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ETERNAL m-SECURITY BONDAGE NUMBERS IN GRAPHS

Hamideh Aram

Department of Mathematics
Gareziaeddin Center, Khoy Branch
Islamic Azad University, Khoy, Iran

e-mail: hamideh.aram@gmail.com

Maryam Atapour

Department of Mathematics
Faculty of Basic Sciences
University of Bonab, Bonab, I.R. Iran

e-mail: m.atapour@bonabu.ac.ir

and

Seyed Mahmoud Sheikholeslami

Department of Mathematics
Azarbaijan Shahid Madani University
Tabriz, I.R. Iran

e-mail: s.m.sheikholeslami@azaruniv.edu

Abstract

An eternal m-secure set of a graph $G = (V, E)$ is a set $S_0 \subseteq V$ that can defend against any sequence of single-vertex attacks by means of multiple guard shifts along the edges of G. The eternal m-security number $\sigma_m(G)$ is the minimum cardinality of an eternal m-secure set in G. The eternal m-security bondage number $b_{\sigma_m}(G)$ of a graph G is the minimum cardinality of a set of edges of G whose removal from G increases the eternal m-security number of G. In this paper, we study properties of the eternal m-security bondage number. In particular, we present some upper bounds on the eternal m-security bondage number in terms of eternal m-security number and edge connectivity number, and we show that the eternal m-security bondage number of trees is at most 2 and we classify all trees attaining this bound.

Keywords: eternal m-secure set, eternal m-security number, eternal m-security bondage number.

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1. Introduction

Throughout this paper, G is a simple connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$ and of order n and size m. For every vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* $\deg(v)$ of v is the number of edges incident with v or, equivalently, $\deg(v) = |N(v)|$. The degree sequence of G is $(\deg(v_1), \deg(v_2), \ldots, \deg(v_n))$, typically written in nondecreasing order. The minimum and maximum degree of vertices in $V(G)$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Let $E(A, B)$ denote the set of all edges with one endpoint in A and the other endpoint in B, $e(A, B)$ be the cardinality of $E(A, B)$, and E_u denote the set of edges incident to u. A *leaf* of a graph G is a vertex of degree 1 and a *support vertex* of G is a vertex adjacent to a leaf. A support vertex is called *strong support vertex* if it is adjacent to at least two leaves. The distance between two vertices x and y is denoted by $d(x, y)$ and the diameter of G is denoted by $\mathrm{diam}(G)$.

A set S of vertices in a graph G is called a *dominating set* if every vertex in V is either an element of S or is adjacent to an element of S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A $\gamma(G)$-*set* is a dominating set of G of size $\gamma(G)$. For a more thorough treatment of domination parameters and for terminology not presented here see [5,11]. The bondage number $b(G)$ of a graph G is the minimum cardinality of a set of edges of G whose removal from G increases the domination number of G. The bondage number was introduced by Fink *et al.* [2] and was studied by several authors, for example [4,6,8–10]. For more information on this topic we refer the reader to the survey article by Xu [12].

An *eternal* 1-*secure* set of a graph G is a set $S_0 \subseteq V$ that can defend against any sequence of single-vertex attacks by means of single-guard shifts along the edges of G. That is, for any k and any sequence $v_1, v_2, \ldots, v_k$ of vertices, there exists a sequence of guards $u_1, u_2, \ldots, u_k$ with $u_i \in S_{i-1}$ and either $u_i = v_i$ or $u_i v_i \in E$, such that each set $S_i = (S_{i-1} - \{u_i\}) \cup \{v_i\}$ is a dominating set. It follows that each S_i can be chosen to be an eternal 1-secure set. The *eternal* 1-*security number* of G, denoted by $\sigma_1(G)$, is the minimum cardinality of an eternal 1-secure set. The eternal 1-security number was introduced by Burger *et al.* [1] using the notation γ_∞. In order to reduce the number of guards needed in an eternal secure set, Goddard *et al.* [3] considered allowing more guards to move. Suppose that in responding to each attack, every guard may shift along an incident edge. The *eternal m-security number* $\sigma_m(G)$ is the minimum number of guards to handle an arbitrary sequence of single attacks using multiple guard shifts. A suitable placement of the guards is called an *eternal m-secure set* (EmSS). An EmSS of size $\sigma_m(G)$ is called a $\sigma_m(G)$-*set.*

The *eternal m-security bondage number* $b_{\sigma_m}(G)$ of a graph G is the minimum cardinality of a set of edges of G whose removal from G increases the eternal m-security number of G. Since in the study of eternal m-security bondage number the assumption $\sigma_m(G) < n$ is necessary, we always assume that when we discuss $b_{\sigma_m}(G)$, all graphs involved satisfy $\sigma_m(G) < n$, i.e., all graphs are nonempty. An edge set B with $\sigma_m(G - B) > \sigma_m(G)$ is called the *eternal m-secure bondage set.* A $b_{\sigma_m}(G)$*-set* is an eternal m-secure bondage set of G of size $b_{\sigma_m}(G)$.

In this paper, we initiate the study of the eternal m-security bondage number in graphs and we establish some bounds on the eternal m-security bondage number in terms of vertex degree, eternal m-security number and edge connectivity number. We also show that the eternal m-security bondage number of trees is at most 2 and we characterize all trees attaining this bound.

2. Preliminaries and Exact Values

The proof of the following four results can be found in [3].

Proposition A. *For any graph G, $\gamma(G) \leq \sigma_m(G)$.*

A set $P \subseteq V(G)$ is called a *k-packing* if $d(u, v) > k$ for each pair of vertices $u, v \in P$, $u \neq v$. The *k-packing number* $\alpha_k(G)$ is the cardinality of a maximum k-packing in G. Note that $\alpha_1(G) = \alpha(G)$ is the independence number of G.

Proposition B. *For any graph G, $\sigma_m(G) \leq \alpha(G)$.*

Proposition C. 1. $\sigma_m(K_n) = 1$.

2. $\sigma_m(K_{r,s}) = 2$ *for* $r, s \geq 1, r + s \geq 3$.
3. $\sigma_m(P_n) = \lceil \frac{n}{2} \rceil$.
4. $\sigma_m(C_n) = \lceil \frac{n}{3} \rceil$.

Proposition D. *For any graph G, $\sigma_m(G) \geq (\mathrm{diam}(G) + 1)/2$.*

Next results are immediate consequences of Propositions C and D.

