d. For convenience, $(d_1, d_2, \ldots, d_n)$ denotes a cycle (or a face) whose boundary vertices are of degree $d_1, d_2, \ldots, d_n$ in the anticlockwise order. Specially, (i, j^+, k^+)-face is a 3-face uvw such that $d(u) = i \leq j \leq d(v) \leq k \leq d(w)$.

***Proof of Theorem* 2.** Let $m = \max\{7, \Delta\}$ and $G = (V, E, F)$ be a minimal counterexample to Theorem 2 with $|V| + |E|$ as small as possible. Then every proper subgraph of G has a total-$(m+1)$-coloring, but G itself does not. First we show some known properties of G.

(a) Every 3-cycle is not adjacent to a 4^--face. It follows that $f_3(v) \leq \left\lfloor \frac{d(v)}{2} \right\rfloor$ for any $v \in V(G)$.

(b) For any edge $uv \in E(G)$, if $\min\{d(u), d(v)\} \leq \lfloor \frac{m}{2} \rfloor$, then $d(u)+d(v) \geq m+2$. So all neighbors of any 2-vertex are 7^+-vertices and all neighbors of any 3-vertex are 6^+-vertices (see [20]).

(c) The subgraph G_2 of G induced by all edges incident with 2-vertices is a forest. So for any component of G_2, we root it at a 7^+-vertex. Then every 2-vertex has exactly one *parent* and exactly one *child* (see [3, 6]).

(d) Each 3-face of G is not incident with two 4^--vertices (see [16]).

(e) If v is a vertex of G with $n_2(v) \geq 1$, then $n_{4^+}(v) \geq 1$ (see [7]).

Lemma 3 [21]**.** *Suppose v is a d-vertex of G with $d \geq 5$. Let $v_1, \ldots, v_d$ be the neighbors of v and $f_1, \ldots, f_d$ be the faces incident with v in clockwise order, where f_i is incident with v_i and v_{i+1}, $i = 1, 2, \ldots, d$. Note that eventually v_1 and v_{d+1} is the same vertex. Then there does not exist an integer i $(2 \leq i \leq d)$ such that $d(v_1) = d(v_i) = 2$, $d(v_k) = 3$ $(2 \leq k \leq i-1)$ and $d(f_t) = 4$ $(1 \leq t \leq i-1)$.*

Lemma 4. *G contains no subgraph isomorphic to one of the configurations in Figure 1, where the vertices marked by $\bullet$ have no other neighbors in G.*

Proof. The proof that G contains no subgraph isomorphic to one of the configurations in Figure 1(1)–(4) can be found in [8]. It remains to prove that G has no configurations depicted in Figure 1(5)–(13).

By the minimality of G, every proper subgraph of G has a total-$(m+1)$-coloring φ with the color set $C = \{1, 2, \ldots, m+1\}$. Erase the colors on all 3^--vertices. Let $C(v) = \{\varphi(uv) : u \in N(v)\} \cup \{\varphi(v)\}$.

Suppose that G contains a configuration depicted in Figure 1(5). Then $G' = G - vv_6$ has a total-8-coloring φ. If $\varphi(v_6x_5) \in C(v)$ or $\varphi(v_6x_6) \in C(v)$, then the forbidden colors for vv_6 is at most 7, so vv_6 can be properly colored. By recoloring the erased vertices, we obtain a total-8-coloring of G, a contradiction. So we can assume that $\varphi(v_6x_5) \notin C(v)$ and $\varphi(v_6x_6) \notin C(v)$. Without loss of generality, assume that $\varphi(v) = 6$, $\varphi(v_6x_5) = 7$, $\varphi(v_6x_6) = 8$, and $\varphi(vv_j) = j$ for $j \in \{1, 2, \ldots, 5\}$. Then we recolor v with 7 or 8, and color vv_6 with 6. By recoloring the erased vertices, we obtain a total-8-coloring of G, a contradiction.

Suppose that G contains a configuration depicted in Figure 1(6)–(13). Then $G' = G - vv_7$ has a total-8-coloring φ. If $\varphi(v_7x_7) \in C(v)$, then the forbidden colors for vv_7 is at most 7, so vv_7 can be properly colored. By recoloring the erased vertices, we obtain a total-8-coloring of G, a contradiction. So we can assume that $\varphi(vv_7) \notin C(v)$. Without loss of generality, assume that $\varphi(v) = 8$, $\varphi(v_7x_7) = 7$, and $\varphi(vv_j) = j$ for $j \in \{1, 2, \ldots, 6\}$. Thus, for each 3^--vertex $v_k (1 \leq k \leq 7)$, there is an edge incident with v_k colored 7, otherwise we can recolor vv_k with 7, and color vv_7 with k to obtain a total-8-coloring of G, a contradiction.

