# THE BIPARTITE-SPLITTANCE OF A BIPARTITE GRAPH ${ }^{1}$ 

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#### Abstract

A bipartite-split graph is a bipartite graph whose vertex set can be partitioned into a complete bipartite set and an independent set. The bipartitesplittance of an arbitrary bipartite graph is the minimum number of edges to be added or removed in order to produce a bipartite-split graph. In this paper, we show that the bipartite-splittance of a bipartite graph depends only on the degree sequence pair of the bipartite graph, and an easily computable formula for it is derived. As a corollary, a simple characterization of the degree sequence pair of bipartite-split graphs is also given.


Keywords: degree sequence pair, bipartite-split graph, bipartite-splittance.
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## 1. Introduction

Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A subset $S$ of $V(G)$ is complete if the subgraph $G[S]$ induced by $S$ is complete, and it is independent if $G[S]$ is a null graph (i.e., a graph without edges). A split graph is a graph whose set of vertices can be partitioned into a complete set and an independent set. Split graphs were introduced by Földes and Hammer [1], who proved that a graph is split if and only if it does not have an induced subgraph isomorphic to $C_{4}, C_{5}$ or $2 K_{2}$, where $C_{k}$ is a cycle on $k$ vertices and $2 K_{2}$ is the disjoint union of two complete graphs $K_{2}$. The splittance $\sigma(G)$ of an arbitrary graph $G$ is the minimum number of edges to be added to, or removed from $G$

[^0]in order to obtain a split graph. An explicit, easily computable formula for the splittance $\sigma(G)$ in terms of the degree sequence of $G$ and a simple characterization of the degree sequence of split graphs were presented by Hammer and Simeone [3].

Analogous problem is also studied for bipartite graphs in this paper. The style of this paper closely follows that of [3]. Let $G$ be a simple bipartite graph with two partite sets $X$ and $Y$, where $|X|=m$ and $|Y|=n$. If $A=\left(a_{1}, \ldots, a_{m}\right)$ (respectively, $\left.B=\left(b_{1}, \ldots, b_{n}\right)\right)$ is the non-increasing sequence of vertex degrees for $X$ (respectively, $Y$ ), then the pair $(A ; B)$ is the degree sequence pair of $G$. We say that $\left(S_{1} ; S_{2}\right)$ is a subset pair of $(X ; Y)$ if $S_{1} \subseteq X$ and $S_{2} \subseteq Y$. A subset pair $\left(S_{1} ; S_{2}\right)$ of ( $X ; Y$ ) is complete bipartite if either $S_{1}=S_{2}=\emptyset$, or $S_{1} \neq \emptyset, S_{2} \neq \emptyset$ and $G\left[S_{1} \cup S_{2}\right]$ is a complete bipartite graph with two partite sets $S_{1}$ and $S_{2}$. A subset pair $\left(S_{1} ; S_{2}\right)$ of $(X ; Y)$ is independent if $G\left[S_{1} \cup S_{2}\right]$ is a null graph. A bipartite-split graph is a bipartite graph whose two partite sets can be partitioned into a complete bipartite subset pair and an independent subset pair. The bipartite-splittance $\tau(G)$ of an arbitrary bipartite graph $G$ is the minimum number of edges to be added to, or removed from $G$ in order to obtain a bipartite-split graph. Clearly, $G$ is bipartite-split if and only if $\tau(G)=0$. The main result of this paper is to give an easily computable formula for the bipartitesplittance $\tau(G)$ of a bipartite graph $G$ in terms of the degree sequence pair of $G$ (Theorem 5). As a corollary, a simple characterization of the degree sequence pair of bipartite-split graphs is also given (Corollary 6).

