

EQUITABLE COLORING AND EQUITABLE  
CHOOSABILITY OF GRAPHS WITH SMALL  
MAXIMUM AVERAGE DEGREE<sup>1</sup>

AIJUN DONG

*School of Science*  
*Shandong Jiaotong University*  
*Jinan, 250023, P.R. China*

**e-mail:** dongaijun@mail.sdu.edu.cn

AND

XIN ZHANG

*School of Mathematics and Statistics*  
*Xidian University*  
*Xi'an, 710071, P.R. China*

**e-mail:** xzhang@xidian.edu.cn

**Abstract**

A graph is said to be equitably  $k$ -colorable if the vertex set  $V(G)$  can be partitioned into  $k$  independent subsets  $V_1, V_2, \dots, V_k$  such that  $||V_i| - |V_j|| \leq 1$  ( $1 \leq i, j \leq k$ ). A graph  $G$  is equitably  $k$ -choosable if, for any given  $k$ -uniform list assignment  $L$ ,  $G$  is  $L$ -colorable and each color appears on at most  $\left\lceil \frac{|V(G)|}{k} \right\rceil$  vertices. In this paper, we prove that if  $G$  is a graph such that  $\text{mad}(G) < 3$ , then  $G$  is equitably  $k$ -colorable and equitably  $k$ -choosable where  $k \geq \max\{\Delta(G), 4\}$ . Moreover, if  $G$  is a graph such that  $\text{mad}(G) < \frac{12}{5}$ , then  $G$  is equitably  $k$ -colorable and equitably  $k$ -choosable where  $k \geq \max\{\Delta(G), 3\}$ .

**Keywords:** graph coloring, equitable choosability, maximum average degree.

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## 1. INTRODUCTION

The terminology and notation used but undefined in this paper can be found in [1]. Let  $G = (V(G), E(G))$  be a graph. Let  $d_G(x)$ , or simply  $d(x)$ , denote the number of edges incident with the vertex (face)  $x$  in  $G$ . If  $d(x) = k$ ,  $d(x) \geq k$  and  $d(x) \leq k$ , then the vertex  $x$  is called a  $k$ -vertex,  $k^+$ -vertex and  $k^-$ -vertex, respectively. We use  $V(G)$ ,  $E(G)$ ,  $\Delta(G)$  and  $\delta(G)$  to denote the vertex set, edge set, maximum degree, and minimum degree of  $G$ , respectively. The *average degree* of a graph  $G$  is  $\frac{\sum_{v \in V(G)} d(v)}{|V(G)|}$ , and denote it by  $ad(G)$ . The *maximum average degree*  $mad(G)$  of  $G$  is the maximum of the average degree of its subgraphs. The *girth* of a planar graph is the length of a smallest cycle in the graph, and denote the girth of a graph  $G$  by  $g(G)$ . We use  $\lceil x \rceil$  to denote a minimum integer which is no less than  $x$ .

A *proper  $k$ -coloring* of a graph  $G$  is a mapping  $\pi$  from the vertex set  $V(G)$  to the set of colors  $\{1, 2, \dots, k\}$  such that  $\pi(x) \neq \pi(y)$  for every edge  $xy \in E(G)$ . A graph  $G$  is *equitable  $k$ -colorable* if  $G$  has a proper  $k$ -coloring such that the size of the color classes differ by at most 1. The *equitable chromatic number* of  $G$ , denoted by  $\chi_e(G)$ , is the smallest integer  $k$  such that  $G$  is equitably  $k$ -colorable. The *equitable chromatic threshold* of  $G$ , denoted by  $\chi_e^*(G)$ , is the smallest integer  $k$  such that  $G$  is equitably  $l$ -colorable (for any  $l \geq k$ ).

In 1970, Hajnal and Szemeredi proved that  $\chi_e^*(G) \leq \Delta(G) + 1$  for any graph  $G$  [9]. This bound is sharp as shown in the example of  $K_{2n+1, 2n+1}$ . In 1973, Meyer introduced the notion of equitable coloring and made the following conjecture.

**Conjecture 1.1** (Meyer [18]). *If  $G$  is a connected graph which is neither a complete graph nor odd cycle, then  $\chi_e(G) \leq \Delta(G)$ .*

In 1994, Chen, Lih and Wu put forth the following conjecture.

