Discussiones Mathematicae Graph Theory 38 (2018) 947–962 doi:10.7151/dmgt.2053

# MAKING A DOMINATING SET OF A GRAPH CONNECTED

Hengzhe Li

College of Mathematics and Information Science Henan Normal University, Xinxiang 453007, P.R. China **e-mail:** lhz@htu.cn

BAOYINDURENG WU

College of Mathematics and System Sciences Xinjiang University, Urumqi 830046, P.R. China e-mail: baoywu@163.com

AND

## WEIHUA YANG

School of Mathematical Science Taiyuan University of Technology Shanxi Taiyuan 030024, P.R. China

e-mail: ywh222@163.com

### Abstract

Let G = (V, E) be a graph and  $S \subseteq V$ . We say that S is a dominating set of G, if each vertex in  $V \setminus S$  has a neighbor in S. Moreover, we say that S is a connected (respectively, 2-edge connected or 2-connected) dominating set of G if G[S] is connected (respectively, 2-edge connected or 2-connected). The domination (respectively, connected domination, or 2edge connected domination, or 2-connected domination) number of G is the cardinality of a minimum dominating (respectively, connected dominating, or 2-edge connected dominating, or 2-connected dominating) set of G, and is denoted  $\gamma(G)$  (respectively  $\gamma_1(G)$ , or  $\gamma'_2(G)$ , or  $\gamma_2(G)$ ). A well-known result of Duchet and Meyniel states that  $\gamma_1(G) \leq 3\gamma(G) - 2$  for any connected graph G. We show that if  $\gamma(G) \geq 2$ , then  $\gamma'_2(G) \leq 5\gamma(G) - 4$  when G is a 2-edge connected graph and  $\gamma_2(G) \leq 11\gamma(G) - 13$  when G is a 2-connected triangle-free graph.

Keywords: independent set, dominating set, connected dominating set. 2010 Mathematics Subject Classification: 05C69.

#### 1. INTRODUCTION

In this paper, all graphs considered are finite, undirected graphs. We follow the notation and terminology of Bondy and Murty [3], unless otherwise stated.

Let G = (V(G), E(G)) be a graph. The order and the size of G are |V(G)|and |E(G)|, respectively. We use c(G) to denote the number of components of G. The graph G is trivial if its order is 1, and nontrivial, otherwise. For  $D \subseteq V(G)$ , the subgraph of G induced by D, denoted by G[D], is the graph with D as the vertex set, in which two vertices are adjacent if and only if they are adjacent in G. D is an independent set of G if G[D] has no edge. The independence number of G, denoted  $\alpha(G)$ , is the maximum cardinality of an independent set of G.

Let G be a nontrivial graph and  $x, y \in V(G)$  be two distinct vertices. An xy-path is a path joining x and y in G. The local connectivity between x and y, denoted  $\kappa_G(x, y)$ , is the maximum number of pairwise internally disjoint xy-paths in G. For a nonnegative integer k, G is k-connected if  $\kappa_G(x, y) \geq k$  for any two distinct vertices x and y. Similarly, the local edge connectivity between x and y, denoted  $\kappa'_G(x, y)$ , is the maximum number of pairwise edge-disjoint xy-paths in G. For two distinct nonadjacent vertices x and y, an xy-vertex cut is a subset S of  $V(G) \setminus \{x, y\}$  such that x and y belong to different components of G - S. We also say that such a subset S separates x and y. The minimum size of a vertex cut separating x and y is denoted by c(x, y).

For a nonnegative integer k, G is k-edge connected if  $\kappa'_G(x, y) \ge k$  for any two distinct vertices x and y of G. An edge cut  $E[X, V(G) \setminus X]$  separates x and y if  $x \in X$  and  $y \in V(G) \setminus X$ . We denote by c'(x, y) the minimum cardinality of such an edge cut. The well-known Menger's Theorem asserts that  $\kappa'_G(x, y) = c'(x, y)$ .

In graph theory, the problem concerning domination of graphs (or networks) is a major area that has attracted a large number of researchers and generated a wealth of important achievements in the past few decades. Let G = (V, E) be a graph and  $D \subseteq V$ . We call D a *dominating set* of G if every vertex in  $V \setminus D$ has a neighbor in D. Furthermore, if G[D] is k-connected (respectively, k-edge connected), D is called a k-connected (respectively, k-edge connected) dominating set. The k-connected domination number (respectively, k-edge connected domination number) of a graph G, denoted by  $\gamma_k(G)$  (respectively, by  $\gamma'_k(G)$ ) is the minimum cardinality of a k-connected (respectively, k-edge connected) dominating set. Clearly, a graph G has a k-connected (respectively, k-edge connected) dominating set if G is k-connected (respectively, k-edge connected). But a graph having a k-connected (respectively, k-edge connected). But a graph having a k-connected (respectively, k-edge connected). It is clear that  $\gamma'_0(G) = \gamma_0(G) = \gamma(G)$  and  $\gamma'_1(G) = \gamma_1(G)$ .

The theory of connected domination of graphs has important applications in communication and computer networks, especially for its role as a virtual backbone in wireless networks, see Du and Wan [6]. Haynes, Hedetniemi and Slater published two monographs [10, 11] concerning domination in graphs, and recently Chellali, Favaron, Hansberg and Volkmann presented a survey paper [4] concerning dominating sets and independent sets. We refer to [1, 2, 5, 13–15, 18] for more results concerning connected dominating sets.

An interesting application of the connected domination of graphs is in minor theory. The well-known Hadwiger's conjecture states that if  $\chi(G) \ge k$ , then Gcontains a  $K_k$ -minor, where  $\chi(G)$  denotes the chromatic number of G. We use  $\alpha(G)$  to denote the independent number of a graph. Since

$$\alpha(G)\chi(G) \ge n$$

for a graph G on n vertices, Hadwiger's conjecture implies that any graph G on n vertices has a  $K_{\lceil \frac{n}{\alpha(G)} \rceil}$ -minor. Duchet and Meyniel in [8] established the following relation between the connected domination number and the independence number of a connected graph, and by applying this result, they proved that any graph G on n vertices has a  $K_{\lceil \frac{n}{2\alpha(G)-1} \rceil}$ -minor.

**Theorem 1** (Duchet and Meyniel [8]). For any connected graph G,  $\gamma_1(G) \leq \min\{2\alpha(G) - 1, 3\gamma(G) - 2\}$ .

In some sense, the above theorem of Duchet and Meyniel is related to the following conjecture in combinatorial optimization.

**Conjecture 1** [20]. For any connected unit disk graph G,  $\alpha(G) \leq 3\gamma_1(G) + 2$ .

There are a number of papers devoted to the relation of the independence number and the connected domination number of unit disk graphs, for instance, [12, 17, 19]. Best known result on Conjecture 1 is  $\alpha(G) \leq 3.399\gamma_1(G) + 4.874$ obtained by Du and Du [7]. So, combining this with Theorem 1, for a connected unit disk graph G,

$$0.5\gamma_1(G) + 0.5 \le \alpha(G) \le 3.399\gamma_1(G) + 4.874.$$

We refer to [20] for more relevant works concerning domination and packing on wireless networks.