## 2. Main Result and Its Proof

Let $\left(S_{1} ; S_{2}\right)$ be a subset pair of $(X ; Y)$, and let

$$
s_{\left(S_{1} ; S_{2}\right)}=\left|S_{1}\right|\left|S_{2}\right|-\left|E\left(G\left[S_{1} \cup S_{2}\right]\right)\right|+\left|E\left(G\left[V(G) \backslash\left(S_{1} \cup S_{2}\right)\right]\right)\right| .
$$

It is easy to see that $\left|S_{1}\right|\left|S_{2}\right|-\left|E\left(G\left[S_{1} \cup S_{2}\right]\right)\right|$ is the number of edges to be added to $G\left[S_{1} \cup S_{2}\right]$ in order to make $G\left[S_{1} \cup S_{2}\right]$ into a complete bipartite graph, and $\left|E\left(G\left[V(G) \backslash\left(S_{1} \cup S_{2}\right)\right]\right)\right|$ is the number of edges to be removed from $G[V(G) \backslash$ $\left.\left(S_{1} \cup S_{2}\right)\right]$ in order to make $G\left[V(G) \backslash\left(S_{1} \cup S_{2}\right)\right]$ into a null graph.
Lemma 1. Let $G$ be a bipartite graph with two partite sets $X$ and $Y$. Then

$$
\tau(G)=\min _{S_{1} \subseteq X, S_{2} \subseteq Y} s_{\left(S_{1} ; S_{2}\right)} .
$$

Proof. Clearly $\tau(G) \leq s_{\left(S_{1} ; S_{2}\right)}$ for all $S_{1} \subseteq X$ and $S_{2} \subseteq Y$. On the other hand, let $G^{\prime}$ be a bipartite-split graph, with the complete bipartite subset pair ( $S_{1}^{\prime} ; S_{2}^{\prime}$ ) and the independent subset pair ( $X \backslash S_{1}^{\prime} ; Y \backslash S_{2}^{\prime}$ ), obtained from $G$ by the addition or the removal of a minimum number of edges. Because of the minimality assumption, no removed edge could have had an end-vertex in $S_{1}^{\prime} \cup S_{2}^{\prime}$
and no added edge could have had an end-vertex in $\left(X \backslash S_{1}^{\prime}\right) \cup\left(Y \backslash S_{2}^{\prime}\right)$, implying that $\tau(G)=s_{\left(S_{1}^{\prime} ; S_{2}^{\prime}\right)}$. Therefore, $\tau(G)=\min _{S_{1} \subseteq X, S_{2} \subseteq Y} s_{\left(S_{1} ; S_{2}\right)}$.

Let $A=\left(a_{1}, \ldots, a_{m}\right)$ and $B=\left(b_{1}, \ldots, b_{n}\right)$ be two integer sequences with $\sum_{i=1}^{m} a_{i}=\sum_{i=1}^{n} b_{i}, n \geq a_{1} \geq \cdots \geq a_{m} \geq 0$ and $m \geq b_{1} \geq \cdots \geq b_{n} \geq 0$. If there is a simple bipartite graph $G$ such that $(A ; B)$ is the degree sequence pair of $G$, then the pair $(A ; B)$ is bigraphic. The following well-known result is the Gale-Ryser characterization of bigraphic pairs.
Theorem $2[2,4]$. The pair $(A ; B)$ is bigraphic if and only if $\sum_{i=1}^{k} a_{i} \leq k \ell+$ $\sum_{i=\ell+1}^{n} b_{i}$ for each $k=1, \ldots, m$ and $\ell=1, \ldots, n$.

By the symmetry, Theorem 2 can be stated that the pair $(A ; B)$ is bigraphic if and only if $\sum_{i=1}^{\ell} b_{i} \leq \ell k+\sum_{i=k+1}^{m} a_{i}$ for each $\ell=1, \ldots, n$ and $k=1, \ldots, m$. Therefore, we have the following.
Corollary 3. If the pair $(A ; B)$ is bigraphic, then $\sum_{i=1}^{k} a_{i}+\sum_{i=1}^{\ell} b_{i} \leq 2 k \ell+$ $\sum_{i=k+1}^{m} a_{i}+\sum_{i=\ell+1}^{n} b_{i}$ for each $k=1, \ldots, m$ and $\ell=1, \ldots, n$.