For each 4-vertex v_i $(1 \leq i \leq 6)$, suppose its adjacent vertices are v, x_{i-1}, x_i, x_j. If $\varphi(v_i) \neq 7$ $(1 \leq i \leq 6)$, then recolor v with 7, and color vv_7 with 8. By recoloring the erased vertices, we obtain a total-8-coloring of G, a contradiction. Otherwise, there is at least one 4-vertex colored with 7. Suppose v is adjacent to only one 4-vertex v_i colored with 7. If $|C(v_i)| < 8$, then we recolor v_i with a color in $C \backslash C(v_i)$, recolor v with 7, and color vv_7 with 8. Otherwise, $|C(v_i)| = 8$. If $i \notin \{\varphi(x_{i-1}), \varphi(x_i), \varphi(x_j)\}$, then we recolor v_i with i, recolor vv_i with 7, and color

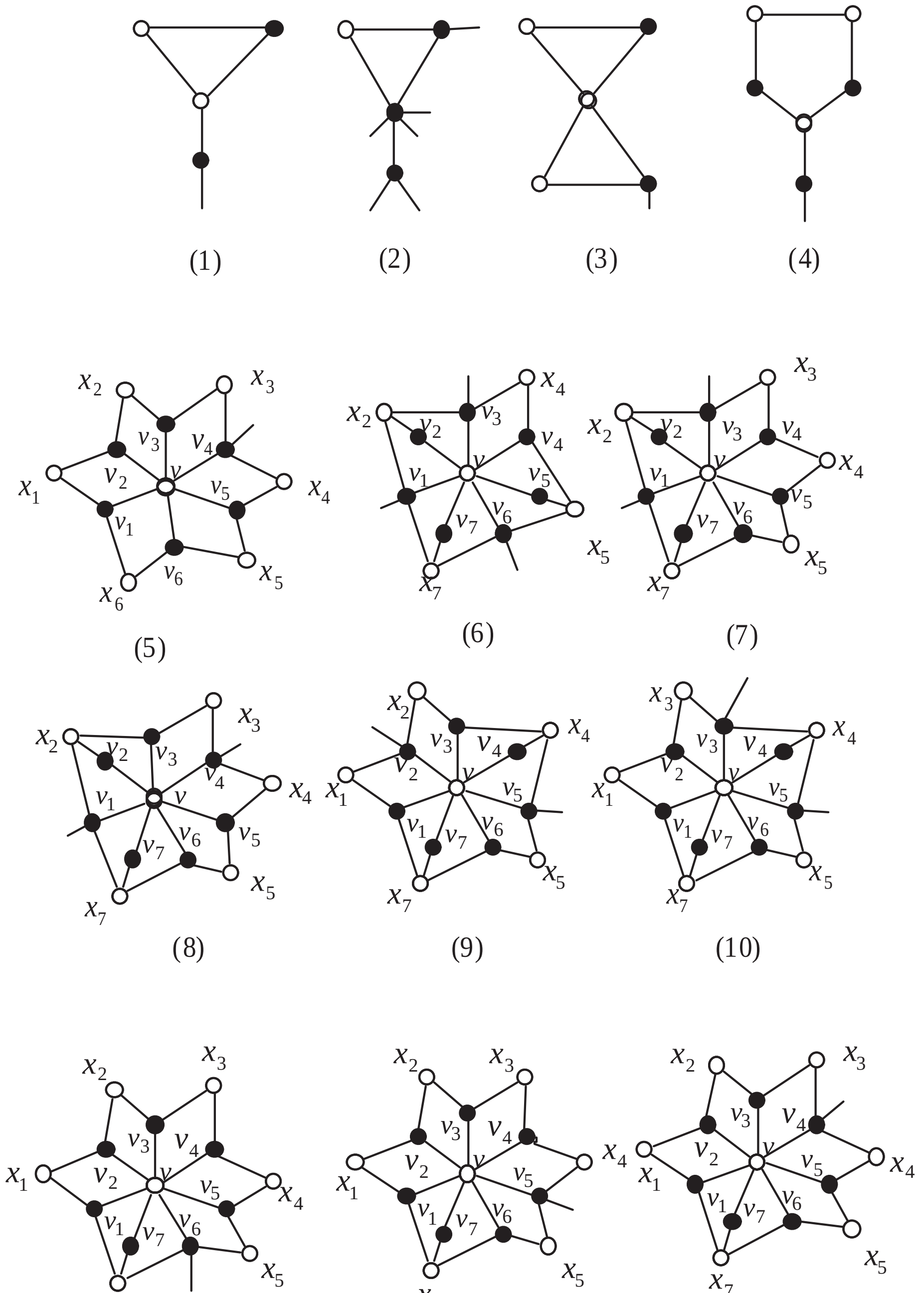

Figure 1. Reducible configurations.