**Conjecture 1.2** (Chen, Lih and Wu [2]). *For any connected graph  $G$ , if it is different from a complete graph, a complete bipartite graph and an odd cycle, then  $\chi_e^*(G) \leq \Delta(G)$ .*

Chen, Lih and Wu [2, 3] proved Conjecture 1.2 for graphs with  $\Delta(G) \leq 3$  or  $\Delta(G) \geq \frac{|V(G)|}{2}$ . In 2012, Chen *et al.* [4] improved the former result and confirmed the Conjecture 1.2 for graphs with  $\Delta(G) \geq \frac{|V(G)|}{3} + 1$ . Yap and Zhang [26, 27] showed that Conjecture 1.2 holds for planar graphs with  $\Delta(G) \geq 13$ . In 2012, Nakprasit [19] confirmed the Conjecture 1.2 for planar graphs with  $\Delta(G) \geq 9$ . Lih and Wu [14] verified  $\chi_e^*(G) \leq \Delta(G)$  for bipartite graphs other than complete bipartite graphs. Wang and Zhang [23] proved Conjecture 1.2 for line graphs, and Kostochka and Nakprasit [12, 13] proved it for graphs with low average degree, and  $d$ -degenerate graphs with  $\Delta(G) \geq 14d + 1$ . Yan and Wang [25] showed that Conjecture 1.2 holds for Kronecker products of complete multipartite graphs and

complete graphs. Wu and Wang [24], Luo *et al.* [17] confirmed Conjecture 1.2 for some planar graphs with large girth, respectively. Li *et al.* [16], Zhu *et al.* [29], Dong *et al.* [5–8], Nakprasit [20] confirmed Conjecture 1.2 for some planar graphs with some forbidden cycles. Zhang and Wu [28], Zhu and Bu [30] verified the Conjecture 1.2 for some series-parallel graphs and outerplanar graphs, respectively.

For a graph  $G$  and a list assignment  $L$  assigning to each vertex  $v \in V(G)$  a set  $L(v)$  of acceptable colors, an  $L$ -coloring of  $G$  is a proper vertex coloring such that for every  $v \in V(G)$  the color on  $v$  belongs to  $L(v)$ . A list assignment  $L$  for  $G$  is  $k$ -uniform if  $|L(v)| = k$  for all  $v \in V(G)$ . A graph  $G$  is *list equitably  $k$ -colorable* (also called *equitably  $k$ -choosable*) if, for any  $k$ -uniform list assignment  $L$ ,  $G$  is  $L$ -colorable and each color appears on at most  $\left\lceil \frac{|V(G)|}{k} \right\rceil$  vertices.

In 2003, Kostochka, Pelsmayer and West investigated the list equitable coloring of graphs. They proposed the following conjectures.

**Conjecture 1.3** (Kostochka, Pelsmayer and West [11]). *Every graph  $G$  is equitably  $k$ -choosable whenever  $k > \Delta(G)$ .*

**Conjecture 1.4** (Kostochka, Pelsmayer and West [11]). *If  $G$  is a connected graph with maximum degree at least 3, then  $G$  is equitably  $\Delta(G)$ -choosable, unless  $G$  is a complete graph or is  $K_{k,k}$  for some odd  $k$ .*

It has been proved that Conjecture 1.3 holds for graphs with  $\Delta(G) \leq 3$  in [21, 22] and then the result was strengthened by Kierstead and Kostochka. They confirmed the Conjecture 1.3 for graphs with  $\Delta(G) \leq 7$  in [10]. Kostochka, Pelsmayer and West proved that a graph  $G$  is equitably  $k$ -choosable if either  $G \neq K_{k+1}, K_{k,k}$  (with  $k$  odd in  $K_{k,k}$ ) and  $k \geq \max\left\{\Delta, \frac{|V(G)|}{2}\right\}$ , or  $G$  is a connected interval graph and  $k \geq \Delta(G)$  or  $G$  is a 2-degenerate graph and  $k \geq \max\{\Delta(G), 5\}$  in [11]. Pelsmayer proved that every graph is equitably  $k$ -choosable for any  $k \geq \frac{\Delta(G)(\Delta(G)-1)}{2} + 2$  in [21]. In 2009, Conjecture 1.4 were proved for planar graphs  $G$  without 4- and 6-cycles and with  $\Delta(G) \geq 6$  by Li *et al.* in [16]. Zhu *et al.* confirmed Conjecture 1.4 for planar graph  $G$  without 3-cycles and with  $\Delta(G) \geq 8$ , planar graph  $G$  without 4- and 5-cycles and with  $\Delta(G) \geq 7$  in [29],  $C_5$ -free planar graph  $G$  without adjacent triangles and with  $\Delta(G) \geq 8$  in [30], outerplanar graphs in [31]. Zhang and Wu proved Conjecture 1.4 for series-parallel graphs in [28]. More results can be seen in [5–8] and [15].