Corollary 1. *For any graph G, $\sigma_m(G) = 1$ if and only if $G \simeq K_n$.*

Corollary 2. *For $n \geq 2$, we have $b_{\sigma_m}(K_n) = 1$.*

Corollary 3. *For $n \geq 5$, $b_{\sigma_m}(C_n) = 1$.*

Corollary 4. *For $n \geq 3$,* $b_{\sigma_m}(P_n) = \begin{cases} 1 & if\ n\ is\ even, \\ 2 & if\ n\ is\ odd. \end{cases}$

Proposition E [7]**.** *For any graph G, $\alpha_2(G) \leq \gamma(G)$.*

Corollary 5. *For any graph G, $\alpha_2(G) \le \sigma_m(G)$.*

Observation 6. *Let G be a graph and H be a spanning subgraph of G such that $\sigma_m(H) = \sigma_m(G)$. If $K = E(G) \setminus E(H)$, then $b_{\sigma_m}(H) \le b_{\sigma_m}(G) \le b_{\sigma_m}(H) + |K|$.*

Proof. Let F be a $b_{\sigma_m}(H)$-set. Then $\sigma_m(G) = \sigma_m(H) < \sigma_m(H - F) = \sigma_m(G - (K \cup F))$, which implies that

$$b_{\sigma_m}(G) \le |K \cup F| = |K| + |F| = b_{\sigma_m}(H) + |K|.$$

Let now T be a $b_{\sigma_m}(G)$-set. Then we have $\sigma_m(H) = \sigma_m(G) < \sigma_m(G - T) \le \sigma_m(H - T)$. Thus $b_{\sigma_m}(H) \le |T| = b_{\sigma_m}(G)$ and the proof is complete. ■

Proposition 7. *If G contains a vertex adjacent to at least three leaves, then $b_{\sigma_m}(G) = 1$.*

Proof. Let u be adjacent to the leaves u_1, u_2, u_3. Consider the graph G' obtained from G by deleting the edge uu_1. Let S be a $\sigma_m(G')$-set which contains u (we may assume that S is a response to an attack on u). Obviously $u_1 \in S$ and $S \setminus \{u_1\}$ is an EmSS of G and so $\sigma_m(G) \le \sigma_m(G') - 1$. Hence, $b_{\sigma_m}(G) = 1$. ■

Next we determine the eternal m-security bondage number of complete bipartite graphs.

Proposition 8. *For $m \ge n \ge 2$, $b_{\sigma_m}(K_{m,n}) = 2$.*

Proof. By Proposition C, $\sigma_m(K_{m,n}) = 2$. If $m = n = 2$, then clearly $b_{\sigma_m}(K_{2,2}) = 2$. Assume that $m \ge 3$. It is not hard to see that for any edge $e = uv \in E(G)$, the set $S = \{u, v\}$ is an EmSS of $K_{m,n} - e$ and so $b_{\sigma_m}(K_{m,n}) \ge 2$.

Now we show that $b_{\sigma_m}(K_{m,n}) \le 2$. Suppose that $X = \{u_1, \ldots, u_m\}$ and $Y = \{v_1, \ldots, v_n\}$ be the partite sets of $K_{m,n}$ and let $F = \{v_1u_1, v_1u_2\}$. Let S be a $\sigma_m(K_{m,n} - F)$-set which contains u_1. To dominate v_1, we have $v_1 \in S$ or $u_i \in S$ for some $i \ge 3$. If $v_1 \in S$, then u_2 is not dominated by $\{u_1, v_1\}$ and so $|S| \ge 3$. Let $v_1 \notin S$. Assume without loss of generality that $u_3 \in S$. Then u_2 is not dominated by $\{u_1, u_3\}$ and this implies that $|S| \ge 3$. Hence, $b_{\sigma_m}(K_{m,n}) \le 2$ and the proof is complete. ■

3. Bounds on the Eternal m-Security Bondage Number

In this section, we present various bounds on the eternal m-security bondage number. We start with an observation.

Observation 9. *Let G be a connected graph. If $\sigma_m(G - v) \ge \sigma_m(G)$ for some vertex $v \in V(G)$, then $b_{\sigma_m}(G) \le \deg(v)$.*

Proof. First, note that $\sigma_m(G-E_v) \geq \sigma_m(G)$. If $\sigma_m(G-E_v) > \sigma_m(G)$, then we are done. Suppose $\sigma_m(G-E_v) = \sigma_m(G)$ and let S be a $\sigma_m(G-E_v)$-set. Clearly, $v \in S$ and $S \setminus \{v\}$ is an EmSS of $G-v$. It follows that

$$\sigma_m(G-E_v) - 1 \geq \sigma_m(G-v) \geq \sigma_m(G),$$

and the proof is complete. ■

Theorem 10. *Let G be a connected graph and $uv \in E(G)$. Then*

$$b_{\sigma_m}(G) \leq \deg(u) + \deg(v) - 1 - |N(u) \cap N(v)|.$$

Proof. Let X be the set consisting of all edges incident with u and v with exception of the edges $E(v, N(u))$. Then $|X| = \deg(u) + \deg(v) - 1 - |N(u) \cap N(v)|$, u is an isolated vertex in $G - X$ and v is only adjacent to the vertices of $N_G(u) \cap N_G(v)$. Let S be a $\sigma_m(G-X)$-set which contains v (we may assume a response to an attack on v). It is easy to verify that $S \setminus \{u\}$ is an EmSS of G and hence $\sigma_m(G) \leq \sigma_m(G-X) - 1$. This completes the proof. ■

Corollary 11. *For any nonempty graph G, $b_{\sigma_m}(G) \leq \delta(G) + \Delta(G) - 1$.*

Theorem 12. *Let G be a connected graph with degree sequence $(d_1, d_2, \ldots, d_n)$. Then*

$$b_{\sigma_m}(G) \leq d_\alpha + d_{\alpha+1} - 1,$$

where α is the independence number of G.

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $\deg(v_i) = d_i$ for each i. Since the set $\{v_1, \ldots, v_{\alpha+1}\}$ is not independent, there is an edge v_iv_j for some $1 \leq i < j \leq \alpha+1$. It follows from Theorem 10 that

$$b_{\sigma_m}(G) \leq \deg(v_i) + \deg(v_j) - 1 \leq \deg(v_\alpha) + \deg(v_{\alpha+1}) - 1,$$

and the proof is complete. ■

Theorem 13. *Let G be a connected graph and u, v be two vertices of G with $d(u,v) = 2$. Then*

$$b_{\sigma_m}(G) \leq \deg(u) + \deg(v).$$

Proof. Let w be a common neighbor of u and v and let X be the set consisting of all edges incident with u and v. Then $|X| = \deg(u) + \deg(v)$ and u, v are isolated vertices in $G - X$. Let S be a $\sigma_m(G-X)$-set which contains w (we may assume a response to an attack on w). Obviously $u, v \in S$ and we can easily check that $S \setminus \{u\}$ is an EmSS of G and so $\sigma_m(G) < \sigma_m(G - |X|)$. Thus $b_{\sigma_m}(G) \leq |X| = \deg(u) + \deg(v)$ as desired. ■

Corollary 14. *Let G be a connected graph of order n with degree sequence* $(\deg(v_1), \deg(v_2), \ldots, \deg(v_n))$. *Then* $b_{\sigma_m}(G) \leq \deg(v_{\alpha_2}) + \deg(v_{\alpha_2+1})$.