vv_7 with i. Otherwise, $i \in \{\varphi(x_{i-1}), \varphi(x_i), \varphi(x_j)\}$. Without loss of generality, $\varphi(x_i) = i$, then $8 \notin \{C(v_i)\backslash\varphi(v)\}$. Therefore, we recolor v_i with 8, recolor v with 7, and color vv_7 with 8. Finally, we recolor the erased vertices, we obtain a total-8-coloring of G, a contradiction. Otherwise, v is adjacent to two or three 4-vertices colored with 7, then we take the same operations as above, respectively. Thus we can also obtain a total-8-coloring of G, a contradiction. ■

Let $G = (V, E, F)$ be a graph which is embedded in a surface of nonnegative Euler characteristic. By Euler's formula $|V| - |E| + |F| = \varepsilon$, we have

$$\sum_{v \in V}(2d(v) - 6) + \sum_{f \in F}(d(f) - 6) = -6(|V| - |E| + |F|) = -6\varepsilon \leq 0.$$

Now we define the initial charge function $ch(x)$ of $x \in V \cup F$ to be $ch(v) = 2d(v)-6$ if $v \in V$ and $ch(f) = d(f)-6$ if $f \in F$. It follows that $\sum_{x \in V \cup F} ch(x) \leq 0$. Now we design appropriate discharging rules and redistribute weights accordingly. Note that any discharging procedure preserves the total charge of G. If we can define suitable discharging rules to charge the initial charge function ch to the final charge function ch' on $V \cup F$ such that $\sum_{x \in V \cup F} ch'(x) > 0$, then we get an obvious contradiction.

Our discharging rules are defined as follows.

R1. Every 2-vertex receives $\frac{3}{2}$ from its child and $\frac{1}{2}$ from its parent.

R2. Let f be a 3-face. If f is incident with a 3^--vertex, then it gets $\frac{3}{2}$ from each of its incident 6^+-vertices. If f is incident with a 4-vertex, then it gets $\frac{1}{2}$ from the 4-vertex and gets $\frac{5}{4}$ from each of its incident 5^+-vertices. If f is not incident with any 4^--vertex, then it gets 1 from each of its incident 5^+-vertices.

R3. Let f be a 4-face. If f is incident with two 3^--vertices, then it gets 1 from each of its two incident 6^+-vertices. If f is incident with only one 3^--vertex and one 4-vertex, then it gets $\frac{1}{2}$ from the incident 4-vertex and gets $\frac{3}{4}$ from each of its two incident 6^+-vertices. If f is incident with only one 3^--vertex and no 4-vertex, then it gets $\frac{2}{3}$ from each of its incident 5^+-vertices. If f is not incident with any 3^--vertex, then it gets $\frac{1}{2}$ from each of its incident vertices.

R4. Every 5-face gets $\frac{1}{3}$ from each of its incident 4^+vertices.

First, we begin to check $ch'(x) \geq 0$ for all $x \in V \cup F$. By our discharging rules, it is easy to check that $ch'(f) \geq 0$ for all $f \in F$ and $ch'(v) \geq 0$ for all 2-vertices $v \in V$. If $d(v) = 3$, then $ch'(v) = ch(v) = 0$. So it suffices to check that $ch'(v) \geq 0$ for all 4^+-vertices G.

Let v be a 4^+-vertex of G. If $d(v) = 4$, then v sends at most $\frac{1}{2}$ to each of its incident faces by R2 and R3, and it follows that $ch'(v) \geq ch(v) - \frac{1}{2} \times 4 = 0$. Suppose $d(v) = 5$. Then v sends at most $\frac{5}{4}$ to each of its incident 3-faces by