Definition. Let $(A ; B)$ be a bigraphic pair. Define

$$
\tau_{k, \ell}^{(A ; B)}=\frac{1}{2}\left(2 k \ell-\sum_{i=1}^{k} a_{i}-\sum_{i=1}^{\ell} b_{i}+\sum_{i=k+1}^{m} a_{i}+\sum_{i=\ell+1}^{n} b_{i}\right)
$$

for $1 \leq k \leq m$ and $1 \leq \ell \leq n, m_{\ell}=\max \left\{i \mid a_{i} \geq \ell\right\}$ for $1 \leq \ell \leq a_{1}$ and $n_{1}=\max \left\{i \mid b_{i} \geq 1\right\}$. If

$$
\min \left\{\tau_{m_{\ell}, \ell}^{(A ; B)} \mid \ell=1, \ldots, a_{1}\right\} \leq \tau_{1, n_{1}}^{(A ; B)}
$$

then we define $L \in\left\{1, \ldots, a_{1}\right\}$ so that $\tau_{m_{L}, L}^{(A ; B)}=\min \left\{\tau_{m_{\ell}, \ell}^{(A ; B)} \mid \ell=1, \ldots, a_{1}\right\}$. If

$$
\min \left\{\tau_{m_{\ell}, \ell}^{(A ; B)} \mid \ell=1, \ldots, a_{1}\right\}>\tau_{1, n_{1}}^{(A ; B)}
$$

then we define $L=n_{1}$ and $m_{L}=1$.
Lemma 4. If the pair $(A ; B)$ is bigraphic, then
(a) $\tau_{k, \ell}^{(A ; B)} \geq 0$ for each $k=1, \ldots, m$ and $\ell=1, \ldots, n$;
(b) for a given $\ell \in\left\{1, \ldots, a_{1}\right\}, \tau_{k, \ell}^{(A ; B)}$ attains its minimum value when $k=m_{\ell}$;
(c) for a given $\ell \in\left\{a_{1}+1, \ldots, n\right\}, \tau_{k, \ell}^{(A ; B)}$ attains its minimum value when $k=1$;
(d) $\min \left\{\tau_{k, \ell}^{(A ; B)} \mid k=1, \ldots, m\right.$ and $\left.\ell=1, \ldots, n\right\}=\min \left\{\min \left\{\tau_{m_{\ell}, \ell}^{(A ; B)} \mid \ell=1, \ldots\right.\right.$, $\left.\left.a_{1}\right\}, \tau_{1, n_{1}}^{(A ; B)}\right\}=\tau_{m_{L}, L}^{(A ; B)}$.