There exist a number of algorithms for constructing maximal independent sets and connected dominating sets. For instance, Vigoda [16] presented a parallel algorithm for constructing a maximal independent set of an input graph on n vertices, in time polynomial in  $\log n$  and in  $\log n$  using a polynomial in nprocessors, Guha and Khuller [9] presented two polynomial time algorithms for constructing a connected dominating set that achieves approximation factors of  $O(h(\Delta))$ , where  $\Delta$  is the maximum degree, and h is the harmonic function. We shall get a connected dominating set if we can make a dominating set connected by adding a small vertex set (with respect to the dominating set). In this paper, we generalize Duchet and Meyniel's theorem by considering the following problems.

**Problem 1.** Given a connected graph G and a dominating set S, what is the least vertex set T such that  $G[S \cup T]$  is connected?

Problem 1 was studied in [8] by Duchet and Meyniel. We are maniply concerned with the following two problems.

**Problem 2.** Given a 2-edge connected graph G and a dominating set S, find a vertex set T with minimum |T| such that  $G[S \cup T]$  is 2-edge connected.

**Problem 3.** Given a 2-connected graph G and a dominating set S, find a vertex set T with minimum |T| such that  $G[S \cup T]$  is 2-connected.

# 2. MINIMUM VERTEX SET JOINING A GIVEN DOMINATING SET

For two vertices  $u, v \in V(G)$ , the distance  $d_G(u, v)$  between u and v is the number of edges in a shortest path connecting u and v in G. In general, for  $X \subseteq V(G)$ and  $Y \subseteq V(G)$ , the distance  $d_G(X, Y)$  between X and Y is  $\min\{d_G(x, y) : x \in X, y \in Y\}$ . Thus  $d_G(X, Y) = d_G(Y, X)$ . If  $Y = \{y\}$  for a vertex  $y \in V(G)$ , we simply write  $d_G(X, y)$  instead of  $d_G(X, \{y\})$ .

## 2.1. Connected dominating set

The idea of the proof of the following theorem is due to Duchet and Meyniel [8].

**Theorem 2.** Let S be a dominating set of a connected graph G. Then there exists a set T such that  $|T| \leq 2|S| - 2$  and  $G[S \cup T]$  is connected.

**Proof.** If c(G[S]) = 1, i.e., S is a connected dominating set, then the assertion of the theorem trivially holds by taking  $T = \emptyset$ . Next we assume that G[S] is disconnected. Since S is a dominating set of G, there exist two components of G[S], say  $G_1$  and  $G_2$ , such that  $d_G(V(G_1), V(G_2)) \leq 3$ . Pick a path P which joins  $V(G_1)$  and  $V(G_2)$  with  $\ell(P) = d_G(V(G_1), V(G_2))$ . Hence  $S \cup V(P)$  is a dominating set of G with  $|S \cup V(P)| \leq |S| + 2$  and  $c(G[S \cup V(P)]) \leq c(G[S]) - 1$ . If  $G[S \cup V(P)]$  is connected, then we are done by letting T = V(P). Otherwise, let  $S := S \cup V(P)$ , and repeat the above operation until G[S] is connected.

Since  $c(G[S]) \leq |S| - 1$ , |S| increases by at most two and the number of components decreases by at least one in each iteration of the above operation, we conclude that the desired set T exists.

So the following is immediate from the above theorem.

**Corollary 1.**  $\gamma_1(G) \leq 3\gamma(G) - 2$  for any connected graph G.

Algorithm 1. An algorithm for constructing a connected dominating set. **Input:** A connected graph G and a dominating set S of G. **Output:** A set T such that  $|T| \leq 2|S| - 2$  and  $G[S \cup T]$  is connected. 1. Set  $T := \emptyset$ ,  $H := G[S \cup T]$ 2. run BFS to get all components of H, say  $H_1, H_2, \ldots, H_c$ , and set  $\mathcal{C} = \{H_i : 1 \leq i \leq l \}$  $i \leq c$  and  $c = |\mathcal{C}|$ 3. if c = 1, then stop 4. else set  $W := V(G) \setminus S$  and F := E(G[W])5. while  $W \neq \emptyset$ 6. pick a vertex  $w \in W$ 7. if  $N(w) \cap V(H_i) \neq \emptyset$  and  $N(w) \cap V(H_i) \neq \emptyset$  for different integers i and j, then set  $H_i := G[\bigcup_{H_i \in \mathcal{H}} V(H_i) \cup \{w\}], \ \mathcal{C} := (\mathcal{C} \setminus \mathcal{H}) \cup \{H_i\}, \ T := T \cup \{w\},$  $H := G[S \cup T]$ , and k := k - h + 1, where  $\mathcal{H} = \{H_i : V(H_i) \cap N_G(w) \neq \emptyset\}$  and  $h = |\mathcal{H}|, \text{ go to step } 3$ else  $W := W \setminus \{w\}$ 8. 9. end if end while 10. 11. while  $F \neq \emptyset$ , pick  $f = uv \in F$ 12.pick  $f = uv \in F$ if  $N(u) \cap V(H_i) \neq \emptyset$  and  $N(v) \cap V(H_i) \neq \emptyset$  for different integers i and j, then 13.set  $H_i := G[\bigcup_{H_i \in \mathcal{H}} V(H_i) \cup \{u, v\}], \ \mathcal{C} := (\mathcal{C} \setminus \mathcal{H}) \cup \{H_i\}, \ T := T \cup \{u, v\},$  $H := G[S \cup T]$ , and  $k := k - |\mathcal{H}| + 1$ , where  $\mathcal{H} = \{H_i : V(H_i) \cap N_G(u) \neq \emptyset$  or  $V(H_i) \cap N_G(v) \neq \emptyset$  and  $h = |\mathcal{H}|$ , go to step 3 14. else  $F := F \setminus \{f\}.$ 15.end if 16. end while 17. end if

**Remark 1.** Let s,  $\Delta$ , n and m be the size of a dominating set S, the maximum degree, order and size of G, respectively. Note that the time complexity of BFS can be expressed as O(n+m). Since the running time of each recursion is at most  $\Delta(n+2m)$  and this algorithm runs at most s-1 recursions, the time complexity of the algorithm is bounded by  $O((s-1)\Delta(n+2m))$ .

#### 2.2. 2-edge connected dominating set

Let G be a connected graph. A subgraph  $F \subseteq G$  is called a *maximal 2-edge* connected subgraph of G if F is trivial or is 2-edge connected, and there exists no other 2-edge connected subgraph  $F' \subseteq G$  such that  $F \subseteq F'$ . It is clear from the

definition that every maximal 2-edge connected subgraph F of G is an induced subgraph of G.