Proof. Clearly, the set $\{v_1, \ldots, v_{\alpha_2+1}\}$ is not a 2-packing. Hence, $d(v_i, v_j) \leq 2$ for some $1 \leq i \neq j \leq \alpha_2 + 1$ and the result follows by Theorems 10 and 13. ■

Next result is an immediate consequence of Corollaries 5 and 14.

Corollary 15. *If G is a connected graph with degree sequence* $(\deg(v_1), \deg(v_2), \ldots, \deg(v_n))$, *then* $b_{\sigma_m}(G) \leq \deg(v_{\sigma_m}) + d(v_{\sigma_m+1})$.

Theorem 16. *For any connected graph G,* $b_{\sigma_m}(G) \leq (\sigma_m(G) - \alpha_2(G) + 1)\Delta(G)$.

Proof. By Corollary 5, $\alpha_2(G) \leq \sigma_m(G)$. Let $s = \sigma_m(G) - \alpha_2(G) + 1$ and $U = \{u_1, \ldots, u_{\alpha_2}\}$ be a maximum 2-packing in G. Clearly, $U \neq V(G)$. Let T be a subset of $V(G) - U$ of size s and let G' be the graph obtained from G by removing all edges incident to the vertices in T. Obviously, $|E(G)| - |E(G')| \leq \Delta s$. Now we have

$$\sigma_m(G') \geq \alpha_2(G') \geq \alpha_2(G) + s = \alpha_2(G) + \sigma_m(G) - \alpha_2(G) + 1 = \sigma_m(G) + 1 > \sigma_m(G)$$

and the proof is complete. ■

The next result is an immediate consequence of Theorem 16.

Corollary 17. *If* $\sigma_m(G) = \alpha_2(G)$, *then* $b_{\sigma_m}(G) \leq \Delta(G)$.

The edge connectivity number $\kappa'(G)$ of a connected graph G is the minimum number of edges that have to be removed out of G to decompose G in two components. The inequality $\kappa'(G) \leq \delta(G)$ is immediate. Next result is an improvement of Corollary 11.

Theorem 18. *If G is a nontrivial connected graph, then*

$$b_{\sigma_m}(G) \leq \Delta(G) + \kappa'(G) - 1.$$

Proof. Let K be a set of edges such that $\kappa'(G) = |K|$ and $G - K$ is disconnected. Assume that G_1 and G_2 are the components of $G - K$. It is easy to see that $\sigma_m(G) \leq \sigma_m(G_1) + \sigma_m(G_2) = \sigma_m(G - K)$. If $\sigma_m(G) < \sigma_m(G_1) + \sigma_m(G_2)$, then $b_{\sigma_m}(G) \leq \kappa'(G)$ and we are done. Let $\sigma_m(G) = \sigma_m(G_1) + \sigma_m(G_2)$. We claim that there is a vertex $v \in V(G_i)$ such that v is incident to an edge of K and $\sigma_m(G_i - E_v) > \sigma_m(G_i)$ for some i. In this case we have $\sigma_m(G - K - E_v) > \sigma_m(G)$, which implies that

$$b_{\sigma_m}(G) \leq \deg(v) + \kappa'(G) - 1 \leq \Delta(G) + \kappa'(G) - 1.$$

Assume, to the contrary, that $\sigma_m(G_1 - E_v) = \sigma_m(G_1)$ for each vertex $v \in V(G_1)$ incident to an edge of K and $\sigma_m(G_2 - E_v) = \sigma_m(G_2)$ for every vertex $v \in V(G_2)$ incident to an edge of K. Let $u_1u_2 \in K$ where $u_i \in V(G_i)$ for $i = 1, 2$. Let S_i be a $\sigma_m(G_i - E_{u_i})$-set for $i = 1, 2$. Clearly, $u_1 \in S_1$ and $u_2 \in S_2$. It is easy to verify that $S = S_1 \cup S_2 \setminus \{u_1\}$ is an eternal m-secure set of G which implies that

$$\sigma_m(G) \leq \sigma_m(G_1 - E_u) + \sigma_m(G_2 - E_v) - 1 = \sigma_m(G_1) + \sigma_m(G_2) - 1 = \sigma_m(G) - 1,$$

a contradiction. This completes the proof. ■

Proposition 19. *If $\sigma_m(G) = 2$, then $b_{\sigma_m}(G) \leq \delta(G) + 1$.*

Proof. Let $u \in V(G)$ be a vertex of minimum degree. If $\sigma_m(G - u) \geq \sigma_m(G)$, then the result follows by Observation 9. Let $\sigma_m(G - u) \leq \sigma_m(G) - 1$. Then obviously $\sigma_m(G - u) = 1$ and so $G - u$ is a complete graph. By Corollary 2, we have $b_{\sigma_m}(G) \leq b_{\sigma_m}(G - u) + \delta(G) = \delta(G) + 1$. ■

4. Complete Multipartite Graphs

In this section we determine the eternal m-security bondage number of complete multipartite graphs yielding that the eternal m-security bondage number can be arbitrary large.

Theorem 20. *Let $t \geq 3$ and $G = K_{n_1,n_2,\ldots,n_t}$ be the complete t-partite graph with $n_1, n_2, \ldots, n_t \geq 2$. Then $b_{\sigma_m}(G) = \left\lceil \frac{3(t-1)}{2} \right\rceil$.*

Proof. Let $X_1, X_2, \ldots, X_t$ be the partite sets of G and let $X_i = \left\{x_1^i, \ldots, x_{n_i}^i\right\}$ for $1 \leq i \leq t$. Clearly $\sigma_m(G) = 2$. Assume

$$X = \left\{x_1^1 x_1^j, x_1^{2s} x_1^{2s+1} : 2 \leq j \leq t,\ 1 \leq s \leq \frac{t-1}{2}\right\}$$

if t is odd and

$$X = \left\{x_1^1 x_1^j, x_1^{2s} x_1^{2s+1}, x_1^1 x_2^t : 2 \leq j \leq t,\ 1 \leq s \leq \frac{t-2}{2}\right\}$$

when t is even. Obviously, $|X| = \left\lceil \frac{3(t-1)}{2} \right\rceil$. It is easy to see that for any eternal m-secure set S of $G - X$ containing x_1^1, we have $|S| \geq 3$ and so $b_{\sigma_m}(G) \leq \left\lceil \frac{3(t-1)}{2} \right\rceil$.

Now we show that $b_{\sigma_m}(G) \geq \left\lceil \frac{3(t-1)}{2} \right\rceil$. Let F be a set of edges of size at most $\left\lceil \frac{3(t-1)}{2} \right\rceil - 1$ and let $G_2 = G - F$.