R2, at most $\frac{2}{3}$ to each of its incident 4^+-faces by R3, at most $\frac{1}{3}$ to each of its incident 5-faces by R4. By (a), $f_3(v) \le 2$. If $f_3(v) = 2$, then v is incident with at least three 5^+-faces, that is, $f_{5^+}(v) \ge 3$, and it follows that $ch'(v) \ge ch(v) - \frac{5}{4} \times 2 - \frac{1}{3} \times 3 = \frac{1}{2} > 0$. If $f_3(v) = 1$, then $f_{5^+}(v) \ge 2$ and $f_{4^+}(v) \le 2$, and it follows that $ch'(v) \ge ch(v) - \frac{5}{4} - \frac{1}{3} \times 2 - \frac{2}{3} \times 2 = \frac{3}{4} > 0$. If $f_3(v) = 0$, then $ch'(v) \ge ch(v) - \frac{2}{3} \times 5 = \frac{2}{3} > 0$. Suppose $d(v) = 6$. Then $f_3(v) \le 3$ and v sends at most $\frac{3}{2}$ to each of its incident 3-faces by R2, at most 1 to each of its incident 4^+-faces by R3, at most $\frac{1}{3}$ to each of its incident 5-faces by R4. Thus, if $1 \le f_3(v) \le 3$, then by the similar argument as above, we have $ch'(v) \ge ch(v) - \max\left\{\frac{3}{2} \times 3 + \frac{1}{3} \times 3, \frac{3}{2} \times 2 + \frac{1}{3} \times 3 + 1, \frac{3}{2} + \frac{1}{3} \times 2 + 1 \times 3\right\} = \frac{1}{2} > 0$. Otherwise, v is adjacent to at least one 4^+-vertex or incident with at least one 5^+-face by Lemma 4, configuration (5). If v is adjacent to at least one 4^+-vertex, then $ch'(v) \ge ch(v) - 2 \times \frac{3}{4} - 4 = \frac{1}{2} > 0$ by R3. If v is incident with at least one 5^+-face, then $ch'(v) \ge ch(v) - \frac{1}{3} - 5 = \frac{2}{3} > 0$ by R4.

Suppose $d(v) = d \ge 7$. Let $N(v) = \{v_1, v_2, \ldots, v_d\}$ and $f_1, f_2, \ldots, f_d$ be faces incident with v in the clockwise order, where f_i is incident with v_i and v_{i+1}, for $i \in \{1, 2, \ldots, d\}$, where all the subscripts here are taken modulo d. If $n_2(v) \ge 1$, then v sends at most $\frac{n_2(v)+2}{2}$ to all its adjacent 2-vertices by R1, at most $\frac{3}{2}$ to each of its incident 3-faces by R2, at most 1 to each of its incident 4^+-faces by R3, at most $\frac{1}{3}$ to each of its incident 5-faces by R4.

Lemma 5. *Suppose that $d(v_i) = d(v_k) = 2$ and $d(v_j) \ge 3$ for all $j = i+1, \ldots, k-1$. If $f_i, f_{i+1}, \ldots, f_{k-1}$ are 4^+-faces, then v sends at most $\frac{3}{2} + (k-3)$ (in total) to $f_i, f_{i+1}, \ldots, f_{k-1}$.*

Proof. By Lemma 3, $\max\{d(v_{i+1}), \ldots, d(v_{k-1})\} \ge 4$ or $\max\{d(f_1), \ldots, d(f_{k-1})\} \ge 5$. If $\max\{d(v_{i+1}), \ldots, d(v_{k-1})\} \ge 4$, then v sends at most $2 \times \frac{3}{4} + (k-1-2)$ to $f_i, \ldots, f_{k-1}$ by R2. If $\max\{d(f_1), \ldots, d(f_{k-1})\} \ge 5$, then v sends at most $\frac{1}{3} + (k-1-1)$ to $f_i, \ldots, f_{k-1}$ by R3 and R4. Since $2 \times \frac{3}{4} > 1 + \frac{1}{3}$, v sends at most $\frac{3}{2} + (k-3)$ to $f_i, f_{i+1}, \ldots, f_{k-1}$. ■

Case 1. $\Delta(G) = 7$. Let v be a 7-vertex. Then $ch(v) = 2 \times 7 - 6 = 8$. We consider the following cases.

Subcase 1.1. $n_2(v) = 6$. By (c), any $2-$vertex is not incident with a 4-face. Moreover, $t = f_3(v) = 0$ and $f_{6^+}(v) \ge 5$ by Lemma 4, configurations (1) and (4). So $ch'(v) \ge ch(v) - \frac{6+2}{2} - 2 = 2 > 0$.

Subcase 1.2. $n_2(v) = 5$. Then $t \le 1$ by Lemma 4, configurations (1) and (4). If $t = 0$, then $f_{6^+}(v) \ge 3$ and $f_{4^+}(v) \le 4$ by Lemma 4, configuration (4), and it follows that $ch'(v) \ge ch(v) - \frac{5+2}{2} - 4 \times 1 = \frac{1}{2} > 0$. Otherwise, $f_{6^+}(v) \ge 4$ and $f_{4^+}(v) \le 2$. Thus $ch'(v) \ge ch(v) - \frac{5+2}{2} - \frac{3}{2} - 2 \times 1 = 1 > 0$.

Subcase 1.3. $n_2(v) = 4$. There are four possible configurations as shown in Figure 2.