For a dominating set S of G, let H = G[S]. We use  $C_H$  to denote the set of all maximal 2-edge connected subgraphs F of H containing at least one vertex of S, and  $c_H = |C_H|$ .

Next we assume that G is a 2-edge connected graph and let S be a dominating set of G with  $|S| \ge 2$ , and let T be an output of Algorithm 1 for G and S. If  $H = G[S \cup T]$  is 2-edge connected, then  $S \cup T$  is a 2-edge connected dominating set of G. Otherwise, we shall decrease  $c_H$  by at least one by adding at most two vertices, see Lemma 3, Corollary 2, and Lemmas 4–5 for details.

**Lemma 3.** Let  $u_1$  and  $u_2$  be two distinct vertices in H. If deleting a cut edge e separates  $u_1$  and  $u_2$  in H, then there exists a vertex  $w \in V(G) \setminus V(H)$  such that  $N_G(w) \cap V(X_e) \neq \emptyset$  and  $N_G(w) \cap V(Y_e) \neq \emptyset$ , or an edge  $uv \in E(G - V(H))$  such that  $N_G(u) \cap V(X_e) \neq \emptyset$  and  $N_G(v) \cap V(Y_e) \neq \emptyset$ , where  $X_e$  and  $Y_e$  are two components of  $H \setminus e$ .

**Proof.** Without loss of generality, let  $u_1 \in V(X_e)$  and  $u_2 \in V(Y_e)$ . Let  $P = x_1x_2\cdots x_k$  be a shortest path joining  $X_e$  and  $Y_e$  in  $G \setminus e$ , where  $x_1 \in V(X_e)$  and  $x_k \in V(Y_e)$ . If  $k \leq 4$ , then  $P - \{x_1, x_k\}$  is a vertex or an edge, as we desired. If  $k \geq 5$ , we consider  $x_3$ . Since  $S \subseteq V(H)$  is a dominating set of G,  $x_3$  has a neighbor  $x'_3 \in S$  in G. If  $x'_3 \in V(X_e)$ , then  $x'_3x_3\cdots x_k$  is a shorter path than P that joins  $X_e$  and  $Y_e$  in  $G \setminus e$ , a contradiction; if  $x'_3 \in V(Y_e)$ , then  $x_1x_2x_3x'_3$  is a shorter path than P joining  $X_e$  and  $Y_e$  in  $G \setminus e$ , a contradiction.

**Corollary 2.** Let  $u_1$  and  $u_2$  be two distinct vertices in S. If  $\kappa'_H(u_1, u_2) = 1$  and  $d_H(u_1, u_2) = 1$ , then there exists a vertex  $w \in V(G) \setminus V(H)$  such that  $c_{H'} \leq c_H - 1$ , where  $H' = G[S \cup T \cup \{w\}]$ , or an edge  $e = uv \in E(G - V(H))$  such that  $c_{H'} \leq c_H - 1$ , where  $H' = G[S \cup T \cup \{w\}]$ .

**Proof.** Note that  $u_1$  and  $u_2$  belong to two distinct maximal 2-edge connected subgraphs of H, while they belong to the same maximal 2-edge connected subgraphs of H' by Lemma 2. Thus  $c_{H'} \leq c_H - 1$ .

**Lemma 4.** Let  $u_1$  and  $u_2$  be two distinct vertices in S such that  $\kappa'_H(u_1, u_2) = 1$ and  $d_H(u_1, u_2)$  is as small as possible. If  $d_H(u_1, u_2) = 2$ , then there exists a vertex  $w \in V(G) \setminus V(H)$  such that  $c_{H'} \leq c_H - 1$ , where  $H' = G[S \cup T \cup \{w\}]$ , or an edge  $e = uv \in E(G - V(H))$  such that  $c_{H'} \leq c_H - 1$ , where  $H' = G[S \cup T \cup \{u, v\}]$ , or a pair of vertices  $u, v \in V(G) \setminus V(H)$  such that  $c_{H'} \leq c_H - 1$ , where  $H' = G[S \cup T \cup \{u, v\}]$ ,  $G[S \cup T \cup \{u, v\}]$ .

**Proof.** Let  $u_1v_1u_2$  be a path of length 2 in H. By the choice of  $u_1$  and  $u_2$ ,  $v_1 \notin S$ . First, we may suppose that  $u_1v_1$  is a cut edge of H and  $u_2v_1$  is not. Let  $a = u_1v_1$ , and let  $X_a$  and  $Y_a$  be two components of  $H \setminus a$  such that  $u_1 \in V(X_a)$ 

and  $v_1 \in V(Y_a)$ . By Lemma 3, there exists a vertex  $w \in V(G) \setminus V(H)$  such that  $N_G(w) \cap V(X_a) \neq \emptyset$  and  $N_G(w) \cap V(Y_a) \neq \emptyset$ , or an edge  $uv \in E(G - V(H))$  such that  $N_G(u) \cap V(X_a) \neq \emptyset$  and  $N_G(v) \cap V(Y_a) \neq \emptyset$ . For the former case, let  $H' = G[S \cup T \cup \{w\}]$ . Clearly  $\kappa'_{H'}(u_1, u_2) \geq 2$ . Thus  $c_{H'} \leq c_H - 1$ . For the latter case, let  $H' = G[S \cup T \cup \{u, v\}]$ . Clearly  $\kappa'_{H'}(u_1, u_2) \geq 2$ . Thus  $c_{H'} \leq c_H - 1$ .

So, we now assume that both  $u_1v_1$  and  $u_2v_1$  are cut edges of H. Let  $a = u_1v_1$ , and let  $X_a, Y_a$  be two components of  $H \setminus a$  such that  $u_1 \in V(X_a)$  and  $v_1 \in V(Y_a)$ . We consider the following cases.

Case 1. There exists a vertex  $w \in V(G) \setminus V(H)$  such that  $N_G(w) \cap V(X_a) \neq \emptyset$ and  $N_G(w) \cap V(Y_a - v_1) \neq \emptyset$ . Then w is the vertex, as we desired.

Case 2. There exists an edge  $uv \in E(G-V(H))$  such that  $N_G(u) \cap V(X_a) \neq \emptyset$ and  $N_G(v) \cap V(Y_a - v_1) \neq \emptyset$ . Then uv is the edge, as we desired.

Case 3. There exists no vertex  $w \in V(G) \setminus V(H)$  such that  $N_G(w) \cap V(X_a) \neq \emptyset$  and  $N_G(w) \cap V(Y_a - v_1) \neq \emptyset$ , and no edge  $uv \in E(G - V(H))$  such that  $N_G(u) \cap V(X_a) \neq \emptyset$  and  $N_G(v) \cap V(Y_a - v_1) \neq \emptyset$ .