Claim. *For each $x \in V(G_2)$, there exists a vertex x' such that $N_{G_2}[x] \cup N_{G_2}[x'] = V(G_2)$.*

Proof. Assume, to the contrary, that there exists a vertex $x \in V(G_2)$ such that $N_{G_2}[x] \cup N_{G_2}[v] \neq V(G_2)$ for each $v \in V(G_2)$. Without loss of generality we may assume that $x = x_1^1$. For $2 \le i \le t$, let $F_i = F \cap \{x_1^1 x_j^i : 1 \le j \le n_i\}$. Let first $F_i = \emptyset$ for each $2 \le i \le t$. Since $N_{G_2}[x_1^1] \cup N_{G_2}[v] \neq V(G_2)$ for each $v \in V(G_2)$, we have $x_j^1 v \in F$ for every $v \in V(G_2) \setminus X_1$ and for some $2 \le j \le n_1$. This implies that $|F| \ge |V(G_2) \setminus X_1| \ge 2t - 2$, a contradiction. Assume that $F_i \neq \emptyset$ for some $2 \le i \le t$. We consider two cases.

Case 1. $|F_i| \le 1$ for each $2 \le i \le t$. Let $I \subseteq \{2, \ldots, t\}$ be the set of all elements such that $|F_i| = 1$ for each $i \in I$ and let $J = \{2, \ldots, t\} \setminus I$. Without loss of generality, assume that $\{x_1^1 x_1^i : i \in I\} \subseteq F$. We estimate the number of edges in F as follows. Since $N_{G_2}[x_1^1] \cup N_{G_2}[x_1^i] \neq V(G_2)$ for $i \in I$, there exists a vertex z^i such that $z^i x_1^i, z^i x_1^1 \notin E(G_2)$. Obviously, $z^i \notin X_i \cup \left(\bigcup_{j \in J} X_j\right)$. If $z^i \in X_1$, then $E_i = \{x_1^1 x_1^i, x_1^i z^i\} \subseteq F$, and if $z^i \in X_\ell$ for some $\ell \in I - \{i\}$, then $z^i = x_1^\ell$ and $E_i = \{x_1^1 x_1^i, x_1^i x_1^\ell, x_1^1 x_1^\ell\} \subseteq F$. Since $N_{G_2}[x_1^1] \cup N_{G_2}[x_s^j] \neq V(G_2)$ for $j \in J$ and $1 \le s \le n_j$, there exists a vertex z_s^j such that $z_s^j x_s^j, z_s^j x_1^1 \notin E(G_2)$. We note that

$$z_s^j \in X_1 \cup \left(\bigcup_{i \in I} X_i\right) \tag{1}$$

for $j \in J$ and $1 \le s \le n_j$. If $z_s^j \in X_1$, then $x_s^j z_s^j \in F \setminus \left(\bigcup_{i \in I} E_i\right)$, and if $z_s^j \in X_i$ for some $i \in I$, then $z_s^j = x_1^i$ and $x_s^j z_s^j \in F \setminus \left(\bigcup_{i \in I} E_i\right)$ again. Since $n_j \ge 2$, we conclude that $\left|F \cap \{z_s^j x_s^j : 1 \le s \le n_j\}\right| \ge 2$ for each $j \in J$. By (1) we have $\{z_s^j x_s^j : 1 \le s \le n_j\} \cap \{z_s^{j'} x_s^{j'} : 1 \le s \le n_{j'}\} = \emptyset$ for $j \neq j'$. Hence, we have

$$\begin{aligned} |F| &\ge \left|\bigcup_{i \in I} E_i\right| + \left|\bigcup_{j \in J} \left(F \cap \{z_s^j x_s^j : 1 \le s \le n_j\}\right)\right| \\ &\ge \tfrac{3|I|}{2} + 2|J| \ge \left\lceil \tfrac{3}{2}(|I| + |J|) \right\rceil = \left\lceil \tfrac{3(t-1)}{2} \right\rceil, \end{aligned}$$

which is a contradiction.

Case 2. $|F_i| \ge 2$ for some $2 \le i \le t$. Let $I \subseteq \{2, \ldots, t\}$ be the set of all elements i such that $|F_i| \ge 2$, $J \subseteq \{2, \ldots, t\}$ be the set of all elements j such that $|F_j| = 1$ and $R = \{2, \ldots, t\} \setminus (I \cup J)$. Without loss of generality, assume that $\{x_1^1 x_1^i, x_1^1 x_2^i, x_1^1 x_1^j : i \in I, j \in J\} \subseteq F$. We estimate the number of edges in F as follows. Obviously, $\left|\bigcup_{i \in I} F_i\right| \ge 2|I|$. Since $N_{G_2}[x_1^1] \cup N_{G_2}[x_1^j] \neq V(G_2)$ for each $j \in J$, there exists a vertex z^j such that $z^j x_1^j, z^j x_1^1 \notin E(G_2)$. Obviously, $z^j \notin X_j \cup \left(\bigcup_{r \in R} X_r\right)$. If $z^j \in X_i$, for some $i \in I$, then $E_j = \{x_1^1 x_1^j, x_1^j z^j\} \subseteq F$, and if $z^j \in X_\ell$ for some $\ell \in J - \{j\}$, then $z^j = x_1^\ell$ and $E_j = \{x_1^1 x_1^j, x_1^j x_1^\ell, x_1^1 x_1^\ell\} \subseteq F$.

As in Case 1, we can see that $\left|F \cap \left\{z_s^r x_s^r : 1 \leq s \leq n_r\right\}\right| \geq 2$ for each $r \in R$, and $\left\{z_s^r x_s^r : 1 \leq s \leq n_r\right\} \cap \left\{z_s^{r'} x_s^{r'} : 1 \leq s \leq n_{r'}\right\} = \emptyset$ for $r \neq r'$.

Hence, we have

$$\begin{aligned} |F| &\geq \left|\bigcup_{i\in I} F_i\right| + \left|\bigcup_{j\in J} E_j\right| + \left|\bigcup_{j\in J}\left(F \cap \left\{z_s^j x_s^j :\ 1 \leq s \leq n_j\right\}\right)\right| \\ &\geq 2|I| + \tfrac{3|J|}{2} + 2|R| \geq \left\lceil \tfrac{3}{2}(|I| + |J| + |R|)\right\rceil = \left\lceil \tfrac{3(t-1)}{2} \right\rceil, \end{aligned}$$

which is a contradiction. □

Now, for each $v \in V(G_2)$, let $x_v \in V(G_2)$ be a vertex such that $N_{G_2}[v] \cup N_{G_2}[x_v] = V(G_2)$. We show that the set $S_v = \{v, x_v\}$ is an EmSS of G_2. Obviously, S_v is a dominating set of G_2. Consider an attack on a vertex u of $V(G_2)$. Then one of v or x_v is adjacent to u. Let $uv \in E(G_2)$. If x_u is adjacent to x_v, then we can shift guards from v and x_v to u and x_u, respectively. Let $x_u x_v \notin E(G_2)$. Then $x_u v, u x_v \in E(G_2)$ and we can shift guards from v and x_v to x_u and u, respectively. Therefore, $\sigma_m(G_2) = 2$ and this implies that $b_{\sigma_m}(G) \geq \left\lceil \frac{3(t-1)}{2} \right\rceil$. Thus $b_{\sigma_m}(G) = \left\lceil \frac{3(t-1)}{2} \right\rceil$ and the proof is complete. ■

5. Trees

In this section, we first prove that for any nontrivial tree T, $b_{\sigma_m}(T) \leq 2$ and then we characterize all trees attaining this bound.