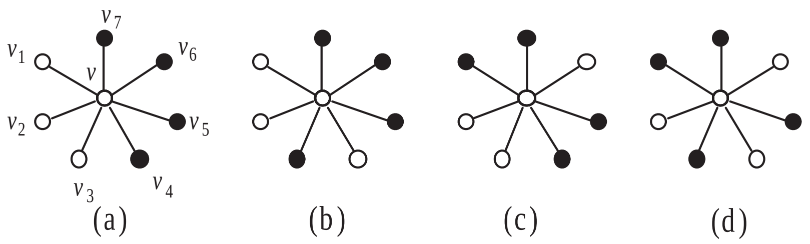

Figure 2. $n_2(v) = 4$.

For Figure 2(a), $t \le 1$ and $f_{6^+}(v) \ge 3$ by Lemma 4, configurations (1) and (4). If $t = 1$, then $f_{5^+}(v) \ge 2$, and it follows that $ch'(v) \ge ch(v) - \frac{4+2}{2} - \frac{3}{2} - 2 \times \frac{1}{3} - 1 = \frac{11}{6} > 0$. Otherwise, $ch'(v) \ge ch(v) - \frac{4+2}{2} - \frac{3}{2} - 2 = \frac{3}{2} > 0$ by Lemma 5.

For Figure 2(b) and (c), $t \le 1$ and $f_{6^+}(v) \ge 2$ by Lemma 4, configurations (1) and (4). If $t = 1$, then $f_{5^+}(v) \ge 2$, and it follows that $ch'(v) \ge ch(v) - \frac{4+2}{2} - \frac{3}{2} - 3 \times \frac{1}{3} - \frac{3}{2} = \frac{4}{3} > 0$ by Lemma 5. Otherwise, $ch'(v) \ge ch(v) - \frac{4+2}{2} - \frac{3}{2} - \frac{3}{2} - 1 = 1 > 0$ by Lemma 5.

For Figure 2(d), $t = 0$ and $f_{6^+}(v) \ge 1$ by Lemma 4, configurations (1) and (4). Then $ch'(v) \ge ch(v) - \frac{4+2}{2} - 3 \times \frac{3}{2} = \frac{1}{2} > 0$ by Lemma 5.

Subcase 1.4. $n_2(v) = 3$. There are four possible configurations as shown in Figure 3.

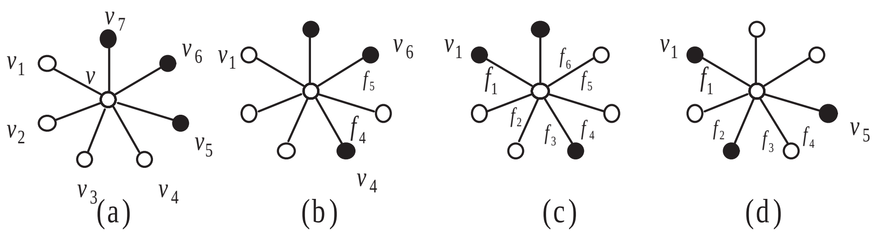

Figure 3. $n_2(v) = 3$.

For Figure 3(a), $t \le 2$ and $f_{6^+}(v) \ge 2$ by Lemma 4, configurations (1) and (4). If $t = 2$, then $f_{5^+}(v) \ge 3$, and it follows that $ch'(v) \ge ch(v) - \frac{3+2}{2} - 2 \times \frac{3}{2} - 3 \times \frac{1}{3} = \frac{3}{2} > 0$. If $t = 1$, then $f_{5^+}(v) \ge 2$, and it follows that $ch'(v) \ge ch(v) - \frac{3+2}{2} - \frac{3}{2} - 2 \times \frac{1}{3} - 2 = \frac{4}{3} > 0$. Otherwise, $ch'(v) \ge ch(v) - \frac{3+2}{2} - \frac{3}{2} - 3 = 1 > 0$ by Lemma 5.

For Figure 3(b), $t \le 1$ and $f_{6^+}(v) \ge 1$ by Lemma 4, configurations (1) and (4). If $t = 1$, then $f_{5^+}(v) \ge 2$, and it follows that $ch'(v) \ge ch(v) - \frac{3+2}{2} - \frac{3}{2} - 2 \times \frac{1}{3} - 1 - \frac{3}{2} = \frac{5}{6} > 0$ by Lemma 5. Otherwise, $ch'(v) \ge ch(v) - \frac{3+2}{2} - \frac{3}{2} - \frac{3}{2} - 2 = \frac{1}{2} > 0$ by Lemma 5.