Let  $b = v_1 u_2$ , and  $X_b$  and  $Y_b$  be two components of  $H \setminus b$  such that  $v_1 \in V(X_b)$  and  $u_2 \in V(Y_b)$ . If there exists a vertex  $w \in V(G) \setminus V(H)$  such that  $N_G(w) \cap V(X_a) \neq \emptyset$  and  $N_G(w) \cap V(Y_a) \neq \emptyset$ , and a vertex  $w' \in V(G) \setminus V(H)$  such that  $N_G(w') \cap V(X_b) \neq \emptyset$  and  $N_G(w') \cap V(Y_b) \neq \emptyset$ , then w and w' are a pair of vertices, as we desired.

Next we show that there exist such a pair of vertices in H. Without loss of generality, suppose that there exists no vertex  $w \in V(G) \setminus V(H)$  such that  $N_G(w) \cap V(X_a) \neq \emptyset$  and  $N_G(w) \cap V(Y_a) \neq \emptyset$ . By Lemma 3, there exists an edge  $uv \in E(G - V(H))$  such that  $N_G(u) \cap V(X_a) \neq \emptyset$  and  $N_G(v) \cap V(Y_a) \neq \emptyset$ . Since  $v_1 \notin S$ ,  $S \subseteq V(H)$  and S is a dominating set of G, it follows that v has a neighbor  $v' \in S$  which belong to  $V(X_a) \cap S$  or  $V(Y_a - v_1)$ . If  $v' \in V(X_a)$ , then  $N_G(v) \cap V(X_a) \neq \emptyset$  and  $N_G(v) \cap V(Y_a) \neq \emptyset$ , a contradiction. Otherwise,  $v' \in V(Y_a - v_1)$ , then uv is an edge with the specified property in the assumption, a contradiction.

So, the proof is completed.

**Lemma 5.** Let  $u_1$  and  $u_2$  be two distinct vertices in S such that  $\kappa'_H(u_1, u_2) = 1$ and  $d_H(u_1, u_2)$  is as small as possible. If  $d_H(u_1, u_2) = 3$ , then there exists a vertex  $w \in V(G) \setminus V(H)$  such that  $c_{H'} \leq c_H - 1$ , where  $H' = G[S \cup T \cup \{w\}]$ , or an edge  $e = uv \in E(G - V(H))$  such that  $c_{H'} \leq c_H - 1$ , where  $H' = G[S \cup T \cup \{u, v\}]$ , or a pair of vertices  $u, v \in V(G) \setminus V(H)$  such that  $c_{H'} \leq c_H - 1$ , where  $H' = G[S \cup T \cup \{u, v\}]$ ,  $G[S \cup T \cup \{u, v\}]$ .

**Proof.** Let  $P = u_1v_1v_2u_2$  be a path of length 3 in H. By the choice of  $u_1$  and  $u_2$ , we have  $v_1 \notin S$  and  $v_2 \notin S$ . If exactly one edge of P is a cut edge of H, then by Lemma 3 the result follows. If exactly two adjacent edges of P are cut edges,

then by a similar argument to the proof of Lemma 6, we may show the assertion of the lemma. So, we consider the remaining cases.

Case 1.  $u_1v_1$  and  $v_2u_2$  are cut edges of H and  $v_1v_2$  is not. Let  $a = u_1v_1$ , and let  $X_a, Y_a$  be two components of  $H \setminus a$  such that  $u_1 \in X_a$  and  $v_1 \in Y_a$ . Similarly, let  $b = u_2v_2$ , and let  $X_b, Y_b$  be two components of  $H \setminus b$  such that  $v_2 \in V(X_b)$ and  $u_2 \in V(Y_b)$ .

Subcase 1.1. There exists a vertex  $w \in V(G) \setminus V(H)$  such that  $N_G(w) \cap V(X_a) \neq \emptyset$  and  $N_G(w) \cap V(Y_a) \neq \emptyset$ , and a vertex  $w' \in V(G) \setminus V(H)$  such that  $N_G(w') \cap V(X_b) \neq \emptyset$  and  $N_G(w') \cap V(Y_b) \neq \emptyset$ . If w = w', then w is a vertex we want, otherwise w and w' are a pair of vertices we want.

Subcase 1.2. There exists no pair of vertices w and w' which satisfies the condition of Subcase 1.1. Without loss of generality, assume that there exists no vertex  $w \in V(G) \setminus V(H)$  such that  $N_G(w) \cap V(X_a) \neq \emptyset$  and  $N_G(w) \cap V(Y_a) \neq \emptyset$ . By Lemma 3, there exists an edge  $uv \in E(G-V(H))$  such that  $N_G(u) \cap V(X_a) \neq \emptyset$  and  $N_G(v) \cap V(Y_a) \neq \emptyset$ . Since  $v_1, v_2 \notin S$ ,  $S \subseteq V(H)$  and S is a dominating set of G, we know that v has a neighbor v' in  $X_a$  or  $Y_a - \{v_1, v_2\}$ . If  $v' \in V(X_a)$ , this contradicts our assumption that there exists no vertex  $w \in V(G) \setminus V(H)$  such that  $N_G(w) \cap V(X_a) \neq \emptyset$  and  $N_G(w) \cap V(Y_a) \neq \emptyset$ . Otherwise,  $v' \in V(Y_a - \{v_1, v_2\})$ , and the edge uv is an our desired edge.

Case 2. All edges of P are cut edges in H. Let  $a = v_1v_2$ , and let  $X_a, Y_a$  be two components of  $H \setminus a$  such that  $v_1 \in V(X_a)$  and  $v_2 \in V(Y_a)$ . Consider the following three subcases.

Subcase 2.1. There exists a vertex  $w \in V(G) \setminus V(H)$  such that  $N_G(w) \cap V(X_a - v_1) \neq \emptyset$  and  $N_G(w) \cap V(Y_a - v_2) \neq \emptyset$ . Then w is a vertex, as we desired.

Subcase 2.2. There exists an edge  $uv \in V(G) \setminus V(H)$  such that  $N_G(u) \cap (X_a - v_1) \neq \emptyset$  and  $N_G(v) \cap (Y_a - v_2) \neq \emptyset$ . Then uv is an edge, as we desired.

Subcase 2.3. There exists no such vertex satisfying the condition of Subcase 2.1, and no such edge satisfying the condition of Subcase 2.2. We shall show that there exists a pair of vertices which satisfies the assertion of this lemma.

**Claim 1.** There exists a vertex  $w \in V(G) \setminus V(H)$  such that  $N_G(w) \cap V(X_a) \neq \emptyset$ and  $N_G(w) \cap V(Y_a - v_2) \neq \emptyset$ , or  $N_G(w) \cap V(Y_a) \neq \emptyset$ , and  $N_G(w) \cap V(X_a - v_1) \neq \emptyset$ .

**Proof.** Assume that there exists a vertex w satisfying  $N_G(w) \cap V(X_a) \neq \emptyset$  and  $N_G(w) \cap V(Y_a) \neq \emptyset$ . If  $N_G(w) \cap V(X_a) = \{v_1\}$  and  $N_G(w) \cap Y_a = \{v_2\}$ , then it contradicts the assumption that S is a dominating set of G. Thus, w is a vertex, as we want.