Theorem 21. *For any tree T of order $n \geq 2$, $b_{\sigma_m}(T) \leq 2$.*

Proof. If $\operatorname{diam}(T) \leq 2$, then T is a star and the result is immediate. Let $\operatorname{diam}(T) \geq 3$. Suppose $P := v_1 v_2 \cdots v_k$ is a diametral path in T and root T at v_k. Obviously, $k \geq 4$. If $\deg(v_2) = 2$, then $b_{\sigma_m}(T) \leq 2$ by Theorem 10. Let $\deg(v_2) \geq 3$. Then v_2 is adjacent to a leaf v' other than v_1. Since $d(v_1, v') = 2$, Theorem 13 implies that $b_{\sigma_m}(T) \leq 2$. This completes the proof. ■

Next, we provide a constructive characterization of all trees attaining the bound of Theorem 21. For this purpose, we describe a procedure to build a family $\mathfrak{T}$ of trees as follows. Let $\mathfrak{T}$ be the family of trees such that a path P_3 is a tree in $\mathfrak{T}$ and if T is a tree in $\mathfrak{T}$, then the tree T' obtained from T by the following four operations which extend the tree T by attaching a tree to a vertex $v \in V(T)$, called an *attacher*, is also a tree in $\mathfrak{T}$ (see Figure 1).

Operation $\mathfrak{T}_1$. If $v \in V(T)$, then $\mathfrak{T}_1$ adds a path vxy to T.

Operation $\mathfrak{T}_2$. If $v \in V(T)$, then $\mathfrak{T}_2$ adds a star $K_{1,3}$ with central vertex y and leaves x, w, z and joins x to v.

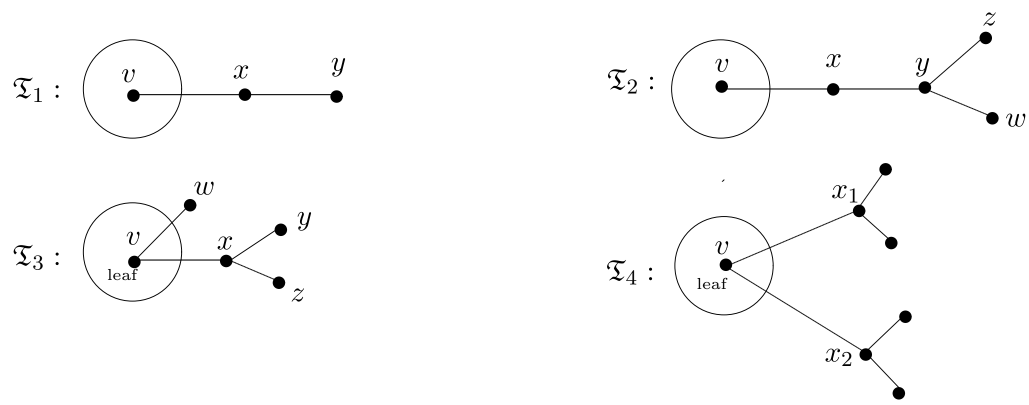

Figure 1. The four operations.

Operation $\mathfrak{T}_3$. If $v \in V(T)$ is a leaf, then $\mathfrak{T}_3$ adds a pendant edge vw and a star $K_{1,2}$ with central vertex x and leaves y, z and joins x to v.

Operation $\mathfrak{T}_4$. If $v \in V(T)$ is a leaf, then $\mathfrak{T}_4$ adds two new stars $K_{1,2}$ centered at x_1 and x_2, and joins v to x_1 and x_2.

We start with some lemmas.

Lemma 22. Let G be a graph and $v \in V(G)$. If G' is the graph obtained from G by attaching a path vxy, then $\sigma_m(G') = \sigma_m(G) + 1$. In particular, $b_{\sigma_m}(G') \leq b_{\sigma_m}(G)$.

Proof. Clearly, adding x to any $\sigma_m(G)$-set yields an EmSS of G' and so $\sigma_m(G') \leq \sigma_m(G)+1$. Let now S' be a $\sigma_m(G')$-set containing y (we may assume a response to an attack on y). If $x \in S'$, then the set $(S' \setminus \{x, y\}) \cup \{w\}$, where $w \in N_G[v] \setminus S'$, is an E$m$SS of G. If $x \notin S'$, then $S' \setminus \{y\}$ is an EmSS of G. Thus $\sigma_m(G) \leq \sigma_m(G')-1$ and so $\sigma_m(G') = \sigma_m(G) + 1$. ∎

Lemma 23. Let G be a graph and $v \in V(G)$. If G' is the graph obtained from G by adding a star $K_{1,3}$ with central vertex y and leaves x, w, z and joining x to v, then $\sigma_m(G') = \sigma_m(G) + 2$. In particular, $b_{\sigma_m}(G') \leq b_{\sigma_m}(G)$.

Proof. Clearly, adding x and y to any $\sigma_m(G)$-set yields an EmSS of G' and so $\sigma_m(G') \leq \sigma_m(G) + 2$. Suppose now S' is a $\sigma_m(G')$-set containing z (we may assume a response to an attack on z). Since S' is a dominating set, we must have $|S' \cap \{y, w\}| \geq 1$. If $x \in S'$ then the set $(S' \setminus \{x, y, z, w\}) \cup \{u\}$, where $u \in N_G[v] \setminus S'$ is an EmSS of G, and if $x \notin S'$ then the set $S' \setminus \{x, y, z, w\}$ is an EmSS of G. Hence $\sigma_m(G) \leq \sigma_m(G') - 2$ and this implies that $\sigma_m(G') = \sigma_m(G) + 2$. ∎

Lemma 24. Let G be a graph and let $v \in V(G)$. If G' is the graph obtained from G by adding a pendant edge vw and a star $K_{1,2}$ with central vertex x

and leaves y, z and joining x to v, then $\sigma_m(G') = \sigma_m(G) + 2$. In particular, $b_{\sigma_m}(G') \leq b_{\sigma_m}(G)$.

Proof. Clearly, adding x, y to any $\sigma_m(G)$-set containing v yields an EmSS of G' and so $\sigma_m(G') \leq \sigma_m(G) + 2$. Assume now that S' is a $\sigma_m(G')$-set. As in the proof of Lemma 23, we may assume that $y \in S'$ and $|S' \cap \{x, z\}| \geq 1$. Since S' is a dominating set, we must have $|S' \cap \{v, w\}| \geq 1$. If $|S' \cap \{x, y, z, w\}| \geq 3$, then let $S'' = (S' \setminus \{x, y, z, w\}) \cup \{u\}$ where $u \in N_G[v] \setminus S'$, and if $|S' \cap \{x, y, z, w\}| = 2$, then let $S'' = S' \setminus \{x, y, z, w\}$. Clearly, S'' is an EmSS of G and hence $\sigma_m(G) \leq \sigma_m(G') - 2$. Thus $\sigma_m(G') = \sigma_m(G) + 2$. ∎

Lemma 25. Let G be a graph and let $v \in V(G)$. If G' is the graph obtained from G by adding two new stars $K_{1,2}$ centered at x_1, x_2 and joining v to x_1, x_2, then $\sigma_m(G') = \sigma_m(G) + 3$. In particular, $b_{\sigma_m}(G') \leq b_{\sigma_m}(G)$.