Proof. (a) is a consequence of Corollary 3. As for (b), a direct computation shows that, for $1 \leq k \leq m$, we have that $\tau_{k, \ell}^{(A ; B)}-\tau_{k-1, \ell}^{(A ; B)}=\ell-a_{k}$. It is easy to see that $\tau_{1, \ell}^{(A ; B)} \geq \tau_{2, \ell}^{(A ; B)} \geq \cdots \geq \tau_{m_{\ell}, \ell}^{(A ; B)}$ and $\tau_{m_{\ell}, \ell}^{(A ; B)} \leq \tau_{m_{\ell}+1, \ell}^{(A ; B)} \leq$ $\cdots \leq \tau_{m, \ell}^{(A ; B)}$. Hence $\tau_{k, \ell}^{(A ; B)}$ attains its minimum value when $k=m_{\ell}$. As for (c), we have that $\tau_{k, \ell}^{(A ; B)}-\tau_{k-1, \ell}^{(A ; B)}=\ell-a_{k}$ for $1 \leq k \leq m$, implying that $\tau_{1, \ell}^{(A ; B)} \leq \tau_{2, \ell}^{(A ; B)} \leq \cdots \leq \tau_{m, \ell}^{(A ; B)}$. Hence $\tau_{k, \ell}^{(A ; B)}$ attains its minimum value when $k=1$. As for (d), we have that $\tau_{1, \ell}^{(A ; B)}-\tau_{1, \ell-1}^{(A ; B)}=1-b_{\ell}$ for $a_{1}+1 \leq \ell \leq n$, implying that $\tau_{1, a_{1}+1}^{(A ; B)} \geq \cdots \geq \tau_{1, n_{1}}^{(A ; B)}$ and $\tau_{1, n_{1}}^{(A ; B)} \leq \cdots \leq \tau_{1, n}^{(A ; B)}$. Hence $\tau_{1, n_{1}}^{(A ; B)}=$ $\min \left\{\tau_{1, \ell}^{(A ; B)} \mid \ell=a_{1}+1, \ldots, n\right\}$. Thus by (b) and (c), $\min \left\{\tau_{k, \ell}^{(A ; B)} \mid k=1, \ldots, m\right.$ and $\ell=1, \ldots, n\}=\min \left\{\min \left\{\tau_{m_{\ell}, \ell}^{(A ; B)} \mid \ell=1, \ldots, a_{1}\right\}, \min \left\{\tau_{1, \ell}^{(A ; B)} \mid \ell=a_{1}+1\right.\right.$, $\ldots, n\}\}=\min \left\{\min \left\{\tau_{m_{\ell}, \ell}^{(A ; B)} \mid \ell=1, \ldots, a_{1}\right\}, \tau_{1, n_{1}}^{(A ; B)}\right\}=\tau_{m_{L}, L}^{(A ; B)}$.

We now give the main result of this paper as follows.
Theorem 5. If $G$ is a bipartite graph with two partite sets $X$ and $Y$ and $(A ; B)$ is the degree sequence pair of $G$, then $\tau(G)=\min \left\{\min \left\{\tau_{m_{\ell}, \ell}^{(A ; B)} \mid \ell=1, \ldots, a_{1}\right\}\right.$, $\left.\tau_{1, n_{1}}^{(A ; B)}\right\}=\tau_{m_{L}, L}^{(A ; B)}$.
Proof. Let $\left(S_{1} ; S_{2}\right)$ be any subset pair of $(X ; Y)$, where $|X|=m$ and $|Y|=n$. Then

$$
\sum_{x \in S_{1}} d_{G}(x)+\sum_{y \in S_{2}} d_{G}(y)=2\left|E\left(G\left[S_{1} \cup S_{2}\right]\right)\right|+e_{G}\left(S_{1} \cup S_{2}, V(G) \backslash\left(S_{1} \cup S_{2}\right)\right)
$$

where $e_{G}\left(S_{1} \cup S_{2}, V(G) \backslash\left(S_{1} \cup S_{2}\right)\right)$ denotes the number of edges in $G$ having one end-vertex in $S_{1} \cup S_{2}$ and the other end-vertex in $V(G) \backslash\left(S_{1} \cup S_{2}\right)$. Thus,

$$
\begin{aligned}
s_{\left(S_{1} ; S_{2}\right)} & =\left|S_{1}\right|\left|S_{2}\right|-\left|E\left(G\left[S_{1} \cup S_{2}\right]\right)\right|+\left|E\left(G\left[V(G) \backslash\left(S_{1} \cup S_{2}\right)\right]\right)\right| \\
& =\frac{1}{2}\left(2\left|S_{1}\right|\left|S_{2}\right|-2\left|E\left(G\left[S_{1} \cup S_{2}\right]\right)\right|+2\left|E\left(G\left[V(G) \backslash\left(S_{1} \cup S_{2}\right)\right]\right)\right|\right) \\
& =\frac{1}{2}\left(2\left|S_{1}\right|\left|S_{2}\right|-\sum_{x \in S_{1}} d_{G}(x)-\sum_{y \in S_{2}} d_{G}(y)+\sum_{x \in X \backslash S_{1}} d_{G}(x)+\sum_{y \in Y \backslash S_{2}} d_{G}(y)\right) .
\end{aligned}
$$