For Figure 3(c), $t \le 2$ and $f_{6^+}(v) \ge 1$ by Lemma 4, configurations (1) and (4). If $t = 2$, then $f_{5^+}(v) \ge 4$, and it follows that $ch'(v) \ge ch(v) - \frac{3+2}{2} - 2 \times$

$\frac{3}{2} - 4 \times \frac{1}{3} = \frac{7}{6} > 0$. If $t = 1$, then $f_{5^+}(v) \geq 2$, and it follows that $ch'(v) \geq ch(v) - \frac{3+2}{2} - \frac{3}{2} - 2 \times \frac{1}{3} - \frac{3}{2} - 1 = \frac{5}{6} > 0$ by Lemma 5. Otherwise, $ch'(v) \geq ch(v) - \frac{3+2}{2} - 2 \times \left(\frac{3}{2} + 1\right) = \frac{1}{2} > 0$ by Lemma 5.

For Figure 3(d), $t \leq 1$ by Lemma 4, configuration (1). If $t = 1$, then $f_{5^+}(v) \geq 2$, and it follows that $ch'(v) \geq ch(v) - \frac{3+2}{2} - \frac{3}{2} - 2 \times \frac{1}{3} - 2 \times \frac{3}{2} = \frac{1}{3} > 0$ by Lemma 5. Otherwise, $t = 0$. If v is adjacent to at least four 4^+-vertices, then $ch'(v) \geq ch(v) - \frac{3+2}{2} - \frac{3}{2} - \frac{3}{2} - 3 \times \frac{3}{4} = \frac{1}{4} > 0$ by Lemma 5. Otherwise, v is adjacent to at least one 5^+-vertex or incident with at least one 5^+-face by Lemma 4, configuration (6), and Lemma 3. If v is adjacent to at least one 5^+-vertex, then $ch'(v) \geq ch(v) - \frac{3+2}{2} - \max\left\{2 \times \frac{2}{3} + 1 + 2 \times \frac{3}{2}, \frac{3}{2} + 1 + 2 \times \frac{2}{3} + \frac{3}{2}\right\} = \frac{1}{6} > 0$ by R3 and Lemma 5. If v is incident with at least one 5^+-face, then $ch'(v) \geq ch(v) - \frac{3+2}{2} - \max\left\{\frac{1}{3} + 2 + 2 \times \frac{3}{2}, \frac{3}{2} + 1 + \frac{1}{3} + \frac{3}{2} + 1\right\} = \frac{1}{6} > 0$ by R4 and Lemma 5.

Subcase 1.5. $n_2(v) = 2$. There are three possible configurations as shown in Figure 4.

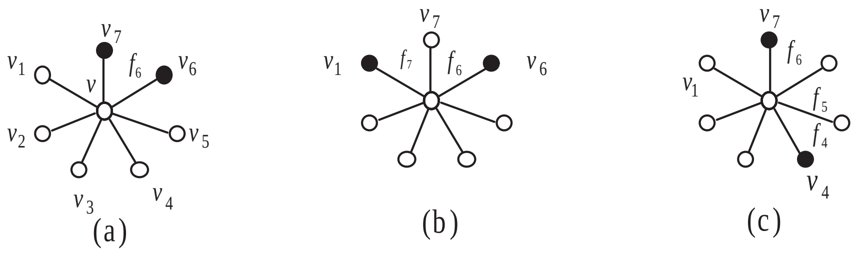

Figure 4. $n_2(v) = 2$.

For Figure 4(a), we have that the face f_6 is a 5^+-face, and it follows that f_6 receives at most $\frac{1}{3}$ from v by R4. So $ch(v) - \frac{2+2}{2} - \frac{1}{3} = 8 - 2 - \frac{1}{3} = \frac{17}{3}$. Moreover, $t \leq 2$ by Lemma 4, configuration (1), and (a). If $1 \leq t \leq 2$, then $f_{5^+}(v) \geq (t+1)$ (except f_6), and it follows that $ch'(v) \geq \frac{17}{3} - t \times \frac{3}{2} - (t+1) \times \frac{1}{3} - (7-1-t-t-1) = \frac{2+t}{6} > 0$. Otherwise, $ch'(v) \geq \frac{17}{3} - (\frac{3}{2} + 4) = \frac{1}{6} > 0$.