Proof. Let y_i, z_i be the leaves adjacent to x_i for $i = 1, 2$. Clearly, adding x_1, x_2, y_1 to any $\sigma_m(G)$-set containing v yields an EmSS of G' and so $\sigma_m(G') \leq \sigma_m(G) + 3$. Let now S' be a $\sigma_m(G')$-set. As above we may assume that $y_1 \in S'$, $|S' \cap \{x_1, z_1\}| \geq 1$ and $|S' \cap \{x_2, y_2, z_2\}| \geq 1$. It is easy to see that $|S' \cap \{x_2, y_2, z_2, v\}| \geq 2$. If $|S' \cap \{x_2, y_2, z_2\}| = 2$, then let $S'' = (S' - \{x_1, y_1, z_1, x_2, y_2, z_2\}) \cup \{u\}$ where $u \in N_G[v] \setminus S'$, and if $|S' \cap \{x_2, y_2, z_2\}| = 1$, then let $S'' = S' \setminus \{x_1, y_1, z_1, x_2, y_2, z_2\}$. Clearly, S'' is an EmSS of G and hence $\sigma_m(G) \leq \sigma_m(G') - 3$. Thus $\sigma_m(G') = \sigma_m(G) + 3$. ∎

Lemma 26. Let $T \in \mathfrak{T}$ and $u \in V(T)$. If T' is a tree obtained from T by adding a pendant edge uu', then $\sigma_m(T') = \sigma_m(T)$.

Proof. Let T' be a tree obtained from T by adding the pendant edge uu'. If S is a $\sigma_m(T')$-set, then let $S' = S$ if $u' \notin S$ and $S' = (S - \{u'\}) \cup \{w\}$, where $w \in N_T[u] \setminus S$, when $u' \in S$. Clearly, S' is an EmSS for G and so $\sigma_m(T) \leq \sigma_m(T')$.

Now we show that $\sigma_m(T') \leq \sigma_m(T)$. Let $P_3 = v_1v_2v_3$ and let T be obtained from P_3 by successive operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^m$, respectively, where $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4\}$ for $1 \leq i \leq m$, if $m \geq 1$, and $T = P_3$ if $m = 0$. The proof is by induction on m. If $m = 0$, then clearly the statement is true. Assume $m \geq 1$ and that the statement holds for all trees which are obtained from P_3 by applying at most $m - 1$ operations. Suppose T_{m-1} is a tree obtained by applying the first $m - 1$ operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^{m-1}$ and let T be obtained from T_{m-1} by adding a new part to the attacher v. Assume that T'_{m-1} is obtained from T_{m-1} by adding a pendant edge uu' when $u \in V(T_{m-1})$. We consider four cases.

Case 1. $\mathfrak{T}^m = \mathfrak{T}_1$. Then T is obtained from T_{m-1} by attaching a path vxy to $v \in V(T_{m-1})$. If $u \in V(T_{m-1})$, then by the inductive hypothesis, $\sigma_m(T_{m-1}) = \sigma_m(T'_{m-1})$ and by Lemma 22 we have $\sigma_m(T') = \sigma_m(T)$. Suppose $u \in \{x, y\}$. Let $T^* = T' - \{y, u'\}$. Then, obviously, T^* is obtained from T_{m-1} by adding the pendant edge xv. By the inductive hypothesis, $\sigma_m(T^*) = \sigma_m(T_{m-1})$. Let S be a $\sigma_m(T^*)$-set containing x. Then $S \cup \{y\}$ is an EmSS of T' and by Lemma 22 we have

$$\sigma_m(T') \leq \sigma_m(T^*) + 1 = \sigma_m(T_{m-1}) + 1 = \sigma_m(T).$$

Case 2. $\mathfrak{T}^m = \mathfrak{T}_2$. Then T is obtained from T_{m-1} by adding a star $K_{1,3}$ with central vertex y and leaves x, w, z and joining x to v. If $u \in V(T_{m-1})$, then the result follows from the induction hypothesis and Lemma 23. Assume that $u \in \{x, y, z, w\}$. Let $T^* = T - \{y, z, w\}$. By the induction hypothesis, we have $\sigma_m(T^*) = \sigma_m(T_{m-1})$. Let S be a $\sigma_m(T^*)$-set containing x. Then the set $S \cup \{y, z\}$ if $u \neq w$ and the set $S \cup \{y, w\}$ if $u = w$, is an EmSS of T' and so $\sigma_m(T') \leq \sigma_m(T^*) + 2$. By Lemma 23, we obtain

$$\sigma_m(T') \leq \sigma_m(T^*) + 2 = \sigma_m(T_{m-1}) + 2 = \sigma_m(T).$$

Case 3. $\mathfrak{T}^m = \mathfrak{T}_3$. Then T is obtained from T_{m-1} by attaching a pendant edge vw at v and adding a star $K_{1,2}$ with central vertex x and leaves y, z and joining x to v. If $u \in V(T_{m-1})$, then we deduce from the induction hypothesis and Lemma 24 that $\sigma_m(T') = \sigma_m(T)$. If $u = x$ or $u = y$ (the case $u = z$ is similar), then let $T^* = T' - \{u', x, y, z\}$. Obviously, T^* is obtained from T_{m-1} by adding the pendant edge vw at v. By the induction hypothesis, we have $\sigma_m(T^*) = \sigma_m(T_{m-1})$. Clearly, adding x, y to any $\sigma_m(T^*)$-set yields an EmSS of T' and so

$$\sigma_m(T') \leq \sigma_m(T^*) + 2 = \sigma_m(T_{m-1}) + 2 = \sigma_m(T).$$

If $u = w$, then let $T^* = T' - \{u', x, y, z\}$. Obviously, T^* is obtained from T_{m-1} by adding the pendant edge vw at v. By the inductive hypothesis, $\sigma_m(T^*) = \sigma_m(T_{m-1})$. Let S be a $\sigma_m(T^*)$-set containing w. Then $S \cup \{v, x\}$ if $v \notin S$ and $S \cup \{x, y\}$ if $v \in S$, is an EmSS of T' and so

$$\sigma_m(T') \leq \sigma_m(T^*) + 2 = \sigma_m(T_{m-1}) + 2 = \sigma_m(T).$$

Case 4. $\mathfrak{T}^m = \mathfrak{T}_4$. Then T is obtained from T_{m-1} by adding two stars $K_{1,2}$ with central vertices x_1 and x_2 and joining x_1, x_2 to $v \in V(T_{m-1})$. Let y_i, z_i be the leaves adjacent to x_i for $i = 1, 2$. If $u \in V(T_{m-1})$, then the result follows from the induction hypothesis and Lemma 25. If $u = x_1$ (the case $u = x_2$ is similar), then adding x_1, y_1, x_2 to any $\sigma_m(T_{m-1})$-set containing v yields an EmSS of T' and we deduce from Lemma 25 that

$$\sigma_m(T') \leq \sigma_m(T_{m-1}) + 3 = \sigma_m(T).$$

Assume that $u = y_1$ (the cases $u = z_1, u = y_2, u = z_2$ are similar). Let $T^* = T' - \{x_1, y_1, z_1, u', y_2, z_2\}$. Obviously, T^* is obtained from T_{m-1} by adding pendant edge vx_2 at v. By the inductive hypothesis, we have $\sigma_m(T^*) = \sigma_m(T_{m-1})$.