Assume that there does not exist a vertex w satisfying  $N_G(w) \cap V(X_a) \neq \emptyset$ and  $N_G(w) \cap V(Y_a) \neq \emptyset$ . By Lemma 3, there exists an edge uv satisfying  $N_G(u) \cap$   $V(X_a) \neq \emptyset$  and  $N_G(v) \cap V(Y_a) \neq \emptyset$ . Since  $v_1 \notin S$ ,  $v_2 \notin S$ ,  $S \subseteq V(H)$  and S is a dominating set of G, we know that u has an neighbor  $u' \in S$  which belong to  $X_a - v_1$  or  $Y_a - v_2$ , and v has an neighbor  $v' \in S$  which belong to  $X_a - v_1$  or  $Y_a - v_2$ . If u' and v' belong to different components of  $H \setminus a$ , then uv is an edge which contradicts the assumption of Subcase 2.3. Thus u' and v' belong to the same component of  $H \setminus a$ . We may suppose that  $u', v' \in V(X_a - v_1)$ . Then v is the vertex, as we want. This proves the claim.  $\Box$ 

By Claim 1, we may assume that there exists a vertex  $w \in V(G) \setminus V(H)$  such that  $N_G(w) \cap V(X_a - v_1) \neq \emptyset$  and  $N_G(w) \cap V(Y_a) \neq \emptyset$ .

Let  $b = v_2 u_2$ , and  $X_b$  and  $Y_b$  be two components of  $H \setminus b$  such that  $v_2 \in V(X_b)$  and  $u_2 \in V(Y_b)$ . If there exists a vertex  $w' \in V(G) \setminus V(H)$  such that  $N_G(w') \cap V(X_b) \neq \emptyset$  and  $N_G(w') \cap V(Y_b) \neq \emptyset$ , then w and w' are a pair of vertices, as we desired. If this is not the case, then by Lemma 3, there is an edge  $uv \in E(G - V(H))$  such that  $N_G(u) \cap V(X_b) \neq \emptyset$  and  $N_G(v) \cap V(Y_b) \neq \emptyset$ . Since  $v_1 \notin S, v_2 \notin S, S \subseteq V(H)$  and S is a dominating set of G, it follows that u has a neighbor  $u' \in S$  which belongs to  $X_b - \{v_1, v_2\}$  or  $Y_b$ . If  $u' \in V(X_b - \{v_1, v_2\})$ , then uv is an edge that contradicts the assumption of Subcase 2.3. So,  $u' \in V(Y_b)$ , which implies that  $N_G(u) \cap V(X_b) \neq \emptyset$  and  $N_G(u) \cap V(Y_b) \neq \emptyset$ . Hence w and u are a pair of vertices, as we desired.

**Theorem 6.** Let G be 2-edge connected graph. If S is a dominating set of G with  $|S| \ge 2$ , then there exists a set  $T \subseteq V(G)$  such that  $|T| \le 4|S| - 4$  and  $G[S \cup T]$  is 2-edge connected.

**Proof.** For G and S, let T be an output of Algorithm 1 and  $H = G[S \cup T]$ . We may suppose that  $c_H \ge 2$  and pick a pair of vertices  $u_1 \in S$  and  $u_2 \in S$  such that  $\kappa'_H(u_1, u_2) = 1$  and  $d_H(u_1, u_2)$  is as small as possible.

Claim 2.  $d_H(u_1, u_2) \le 3$ .

**Proof.** Suppose that the claim is not true, and let  $P = x_1 x_2 \cdots x_k$  be a shortest path joining  $u_1$  and  $u_2$  in H, where  $k \ge 5$ ,  $x_1 = u_1$  and  $x_k = u_2$ . We consider  $x_3$ . Since S is a dominating set of H,  $x_3$  has a neighbor  $x'_3 \in S$  in H.

If at least one of  $x_1x_2$  and  $x_2x_3$  is a cut edge of H, then  $u_1$  and  $x'_3$  are a pair of vertices such that  $\kappa'_H(u_1, x'_3) = 1$  and  $d_H(u_1, x'_3) < d_H(u_1, u_2)$ , a contradiction; otherwise, at least one edge of the path  $x_3x_4 \cdots x_k$  is a cut edge of H. Thus  $u_2$ and  $x'_3$  are a pair of vertices of S such that  $\kappa'_H(x'_3, u_2) = 1$  and  $d_H(x'_3, u_2) < d_H(u_1, u_2)$ , a contradiction. Thus  $d_H(u_1, u_2) \leq 3$ .

By Lemmas 3, 4 and 5, there exists a vertex set T' such that  $|T'| \leq 2$  and  $c_{H'} \leq c_H - 1$ , where  $H' = G[S \cup T \cup T']$ . If H' is 2-edge connected, then we are done by letting  $T := T \cup T'$ . Otherwise, let  $T := T \cup T'$ , and repeat the above operation until  $G[S \cup T]$  is 2-edge connected.

Since  $c_H \leq |S|, |T|$  increases by at most two and  $c_H$  decreases by at least one in each iteration of the above operation, we conclude that the desired set Texists.

**Corollary 3.** For a 2-edge connected graph G, if  $\gamma(G) \geq 2$ , then  $\gamma'_2(G) \leq 2$  $5\gamma(G) - 4.$ 

Algorithm 2. An algorithm for constructing a 2-edge connected dominating set. Input: A 2-edge connected graph G, a dominating set S with at least 2 vertices. **Output:** A set T such that  $|T| \leq 4|S| - 4$  and  $G[S \cup T]$  is 2-edge connected.

**I.** run Algorithm 1 to get set T

- **II.** 1. for  $G[S \cup T]$ , run DFS to get all blocks, say  $B_1, B_2, \ldots, B_k$ , and all cut vertices, say  $w_1, w_2, \ldots, w_\ell$ 
  - 2. set  $H := G[S \cup T]$ ,  $W = \{w_1, w_2, \ldots, w_\ell\}$ , and  $\mathcal{B}$  the set of blocks  $B_i$  in H such that  $|V(B_i)| \geq 3$
  - 3. if  $W = \emptyset$ , then stop
  - 4. else pick  $w \in W$
  - if  $B_{i_1}, B_{i_2}, \ldots, B_{i_r}$  are blocks in G such that  $w \in V(B_{i_1}) \cap V(B_{i_2}) \cap \cdots \cap V(B_{i_r})$ , 5. **then** set  $B_{i_1} = B_{i_1} \cup B_{i_2} \cup \cdots \cup B_{i_r}$ ,  $W = W \setminus \{w\}$ , go to Step 3
  - 6. else  $W = W \setminus \{w\}$
  - 7. end if
  - 8. end if