For Figure 4(b), $t \leq 2$ by Lemma 4, configuration (1), and (a). If $t = 2$, then $f_{5^+}(v) \geq 3$, it follows that $ch'(v) \geq ch(v) - \frac{2+2}{2} - 2 \times \frac{3}{2} - 3 \times \frac{1}{3} - \frac{3}{2} = \frac{1}{2} > 0$. If $t = 1$, then $f_{5^+}(v) \geq 2$, and it follows that $ch'(v) \geq ch(v) - \frac{2+2}{2} - \frac{3}{2} - \frac{1}{3} \times 2 - 2 \times 1 - \frac{3}{2} = \frac{1}{3} > 0$. Suppose $t = 0$. If v is adjacent to at least three 4^+-vertices, then $ch'(v) \geq ch(v) - \frac{2+2}{2} - \frac{3}{2} - 3 \times \frac{3}{4} - 2 = \frac{1}{4} > 0$ by Lemma 5. Otherwise, v is adjacent to at least one 5^+-vertex or incident with at least one 5^+-face by Lemma 4, configurations (7)–(8). By the same argument as above, we can obtain $ch'(v) \geq ch(v) - \frac{2+2}{2} - \max\left\{\frac{3}{2} + 3 + 2 \times \frac{2}{3}, \frac{3}{2} + 4 + \frac{1}{3}\right\} = \frac{1}{6} > 0$.

For Figure 4(c), $t \leq 2$ by Lemma 4, configuration (1), and (a). If $t = 2$, then $f_{5^+}(v) \geq 4$, it follows that $ch'(v) \geq ch(v) - \frac{2+2}{2} - 2 \times \frac{3}{2} - 4 \times \frac{1}{3} - 1 = \frac{2}{3} > 0$. If $t = 1$, then $f_{5^+}(v) \geq 2$, and it follows that $ch'(v) \geq ch(v) - \frac{2+2}{2} -$

$\max\left\{\frac{3}{2}+\frac{1}{3}\times 2+\frac{3}{2}+2,\ \frac{3}{2}+\frac{1}{3}\times 2+1+\frac{3}{2}+1\right\} = \frac{1}{3} > 0$. Suppose $t=0$. If v is adjacent to at least three 4^+-vertices, then $ch'(v) \geq ch(v) - \frac{2+2}{2} - \frac{3}{2} - 3\times\frac{3}{4} - 2 = \frac{1}{4} > 0$ by Lemma 5. Otherwise, v is adjacent to at least one 5^+-vertex or incident with at least one 5^+-face by Lemma 4, configurations (9)–(10). By the same argument as above, we can obtain $ch'(v) \geq ch(v) - \frac{2+2}{2} - \max\left\{\frac{3}{2}+3+2\times\frac{2}{3}, \frac{3}{2}+4+\frac{1}{3}\right\} = \frac{1}{6} > 0$.

Subcase 1.6. $n_2(v) = 1$. Note that $n_{4^+}(v) \geq 1$ by (e) and $t \leq 3$ by (a). Suppose $t=0$. If v is adjacent to at least two 4^+-vertices, then $ch'(v) \geq ch(v) - \frac{1+2}{2} - 3\times\frac{3}{4} - 4 = \frac{1}{4} > 0$. Otherwise, v is adjacent to at least one 5^+-vertex or incident with at least one 5^+-face by Lemma 4, configurations (11)–(13), then $ch'(v) \geq ch(v) - \frac{1+2}{2} - 2\times\frac{2}{3} - 5 = \frac{1}{6} > 0$ by R3 or $ch'(v) \geq ch(v) - \frac{1+2}{2} - \frac{1}{3} - 6 = \frac{1}{6} > 0$ by R4. Suppose $1 \leq t \leq 3$. If v is incident with a $(2,7,7)$-face, then the other face incident with the 2-vertex is a 6^+-face. Moreover, the other 3-faces incident with v are $(4,5^+,5^+)$-faces by Lemma 4, configuration (3), and v is incident with at least t 5^+-faces. Then we can obtain that $ch'(v) \geq ch(v) - \frac{1+2}{2} - \frac{3}{2} - (t-1)\times\frac{5}{4} - \frac{1}{3}\times t - (7-1-t-t) = \frac{3+5t}{12} > 0$, where v sends at most $\frac{5}{4}$ to each of its incident $(4,5^+,5^+)$-faces by R2. Otherwise, 2-vertex is not incident with any 3-face. Then v is incident with at least $(t+1)$ 5^+-faces, and it follows that $ch'(v) \geq ch(v) - \frac{1+2}{2} - t\times\frac{3}{2} - \frac{1}{3}\times(t+1) - (7-t-t-1) = \frac{1+t}{6} > 0$.

Subcase 1.7. $n_2(v) = 0$. Note that $t \leq 3$ by (a). If $t=0$, then $ch'(v) \geq ch(v) - 7\times 1 = 1 > 0$. Otherwise, by the same argument as above, we can obtain that $ch'(v) \geq ch(v) - t\times\frac{3}{2} - \frac{1}{3}\times(t+1) - (7-t-t-1) = \frac{10+t}{6} > 0$.

Case 2. $\Delta(G) \geq 8$. In [23], Theorem 1 was established for $\Delta \geq 9$. So we assume that $\Delta = 8$. Then $ch(v) = 2\times 8 - 6 = 10$. By the same argument as above, we consider the following cases.