Clearly, adding x_1, y_1, y_2 to any $\sigma_m(T^*)$-set containing x_2, yields an EmSS of T' and this implies that

$$\sigma_m(T') \le \sigma_m(T^*) + 3 = \sigma_m(T_{m-1}) + 3 = \sigma_m(T).$$

Hence $\sigma_m(T') \le \sigma_m(T)$. Thus $\sigma_m(T') = \sigma_m(T)$ and the proof is complete. ■

Theorem 27. If $T \in \mathfrak{T}$, then $b_{\sigma_m}(T) = 2$.

Proof. Let $T \in \mathfrak{T}$, $e \in E(T)$ and $T' = T - e$. Clearly $\sigma_m(T') \ge \sigma_m(T)$. Now we show that $\sigma_m(T') \le \sigma_m(T)$. Let $P_3 := v_1v_2v_3$ and let T be obtained from P_3 by successive operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^m$, respectively, where $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4\}$ for $1 \le i \le m$ if $m \ge 1$ and $T = P_3$ if $m = 0$. The proof is by induction on m. If $m = 0$, then the statement is true by Corollary 4. Assume $m \ge 1$ and that the statement holds for all trees obtained from P_3 by applying at most $m - 1$ operations. Suppose T_{m-1} is a tree obtained by applying the first $m - 1$ operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^{m-1}$. We consider four cases.

Case 1. $\mathfrak{T}^m = \mathfrak{T}_1$. Then T is obtained from T_{m-1} by attaching a path vxy at $v \in V(T_{m-1})$. If $e \in E(T_{m-1})$, then we deduce from the induction hypothesis and Lemma 22 that

$$\sigma_m(T') = \sigma_m(T_{m-1} - e) + 1 = \sigma_m(T_{m-1}) + 1 = \sigma_m(T).$$

If $e = vx$, then clearly $\sigma_m(T') = \sigma_m(T_{m-1}) + 1 = \sigma_m(T)$. Assume that $e = xy$. Let $T^* = T' - \{y\}$. Then T^* is obtained from T_{m-1} by adding a pendant path vx at v. Clearly $\sigma_m(T') = \sigma_m(T^*) + 1$. It follows from Lemmas 26 and 22 that

$$\sigma_m(T') = \sigma_m(T^*) + 1 = \sigma_m(T_{m-1}) + 1 = \sigma_m(T).$$

Case 2. $\mathfrak{T}^m = \mathfrak{T}_2$. Then T is obtained from T_{m-1} by adding a star $K_{1,3}$ with central vertex y and leaves x, w, z and joining x to v. If $e \in E(T_{m-1})$, then by the inductive hypothesis and Lemma 23 we have

$$\sigma_m(T') = \sigma_m(T_{m-1} - e) + 2 = \sigma_m(T_{m-1}) + 2 = \sigma_m(T).$$

If $e = vx$, then clearly $\sigma_m(T') = \sigma_m(T_{m-1}) + 2 = \sigma_m(T)$. If $e = xy$, then let $T^* = T - \{y, z, w\}$. By Lemma 26, we have $\sigma_m(T^*) = \sigma_m(T_{m-1})$. Clearly $\sigma_m(T') = \sigma_m(T^*) + 2$ and by Lemma 23 we have

$$\sigma_m(T') = \sigma_m(T^*) + 2 = \sigma_m(T_{m-1}) + 2 = \sigma_m(T).$$

Assume that $e = yz$. Let $T^* = T' - \{z, w\}$. Then T^* is obtained from T_{m-1} by Operation $\mathfrak{T}_1$ and so $T^* \in \mathfrak{T}$ and $\sigma_m(T^*) = \sigma_m(T_{m-1}) + 1$. By Lemma 26, we have

$\sigma_m(T^*+yw)=\sigma_m(T^*)$. Now it is easy to check that $\sigma_m(T')\le\sigma_m(T^*+yw)+1$ and by Lemma 23 we have

$$\sigma_m(T')\le\sigma_m(T^*)+1=\sigma_m(T_{m-1})+2=\sigma_m(T).$$

Case 3. $\mathfrak{T}^m=\mathfrak{T}_3$. Then T is obtained from T_{m-1} by adding a pendant edge vw at a leaf $v\in V(T_{m-1})$ and adding a star $K_{1,2}$ with central vertex x and leaves y,z and joining x to v. If $e\in E(T_{m-1})$, then we conclude from the induction hypothesis and Lemma 24 that

$$\sigma_m(T')=\sigma_m(T_{m-1}-e)+2=\sigma_m(T_{m-1})+2=\sigma_m(T).$$

If $e=vw$, then let $T^*=T-\{y,z,w\}$. Then we have $\sigma_m(T^*)=\sigma_m(T_{m-1})$ by Lemma 26. On the other hand, adding y,w to any $\sigma_m(T^*)$-set containing x, yields an EmSS of T' and we deduce from Lemma 24 that

$$\sigma_m(T')\le\sigma_m(T^*)+2=\sigma_m(T_{m-1})+2=\sigma_m(T).$$

If $e\in\{xv,xy,xz\}$, then let $T^*=T'-\{x,y,z\}$. Then T^* is obtained from T_{m-1} by attaching a pendant edge vw. By Lemma 26, we have $\sigma_m(T^*)=\sigma_m(T_{m-1})$. On the other hand, adding x,y to any $\sigma_m(T^*)$-set yields an EmSS of T' and it follows from Lemma 24 that

$$\sigma_m(T')\le\sigma_m(T^*)+2=\sigma_m(T_{m-1})+2=\sigma_m(T).$$

Case 4. $\mathfrak{T}^m=\mathfrak{T}_4$. Then T is obtained from T_{m-1} by adding two stars $K_{1,2}$ with central vertices x_1 and x_2 and joining x_1,x_2 to a leaf v. Let y_i,z_i be the leaves adjacent to x_i for $i=1,2$. If $e\in E(T_{m-1})$, then by the inductive hypothesis and Lemma 25 we have