**III.** 1. set 
$$\mathcal{B} = \mathcal{B} \cup (S \setminus \bigcup_{B_i \in \mathcal{B}} V(B_i)), b := |\mathcal{B}|$$

- 2. if b = 1, then stop
- 3. else set  $W := V \setminus V(H)$ , F := E(G[W]), and  $R := W \times W$
- while  $F \neq \emptyset$ 4.
- 5.pick  $f = uv \in F$
- 6. if  $N_G(u) \cap V(B_i) \neq \emptyset$  and  $N_G(u) \cap V(B_j) \neq \emptyset$  for different integers *i* and *j*, then set  $B_i := G[\bigcup_{B_i \in \mathcal{H}} V(B_i) \cup \{u, v\}], \mathcal{B} := (\mathcal{B} \setminus \mathcal{H}) \cup \{B_i\}, T := T \cup \{u, v\},$  $H := G[S \cup T]$ , and b := b - h + 1, where  $\mathcal{H} = \{H_i : N_G(B_i) \cap N_G(u) \neq \emptyset$ or  $N_G(B_i) \cap N_G(v) \neq \emptyset$  and  $h = |\mathcal{H}|$ , go to Step 2 7.
  - else  $F := F \setminus \{f\}$
- 8. end if
- end while 9.
- while  $W \neq \emptyset$ . 11.
- 12.pick  $w \in W$
- if  $N_G(w) \cap V(B_i) \neq \emptyset$  and  $N_G(w) \cap V(B_j) \neq \emptyset$  for different integers *i* and *j*, 13. then set  $B_i := G[\bigcup_{B_i \in \mathcal{H}} V(B_i) \cup \{w\}], \mathcal{B} := (\mathcal{B} \setminus \mathcal{H}) \cup \{B_i\}, T := T \cup \{w\},$  $H := G[S \cup T]$ , and b := b - h + 1, where  $\mathcal{H} = \{B_i : N_G(B_i) \cap N_G(w) \neq \emptyset\}$ and  $h = |\mathcal{H}|$ , go to Step 2
- 14. else  $W := W \setminus \{w\}$
- 15.end if
- end while 16.
- 17. while  $R \neq \emptyset$ ,
- 18. pick  $r = (u, v) \in R$

956

19. **if**  $N_G(u) \cap V(B_i) \neq \emptyset$ ,  $N_G(u) \cap N_H(B_j) \neq \emptyset$ ,  $N_G(v) \cap V(B_i) \neq \emptyset$ , and  $N_G(v) \cap N_H(B_i) \neq \emptyset$  for different integers *i* and *j*, **then** set  $B_i := G[\bigcup_{B_i \in \mathcal{H}} V(B_i) \cup \{u, v\}]$ ,  $\mathcal{B} := (\mathcal{B} \setminus \mathcal{H}) \cup B_i$ ,  $T := T \cup \{u, v\}$ ,  $H := G[S \cup T]$ , and b := b - h + 1, where  $\mathcal{H} = \{H_i : N_G(H_i) \cap N_G(u) \neq \emptyset$  or  $N_G(H_i) \cap N_G(v) \neq \emptyset\}$  and  $h = |\mathcal{H}|$ , go to Step 2 20. **else**  $R := R \setminus \{r\}$ 21. **end if** 22. **end while** 23. **end if** 

**Remark 2.** Let  $s, \Delta, n$  and m be the size of a dominating set S, the maximum degree, order and size of G, respectively. Note that the time complexity of stage I can be expressed as  $O((s-1)\Delta(n+2m))$ , and the time complexity of II can be expressed as  $O(m+k\ell)$ . In III, since the running time of each recursion is at most  $\Delta(n+2m+n^2)$  and III runs at most s-1 recursions. Thus the time complexity of this algorithm is bounded by  $O((s-1)\Delta(m+n^2))$ .

## 2.3. 2-connected dominating set

Let G be a connected graph which is not complete, let X be a vertex cut of G, and let Y be the vertex set of a component of G - X. The subgraph H of G induced by  $X \cup Y$  is called an X-component of G. We simply write x-component if  $X = \{x\}$ .

**Lemma 7.** Let S be a dominating set of a 2-edge connected graph G with  $|S| \ge 2$ . If T is an output of Algorithm 2 for G and S, and  $T' \subseteq T$  is an output of stage I of Algorithm 2 for G and S, then the following is true for  $H = G[S \cup T]$ :

- (i) if u is a cut vertex in H, then  $u \in S \cup T'$ ,
- (ii)  $b(H) \leq 2|S| 2$ , where b(H) is the number of blocks in H.

**Proof.** To show (i), it suffices to show that each vertex  $u \in T \setminus T'$  is not a cut vertex of H. Since T' is an output of stage I of Algorithm 2 for G and  $S, S \cup T'$  is a connected dominating set of G, and thus  $S \cup T'$  is also a connected dominating set of H. Therefore H - u is connected, i.e., u is not a cut vertex of H.

Suppose that (ii) is not true, and G is a graph of minimum order satisfying the conditions of this lemma but  $b(H) > 2|S| - 2 \ge 2$ . If |S| = 2, then  $b(H) \le 2$ , and thus  $b(H) = 2 \le 2|S| - 2$ , a contradiction. So,  $|S| \ge 3$ . Let u be a cut vertex of H. We consider the following two cases according to (i).

Case 1.  $u \in S$ . Let  $H_1, H_2, \ldots, H_k$  be the *u*-components of *H*. Clearly  $H_i$  is 2-edge connected. Let  $S_i = V(H_i) \cap S$  and  $T_i = V(H_i) \setminus S_i$  for  $i = 1, 2, \ldots, k$ . Since  $T_i$  is a possible output of Algorithm 2 for  $H_i$  and  $S_i$ , we have  $b(H_i) =$ 

 $b(G[S_i \cup T_i]) \le 2|S_i| - 2$  by the minimality of G. Thus  $b(H) = \sum_{i=1}^k b(H_i) \le \sum_{i=1}^k (2|S_i|-2) \le 2\sum_{i=1}^k |S_i| - 2k = 2(|S|+k-1) - 2k = 2|S|-2$ , a contradiction.

Case 2.  $u \in T'$ . Let  $H_1, H_2, \ldots, H_k$  be the *u*-components of *H*. Clearly  $H_i$  is 2-edge connected. Let  $S_i = V(H_i) \cap S$  and  $T_i = V(H_i) \setminus S_i$  for  $i = 1, 2, \ldots, k$ . Without loss of generality, let  $N_{H_i}(u) \cap S_i \neq \emptyset$  for  $1 \leq i \leq r$  for an integer *r* and  $N_{H_j}(u) \cap S_j = \emptyset$ ,  $r < j \leq k$ . Since *S* is a dominating set of *H*,  $r \geq 1$ .

When  $1 \le i \le r$ , since  $T_i$  is a possible output of Algorithm 2 for  $H_i$  and  $S_i$ , we have  $b(H_i) = b(G[S_i \cup T_i]) \le 2|S_i| - 2$  by the minimality of G.