By putting $k=\left|S_{1}\right|$ and $\ell=\left|S_{2}\right|$, we have that $\sum_{x \in S_{1}} d_{G}(x) \leq \sum_{i=1}^{k} a_{i}$, $\sum_{y \in S_{2}} d_{G}(y) \leq \sum_{i=1}^{\ell} b_{i}, \quad \sum_{x \in X \backslash S_{1}} d_{G}(x) \geq \sum_{i=k+1}^{m} a_{i}$ and $\sum_{y \in Y \backslash S_{2}} d_{G}(y) \geq$ $\sum_{i=\ell+1}^{n} b_{i}$. It follows that

$$
s_{\left(S_{1} ; S_{2}\right)} \geq \frac{1}{2}\left(2 k \ell-\sum_{i=1}^{k} a_{i}-\sum_{i=1}^{\ell} b_{i}+\sum_{i=k+1}^{m} a_{i}+\sum_{i=\ell+1}^{n} b_{i}\right)=\tau_{k, \ell}^{(A ; B)} .
$$

We notice that if we take $S_{1}$ to be the set of vertices with degree $a_{1}, \ldots, a_{k}$ and $S_{2}$ to be the set of vertices with degree $b_{1}, \ldots, b_{\ell}$, then $s_{\left(S_{1} ; S_{2}\right)}=\tau_{k, \ell}^{(A ; B)}$. Therefore, by Lemmas 1 and 4, we have that $\tau(G)=\min _{S_{1} \subseteq X, S_{2} \subseteq Y} s_{\left(S_{1} ; S_{2}\right)}=$ $\min _{1 \leq k \leq m, 1 \leq \ell \leq n} \tau_{k, \ell}^{(A ; B)}=\min \left\{\min \left\{\tau_{m_{\ell}, \ell}^{(A ; B)} \mid \ell=1, \ldots, a_{1}\right\}, \tau_{1, n_{1}}^{(A ; B)}\right\}=\tau_{m_{L}, L}^{(A ; B)}$. The proof of Theorem 5 is completed.

Theorem 5 yields an easily computable formula for the bipartite-splittance of a bipartite graph. For example, for $1 \leq r \leq m$, let $G$ be an $r$-regular bipartite graph on $2 m$ vertices with two partite sets $X$ and $Y$, and let $(A ; B)$ be the degree sequence pair of $G$. Then $|X|=|Y|=m, a_{1}=\cdots=a_{m}=r$ and $b_{1}=\cdots=b_{m}=r$. It is easy to compute that $m_{\ell}=m$ for $1 \leq \ell \leq r$ and $n_{1}=m$, and so $\tau_{m_{\ell}, \ell}^{(A ; B)}=\frac{1}{2}(2 m \ell-m r-\ell r+(m-\ell) r)=(m-r) \ell$ for $1 \leq \ell \leq r$ and $\tau_{1, n_{1}}^{(A ; B)}=\frac{1}{2}(2 m-r-m r+(m-1) r)=m-r$. Thus, $\tau(G)=\min \left\{\min \left\{\tau_{m_{\ell} \ell}^{(A ; B)} \mid \ell=\right.\right.$ $\left.1, \ldots, r\}, \tau_{1, n_{1}}^{(A ; B)}\right\}=m-r$.

Let $G$ be a bipartite graph with two partite sets $X$ and $Y$, where $|X|=m$ and $|Y|=n$. The proof of Theorem 5 yields a simple procedure (see Algorithm 1 on next page) for obtaining a bipartite-split graph from $G$ with a minimum number of additions or removals of edges. Moreover, we can easily analyze the complexity of Algorithm 1 is $O(\max \{m \log m, n \log n, m n\})$.

By the fact that $G$ is bipartite-split if and only if $\tau(G)=0$, a simple characterization of the degree sequence pair of bipartite-split graphs is an immediate consequence of Theorem 5 .