$$\sigma_m(T')=\sigma_m(T_{m-1}-e)+3=\sigma_m(T_{m-1})+3=\sigma_m(T).$$

If $e=x_1v$ or $e=x_1y_1$, then let $T^*=T'-\{x_1,y_1,z_1,y_2,z_2\}$. Then T^* is obtained from T_{m-1} by attaching a pendant edge vx_2 at v. By Lemma 26, we have $\sigma_m(T^*)=\sigma_m(T_{m-1})$. On the other hand, adding x_1,y_1,y_2 to any $\sigma_m(T^*)$-set containing x_2 yields an EmSS of T' and it follows from Lemma 25 that

$$\sigma_m(T')\le\sigma_m(T^*)+3=\sigma_m(T_{m-1})+3=\sigma_m(T).$$

In the other cases, we can see that $\sigma_m(T')\le\sigma_m(T)$ as above. Hence $\sigma_m(T')\le\sigma_m(T)$. Thus $\sigma_m(T')=\sigma_m(T)$ and this implies that $b_{\sigma_m}(T)\ge 2$. Now the result follows from Theorem 21. ■

Now we are ready to prove the main theorem of this section.

Theorem 28. Let T be a tree of order $n \geq 3$. Then $b_{\sigma_m}(T) = 2$ if and only if $T \in \mathfrak{T}$.

Proof. According to Theorem 27, we only need to prove the necessity. We proceed by the induction on n. If $n = 3$, then the result is trivial. Assume that $n \geq 4$ and the statement holds for all trees T of order less than n. Let T be a tree of order n with $b_{\sigma_m}(T) = 2$. Since $b_{\sigma_m}(K_{1,n-1}) = 1$, we have $\operatorname{diam}(T) \geq 3$. Suppose $P := v_1 \cdots v_k$ is a diametral path in T such that $\deg(v_2)$ is as small as possible and root T at v_k. If $\deg(v_2) = 2$, then let $T' = T - \{v_1, v_2\}$. By Lemma 22, we have $\sigma_m(T) = \sigma_m(T') + 1$ and $b_{\sigma_m}(T') = 2$. It follows from the induction hypothesis that $T' \in \mathfrak{T}$. Now T can be obtained from T' by Operation $\mathfrak{T}_1$ and hence $T \in \mathfrak{T}$. Let $\deg(v_2) \geq 3$. We conclude from Proposition 7 that $\deg(v_2) = 3$. Let $w \neq v_1$ be a leaf adjacent to v_2. If $\deg(v_3) = 2$, then let $T' = T - \{v_1, v_2, v_3, w\}$. By Lemma 23, we have $\sigma_m(T) = \sigma_m(T') + 2$ and $b_{\sigma_m}(T') = 2$. By the induction hypothesis, we obtain $T' \in \mathfrak{T}$. Now T can be obtained from T' by Operation $\mathfrak{T}_2$ and so $T \in \mathfrak{T}$. Let $\deg(v_3) \geq 3$. We consider the following cases.

Case 1. There exists a path v_3xy in T such that $x \notin \{v_2, v_4\}$. By the choice of diametral path and Proposition 7, we have $\deg(x) = 3$. If v_3 is a support vertex and u is a leaf adjacent to v_3, then it is not hard to see that deleting the edge v_3u increases the eternal m-security number which leads to a contradiction. Suppose v_3 is not a support vertex. If v_3 is adjacent to a support vertex w other than x, v_2, v_4, then as above we may assume that $\deg(w) = 3$. It is easy to see that deleting the edge v_3w increases the eternal m-security number which leads to a contradiction. Hence, $\deg(v_3) = 3$. Let $T' = T - \{v_1, v_2, w, x, y, z\}$ where y and z are the leaves adjacent to x. Then $\sigma_m(T) = \sigma_m(T') + 3$ and $b_{\sigma_m}(T') = 2$ by Lemma 25. We deduce from the induction hypothesis that $T' \in \mathfrak{T}$ and so T can be obtained from T' by Operation $\mathfrak{T}_4$. Hence $T \in \mathcal{T}$.

Case 2. Any neighbor of v_3, except v_2, v_4, is a leaf. Let u be a leaf adjacent to v_3. If $\deg(v_3) \geq 4$, then it is easy to see that deleting the edge v_3u increases the eternal m-security number and so $b_{\sigma_m}(T) = 1$, a contradiction. Thus $\deg(v_3) = 3$. Let $T' = T - \{v_1, v_2, u, w\}$. By Lemma 24 we have $\sigma_m(T) = \sigma_m(T') + 2$ and $b_{\sigma_m}(T') = 2$. It follows from the inductive hypothesis that $T' \in \mathfrak{T}$. By Operation $\mathcal{T}_3$, T can be obtained from T' and so $T \in \mathcal{T}$. This completes the proof. ■

References

[1] A.P. Burger, E.J. Cockayne, W.R. Gröndlingh, C.M. Mynhardt, J.H. van Vuuren and W. Winterbach, *Infinite order domination in graphs*, J. Combin. Math. Combin. Comput. **50** (2004) 179–194.

[2] J.F. Fink, M.S. Jacobson, L.F. Kinch and J. Roberts, *The bondage number of a graph*, Discrete Math. **86** (1990) 47–58.
doi:10.1016/0012-365X(90)90348-L

[3] W. Goddard, S.M. Hedetniemi and S.T. Hedetniemi, *Eternal security in graphs*, J. Combin. Math. Combin. Comput. **52** (2005) 160–180.

[4] B.L. Hartnell and D.F. Rall, *Bounds on the bondage number of a graph*, Discrete Math. **128** (1994) 173–177.
doi:10.1016/0012-365X(94)90111-2

[5] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in graphs (New York, Marcel Dekker, Inc., 1998).

[6] J. Huang and J. Ming Xu, *The bondage numbers and efficient dominations of vertex-transitive graphs*, Discrete Math. **308** (2008) 571–582.
doi:10.1016/j.disc.2007.03.027

[7] A. Meir and J.W. Moon, *Relations between packing and covering numbers of a tree*, Pacific J. Math. **61** (1975) 225–233.
doi:10.2140/pjm.1975.61.225

[8] U. Teschner, *The bondage number of a graph G can be much greater than $\Delta(G)$*, Ars Combin. **43** (1996) 81–87.

[9] U. Teschner, *New results about the bondage number of a graph*, Discrete Math. **171** (1997) 249–259.
doi:10.1016/S0012-365X(96)00007-6

[10] U. Teschner, *A counterexample to a conjecture on the bondage number of a graph*, Discrete Math. **122** (1993) 393–395.
doi:10.1016/0012-365X(93)90317-M

[11] D.B. West, Introduction to Graph Theory (Prentice-Hall, Inc., 2000).

[12] Jun-Ming Xu, *On bondage numbers of graphs: a survey with some comments*, Int. J. Combin. **2013**, article ID 595210, 34 pages.
doi:10.1155/2013/595210

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