When  $r < j \le k$ , let  $S'_j = S_j \cup \{u\}$  and  $T'_j = (T_j \setminus u)$ . Since  $T'_j$  is a possible output of Algorithm 2 for  $H_j$  and  $S'_j$ , we have  $b(H_j) = b(G[S_j \cup T_j]) \le 2|S'_j| - 2 = 2(|S_j| + 1) - 2$  by the choice of G.

Thus  $b(H) = \sum_{i=1}^{k} b(H_i) = \sum_{i=1}^{r} (2|S_i| - 2) + \sum_{j=r+1}^{k} (2(|S_j| + 1) - 2) \le \sum_{i=1}^{r} (2|S_i| - 2) + \sum_{j=r+1}^{k} (2|S_j|) \le 2|S| - 2r \le 2|S| - 2$ , a contradiction. This shows (ii).

**Theorem 8.** Let G be a 2-connected triangle-free graph G. If S is a dominating set of G with  $|S| \ge 2$ , then there exists a set  $T \subseteq V(G)$  such that  $|T| \le 10|S| - 13$  and  $G[S \cup T]$  is 2-connected.

**Proof.** Let T be an output of Algorithm 2 for G and S. We may suppose that  $G[S \cup T]$  is not 2-connected and let b(H) be the number of blocks in  $H = G[S \cup T]$ . Since H is 2-edge connected, each block of H is 2-edge connected. Let u be a cut vertex in H, let  $B_1$  and  $B_2$  be two blocks of H such that  $u \in V(B_1) \cap V(B_2)$ , and let  $H_1$  and  $H_2$  be u-components such that  $B_i \subseteq H_i$  for i = 1, 2.

Let  $P = x_1 x_2 \cdots x_k$  be a shortest path connecting  $V(H_1)$  and  $V(H_2)$  in  $G \setminus u$ where  $x_1 \in V(H_1)$ ,  $x_k \in V(H_2)$  and  $x_2, x_3, \ldots, x_{k-1} \notin V(H_1) \cup V(H_2)$ . Suppose  $k \ge 6$ . Then  $ux_3, ux_4 \in E(G)$  since  $S \subseteq V(H)$  is a dominating set of G, and P is a shortest path connecting  $V(H_1)$  and  $V(H_2)$  in  $G \setminus u$ . Thus  $ux_3x_4$  is a triangle, a contradiction. Thus  $k \le 5$ . Let  $T' = V(P) \setminus \{x_1, x_k\}$ . Then  $|T'| \le 3$ .

Hence  $S \cup T \cup T'$  is a 2-edge connected dominating set of G with  $|T \cup T'| \le |T| + 2 \le 4|S| - 4 + 3$  and  $b(G[S \cup T \cup T']) \le b(G[S \cup T]) - 1 = b(H) - 1$ . If  $G[S \cup T \cup T']$  is 2-connected, then we are done by letting  $T := T \cup T'$ . Otherwise, let  $T := T \cup T'$ , and repeat the above operation until  $G[S \cup T]$  is 2-connected.

Since  $|T| \leq 4|S|-4$ ,  $b(H) \leq 2|S|-2$ , |T| increases by at most three and b(H) decreases by at least one in each iteration of the above operation, we conclude that the desired set T exists since  $|T| \leq 4|S|-4+3(b(H)-1)=10|S|-13$ .

**Corollary 4.** For a 2-connected triangle-free graph G, if  $\gamma(G) \ge 2$ , then  $\gamma_2(G) \le 11\gamma(G) - 13$ .

**Remark 3.** For a graph with triangle, Theorem 8 does not holds. For example, let G be the graph in Figure 1. Since  $\{u, v, w\}$  is a smallest dominating set

and any proper subgraph of G is not 2-connected, we have that  $\gamma(G) = 3$  but  $\gamma_2(G) = V(G)$ , that is, there is not a constant k such that  $\gamma_2(G) \leq k\gamma(G)$  for graphs with triangle. So the condition that G is triangle-free is indispensable.



Figure 1. A graph with  $\gamma(G) = 3$  but  $\gamma_2 = V(G)$ .

Algorithm 3. An algorithm for constructing a 2-connected dominating set. **Input:** A 2-connected graph G, a dominating set S with at least 2 vertices. **Output:** A set T such that  $|T| \leq 10|S| - 13$  and  $G[S \cup T]$  is 2-connected. I. run Algorithm 2. **II.** run DFS to get all blocks of  $G[S \cup T]$ , say  $B_1, B_2, \ldots, B_k$ **III.** 1. set  $H := G[S \cup T], \mathcal{B} = \{B_i : 1 \le i \le k\}, b := |\mathcal{B}|$ 2. if b = 1, then stop 3. else set  $W := V \setminus V(H)$  and F := E(G[W])4. while  $F \neq \emptyset$ 5.pick  $f = uv \in F$ if  $N_G(u) \cap V(B_i) \neq \emptyset$  and  $w \in N_G(v) \cap N_G(B_j) \neq \emptyset$ , then set  $B_i :=$ 6.  $G[\bigcup_{B_i \in \mathcal{H}} V(B_i) \cup \{u, v\}], \mathcal{B} := (\mathcal{B} \setminus \mathcal{H}) \cup \{B_i\}, T := T \cup \{u, v, w\}, H := G[S \cup \mathcal{H}) \cup \{u, v\}, H := G[S \cup \mathcal{H$  $T], b := b - h + 1, \text{ where } \mathcal{H} = \{B_i : V(B_i) \cap N_G(u) \neq \emptyset \text{ or } V(B_i) \cap N_G(v) \neq \emptyset\},\$ and  $h = |\mathcal{H}|$ , go to Step 2 else  $F := F \setminus \{f\}$ 7. end if 8. 9. end while while  $W \neq \emptyset$ 10. pick  $w \in W$ 11. 12.if  $N_G(w) \cap V(B_i) \neq \emptyset$  and  $N_G(w) \cap V(B_j) \neq \emptyset$ , then set  $B_i :=$  $G[\bigcup_{B_i \in \mathcal{H}} V(B_i) \cup \{w\}], \ \mathcal{B} := (\mathcal{B} \setminus \mathcal{H}) \cup \{B_i\}, \ T := T \cup \{w\}, \ H := G[S \cup T],$ b := b - h + 1, where  $\mathcal{H} = \{B_i : V(B_i) \cap N_G(w) \neq \emptyset\}$ , and  $h = |\mathcal{H}|$ , go to Step 2 13.else  $F := F \setminus \{f\}$ end if 14. 15. end while 16. end if

**Remark 4.** Let s,  $\Delta$ , n and m be the size of a dominating set S, the maximum degree, order and size of G, respectively. Note the time complexity of stage I is  $O((s-1)\Delta(m+n^2))$ , and the time complexity of II is O(m). In III, since the running time of each recursion is at most  $2\Delta n^2$  and III implements at most s-1 recursions. Thus the time complexity of the algorithm is bounded by  $O((s-1)\Delta(m+n^2))$ .