Subcase 2.1. $n_2(v) = 7$. Then $t = 0$ and $f_{6^+}(v) \geq 6$ by Lemma 4, configurations (1) and (4). So $ch'(v) \geq ch(v) - \frac{7+2}{2} - 2 = \frac{7}{2} > 0$.

Subcase 2.2. $n_2(v) = 6$. Then $t \leq 1$ and $f_{6^+}(v) \geq 4$ by Lemma 4, configurations (1) and (4). If $t=0$, then $ch'(v) \geq ch(v) - \frac{6+2}{2} - 4 = 2 > 0$. Otherwise, $ch'(v) \geq ch(v) - \frac{6+2}{2} - \frac{3}{2} - 2\times\frac{1}{3} = \frac{23}{6} > 0$.

Subcase 2.3. $n_2(v) = 5$. Then $t \leq 1$ and $f_{6^+}(v) \geq 2$ by Lemma 4, configurations (1) and (4). If $t=0$, then $ch'(v) \geq ch(v) - \frac{5+2}{2} - 6 = \frac{1}{2} > 0$. Otherwise, $f_{5^+}(v) \geq 2$. Thus $ch'(v) \geq ch(v) - \frac{5+2}{2} - \frac{3}{2} - 2\times\frac{1}{3} - 3 = \frac{4}{3} > 0$.

Subcase 2.4. $n_2(v) = 4$. Then $t \leq 2$ by Lemma 4, configuration (1). If $1 \geq t \geq 2$, then $f_{6^+}(v) \geq 1$ and $f_{5^+}(v) \geq t+1$, and it follows that $ch'(v) \geq ch(v) - \frac{4+2}{2} - t\times\frac{3}{2} - (t+1)\times\frac{1}{3} - (8-1-t-t-1) = \frac{4+t}{6} > 0$. Otherwise, $ch'(v) \geq ch(v) - \frac{4+2}{2} - \max\left\{\frac{3}{2}+3, \frac{3}{2}\times 2+2, \frac{3}{2}\times 3+1, \frac{3}{2}\times 4\right\} = 1 > 0$.

Subcase 2.5. $n_2(v) = 3$. Then $t \leq 2$ by Lemma 4 configuration (1). If $1 \geq t \geq 2$, then $f_{5^+}(v) \geq t+1$, and it follows that $ch'(v) \geq ch(v) - \frac{3+2}{2} - t\times\frac{3}{2} -$

$(t+1)\times\frac{1}{3}-(8-t-t-1)=\frac{1+t}{6}>0$. Otherwise, $ch'(v)\geq ch(v)-\frac{3+2}{2}-\max\left\{\frac{3}{2}+4,\frac{3}{2}\times 2+3,\frac{3}{2}\times 3+2\right\}=1>0$.

Subcase 2.6. $n_2(v)=2$. Then $t\leq 2$ by Lemma 4 configuration (1). If $1\geq t\geq 2$, then $f_{5^+}(v)\geq t+1$, and it follows that $ch'(v)\geq ch(v)-\frac{2+2}{2}-t\times\frac{3}{2}-(t+1)\times\frac{1}{3}-(8-t-t-1)=\frac{4+t}{6}>0$. Otherwise, $ch'(v)\geq ch(v)-\frac{2+2}{2}-\max\left\{\frac{3}{2}+5,\frac{3}{2}\times 2+4\right\}=1>0$.

Subcase 2.7. $n_2(v)=1$. Then $t\leq 4$ by (a). If $1\geq t\geq 4$, then $f_{5^+}(v)\geq t$, and it follows that $ch'(v)\geq ch(v)-\frac{1+2}{2}-t\times\frac{3}{2}-t\times\frac{1}{3}-(8-t-t)=\frac{3+t}{6}>0$. Otherwise, $ch'(v)\geq ch(v)-\frac{1+2}{2}-8=\frac{1}{2}>0$.

Subcase 2.8. $n_2(v)=0$. Then $t\leq 4$ by (a). If $1\geq t\geq 4$, then $f_{5^+}(v)\geq t$, and it follows that $ch'(v)\geq ch(v)-t\times\frac{3}{2}-t\times\frac{1}{3}-(8-t-t)=\frac{12+t}{6}>0$. Otherwise, $ch'(v)\geq ch(v)-8=2>0$.

Finally, according to the above argument, we have checked $ch'(x)\geq 0$ for all $x\in V\cup F$ and $ch'(x)>0$ for any 5^+-vertex $x\in V$. By Lemma 1, we have $\Delta(G)\geq 6$. So $\sum_{x\in V\cup F}ch'(x)>0$. Hence we complete the proof of Theorem 2.

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