Corollary 6. Let $(A ; B)$ be a bigraphic pair, and let $L$ and $m_{L}$ be defined as in Definition. Then $(A ; B)$ is the degree sequence pair of a bipartite-split graph $G$ if and only if $\tau_{m_{L}, L}^{(A ; B)}=0$, that is, $\sum_{i=1}^{m_{L}} a_{i}+\sum_{i=1}^{L} b_{i}=2 m_{L} L+\sum_{i=m_{L}+1}^{m} a_{i}+$ $\sum_{i=L+1}^{n} b_{i}$.

The following Corollary 7 is an immediate consequence of Corollary 6 .
Corollary 7. If a bipartite graph $G$ is bipartite-split, then every bipartite graph with the same degree sequence pair as $G$ is also bipartite-split.

Remark 8. The problem in this paper can directly be considered in general graphs and is clearly hard in general graphs (for instance using a reduction from minimum edge removing to make the graph bipartite). Tighter hardness results in super classes of bipartite graphs would provide a nice motivation of the explicit formula in the bipartite case.

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Algorithm 1:
    Input: Bipartite graph \(G\);
    Output: Bipartite-split graph from \(G\);
    1 Let two partite sets of \(G\) be \(X\) and \(Y ; m=\) number of vertices in \(X ; n=\)
        number of vertices in \(Y\);
    2 Determine the degree sequence pair \((A ; B)\) of \(G\) so that \(A=\left(a_{1}, \ldots, a_{m}\right)\)
        (respectively, \(\left.B=\left(b_{1}, \ldots, b_{n}\right)\right)\) is the non-increasing sequence of vertex
        degrees for \(X\) (respectively, \(Y\) );
    3 Index the vertices of \(G\) so that \(X=\left\{x_{1}, \ldots, x_{m}\right\}\) with \(d_{G}\left(x_{i}\right)=a_{i}\), for
        \(1 \leq i \leq m\) and \(Y=\left\{y_{1}, \ldots, y_{n}\right\}\) with \(d_{G}\left(y_{j}\right)=b_{j}\), for \(1 \leq j \leq n ;\)
    \(4 m_{\ell}=\max \left\{i \mid a_{i} \geq \ell\right\}\), for \(1 \leq \ell \leq a_{1} ; n_{1}=\max \left\{i \mid b_{i} \geq 1\right\}\);
    \({ }_{5} \tau_{m_{\ell}, \ell}^{(A ; B)}=\frac{1}{2}\left(2 m_{\ell} \ell-\sum_{i=1}^{m_{\ell}} a_{i}-\sum_{i=1}^{\ell} b_{i}+\sum_{i=m_{\ell}+1}^{m} a_{i}+\sum_{i=\ell+1}^{n} b_{i}\right)\), for
        \(1 \leq \ell \leq a_{1} ; \tau_{1, n_{1}}^{(A ; B)}=\frac{1}{2}\left(2 n_{1}-a_{1}-\sum_{i=1}^{n_{1}} b_{i}+\sum_{i=2}^{m} a_{i}+\sum_{i=n_{1}+1}^{n} b_{i}\right) ;\)
    6 \(\tau_{m_{r}, r}^{(A ; B)}=\min \left\{\tau_{m_{1}, 1}^{(A ; B)}, \ldots, \tau_{m_{a_{1}}, a_{1}}^{(A ; B)}\right\}\);
    if \(\tau_{m_{r}, r}^{(A ; B)} \leq \tau_{1, n_{1}}^{(A ; B)}\) then
        \(L=r, m_{L}=m_{r} ;\)
    else
        \(L=n_{1}, m_{L}=1 ;\)
    for \(i=1, \ldots, m_{L}\) and \(j=1, \ldots, L\) do
        Add edges to \(E(G)\) so that \(x_{i}\) and \(y_{j}\) are adjacent;
    for \(i=m_{L}+1, \ldots, m\) and \(j=L+1, \ldots, n\) do
        Remove edges from \(E(G)\) so that \(x_{i}\) and \(y_{j}\) are not adjacent.
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