## 3. Concluding Remarks

Let  $P = u_0 u_1 \cdots u_{3k}$  and  $Q = v_0 v_1 \cdots v_{3k}$  be two path of length 3k. The symbol G denotes the graph obtained from P and Q by identifying  $u_{3i}$  and  $v_{3i}$  (denote the resulting vertex by  $w_{3i}$ ), where  $0 \le i \le n$ . It is easy to check that G is 2-edge connected and  $S = \{w_{3i} : 0 \le i \le n\}$  is a dominating set. Note that  $T = \{u_{3i+1}, u_{3i+2} : 0 \le i \le n-1\}$  and  $T' = \{v_{3i+1}, v_{3i+2} : 0 \le i \le n-1\}$  are minimum sets of vertices such that  $G[S \cup T]$  and  $G[S \cup T']$  are connected, and  $Q = T \cup T'$  is the unique set of vertices such that  $G[S \cup Q]$  is 2-edge connected. Thus the bounds given in Theorem 2, 6 and Corollary 3 are sharp.

We suspect that the bound of Theorem 8 is not sharp and the best possible bound might be the following.

**Conjecture 2.** For a dominating set S of a 2-connected triangle-free graph G with  $|S| \ge 2$ , there exists a vertex set  $T \subseteq V(G)$  with  $|T| \le 5|S|$  such that  $G[S \cup T]$  is 2-connected.

Inspired by Corollaries 1, 3 and 4, one may ask the following two problems.

**Problem 4.** Does there exist an absolute constant  $c'_k$  for a given integer  $k \ge 1$  such that  $\gamma'_k(G) \le c'_k \gamma(G)$  for any k-edge connected graph G?

**Problem 5.** Does there exist an absolute constant  $c_k$  for a given integer  $k \ge 1$  such that  $\gamma_k(G) \le c_k \gamma(G)$  for any k-connected graph G?

By our main results,  $c'_k$  and  $c_k$  exist for  $1 \le k \le 2$ . But,  $c'_k$  and  $c_k$  do not exist for an integer  $k \ge 3$ . Let  $C_n$  and  $K_{k-2}$  be the cycle of order n and the complete graph of order k-2. Let  $G_{n,k} = C_n \lor K_{k-2}$ , be the graph obtained from  $C_n$  and  $K_{k-2}$  by joining every vertex of  $C_n$  to all vertices of  $K_{k-2}$ . It is clear that  $G_{n,k}$  is k-connected, and thus k-edge connected. But,  $\gamma(G_{n,k}) = 1$  and  $\gamma'_k(G_{n,k}) = \gamma_k(G_{n,k}) = n + k$ .

## Acknowledgments

We would like to thank the referees for helpful suggestions and comments. The first author gratefully acknowledges the support by NSFC No. 1140118 and Foun-

960

dation of Henan Educational Committee 15A110032. The second author gratefully acknowledges the support by NSFC No. 11571294. The third author gratefully acknowledges the support by NSFC No. 11671296.

#### References

- L. Arseneau, A. Finbow, B. Hartnell, A. Hynick, D. MacLean and L. O'Sullivan, On minimal connected dominating sets, J. Combin. Math. Combin. Comput. 24 (1997) 185–191.
- C. Bo and B. Liu, Some inequalities about the connected domination number, Discrete Math. 159 (1996) 241-245. doi:10.1016/0012-365X(95)00088-E
- [3] J.A. Bondy and U.S.R. Murty, Graph Theory (GTM 244, Springer, London, 2008).
- M. Chellali, O. Favaron, A. Hansberg and L. Volkmann, k-domination and k-independence in graphs: A survey, Graphs Combin. 28 (2012) 1–55. doi:10.1007/s00373-011-1040-3
- C.J. Colbourn and L.K. Stewart, Permutaion graphs: Connected domination and Steiner trees, Discrete Math. 86 (1990) 179–189. doi:10.1016/0012-365X(90)90359-P
- [6] D.-Z. Du and P.-J. Wan, Connected Dominating Set: Theory and Applications (Springer, New York, 2013). doi:10.1007/978-1-4614-5242-3
- Y.L. Du and H.W. Du, A new bound on maximum independent set and minimum connected dominating set in unit disk graphs, J. Comb. Optim. **30** (2015) 1173–1179. doi:10.1007/s10878-013-9690-0
- [8] P. Duchet and H. Meyniel, On Hadwiger's number and the stability number, North-Holland Math. Studies 62 (1982) 71–73. doi:10.1016/S0304-0208(08)73549-7
- S. Guha and S. Khuller, Approximation algorithms for connected dominating sets, Algorithmica 20(1998) 374–387. doi:10.1007/PL00009201
- [10] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in Graphs: The Theory (Marcel Dekker, New York, 1997).
- [11] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in Graphs: Selected Topics (Marcel Dekker, New York, 1997).
- [12] M. Li, P.-J. Wan and F. Yao, Tighter approximation bounds for minimum CDS in unit disk graphs, Algorithmica 61 (2011) 1000–1021. doi:10.1007/s00453-011-9512-7
- X. Li and Z. Zhang, Two algorithms for minimum 2-connected r-hop dominating set, Inform. Process. Lett. 110 (2010) 986–991. doi:10.1016/j.ipl.2010.08.008

- M. Moscarini, Doubly chordal graphs, Steiner trees, and connected domination, Networks 23 (1993) 59–69. doi:10.1002/net.3230230108
- [15] E. Sampathkumar and H.B. Walikar, The connected domination number of a graphs, J. Math. Phys. Sci. 13 (1979) 607–613.
- [16] E. Vigoda, Lecture Notes on a Parallel Algorithm for Generating a Maximal Independent Set, Georgia Institute of Technology, last updated for 7530 – Randomized Algorithms, Spring 2010.
- [17] P.-J. Wan, L. Wang and F. Yao, Two-phased approximation algorithms for minimum CDS in wireless ad hoc networks, in: IEEE ICDCS (2008) 337–344. doi:10.1109/ICDCS.2008.15
- [18] K. White, M. Farber and W.R. Pulleyblank, Steiner trees, connected domination and strongly chordal graphs, Networks 15 (1985) 109–124. doi:10.1002/net.3230150109
- [19] W. Wu, H. Du, X. Jia, Y. Li and S. Huang, Minimum connected dominating sets and maximal independent sets in unit disk graphs, Theoret. Comput. Sci. 352 (2006) 1–7. doi:10.1016/j.tcs.2005.08.037
- [20] W. Wu, X. Gao, P.M. Pardalos and D.-Z. Du, Wireless networking, dominating and packing, Optim. Lett. 4 (2010) 347–358. doi:10.1007/s11590-009-0151-8

Received 20 February 2016 Revised 3 March 2017 Accepted 8 March 2017