As for the sparse graph  $G$  with  $\Delta(G) = 2$ , it is clear that  $G$  is equitably  $k$ -colorable and equitably  $k$ -choosable where  $k \geq \max\{\Delta(G), 3\}$ , if  $G$  is an odd cycle. Otherwise,  $G$  is equitably  $k$ -colorable and equitably  $k$ -choosable where  $k \geq \max\{\Delta(G), 2\}$ . In this paper, we consider the sparse graph  $G$  with  $\Delta(G) \geq 3$  and show that if  $G$  is a graph such that  $\text{mad}(G) < 3$ , then  $G$  is equitably  $k$ -colorable and equitably  $k$ -choosable where  $k \geq \max\{\Delta(G), 4\}$ . Moreover, if  $G$  is

a graph such that  $\text{mad}(G) < \frac{12}{5}$ , then  $G$  is equitably  $k$ -colorable and equitably  $k$ -choosable where  $k \geq \max\{\Delta(G), 3\}$ .

## 2. SOME IMPORTANT LEMMAS

**Lemma 2.1** (Kostochka, Pelsmayer and West [11]). *Let  $G$  be a graph with a  $k$ -uniform list assignment  $L$ . Let  $S = \{v_1, v_2, \dots, v_k\}$ , where  $\{v_1, v_2, \dots, v_k\}$  are distinct vertices in  $G$ . If  $G - S$  has an equitable  $L$ -coloring and  $|N_G(v_i) - S| \leq k - i$  for  $1 \leq i \leq k$ , then  $G$  has an equitable  $L$ -coloring.*

**Lemma 2.2** (Zhu and Bu [29]). *Let  $S = \{v_1, v_2, \dots, v_k\}$  be a set of  $k$  different vertices in  $G$  such that  $G - S$  has an equitable  $k$ -coloring. If  $|N_G(v_i) - S| \leq k - i$  for  $1 \leq i \leq k$ , then  $G$  has an equitable  $k$ -coloring.*

**Lemma 2.3** (Hajnal and Szemerédi [9]). *Every graph has an equitable  $k$ -coloring whenever  $k \geq \Delta(G) + 1$ .*

**Lemma 2.4** (Pelsmayer, Wang and Lih [21, 22]). *Every graph  $G$  with maximum degree  $\Delta(G) \leq 3$  is equitably  $k$ -choosable whenever  $k \geq \Delta(G) + 1$ .*

**Lemma 2.5.** *Let  $G$  be a graph with  $\text{mad}(G) < 3$ . Then  $G$  is 2-degenerate.*

**Proof.** By contradiction, there is subgraph  $G'$  of  $G$  such that  $\delta(G') \geq 3$ . It is clear that  $\text{mad}(G') \geq 3$ , a contradiction. ■

**Lemma 2.6** (Dong, Zou and Li [8]). *If  $G$  is a graph such that  $\text{mad}(G) \leq 3$ , then  $G$  is equitably  $k$ -colorable and equitably  $k$ -choosable where  $k \geq \max\{\Delta(G), 5\}$ .*

## 3. GRAPHS WITH $\text{mad}(G) < 3$

**Lemma 3.1.** *Let  $G$  be a connected graph with order at least 4 and  $\delta(G) \geq 1$ . If  $\Delta(G) \leq 4$  and  $\text{mad}(G) < 3$ , then  $G$  has at least one of the structures in Figure 1.*

**Proof.** Let  $G$  be a counterexample. Then  $G$  does not contain any configuration  $H_1 \sim H_6$  presented in Figure 1.

For each  $v \in V(G)$ , if  $d(v) = 2$ , then  $v$  is adjacent to at least one 4-vertex for the reason that  $G$  contains no structure  $H_1$ . If  $d(v) = 4$ , then  $v$  is adjacent to at most one 2-vertex for the reason that  $G$  contains no structure  $H_2$ . For convenience, let  $r$  denote the number of 4-vertices which are not adjacent to any 2-vertex. Obviously,  $G$  has the following property.

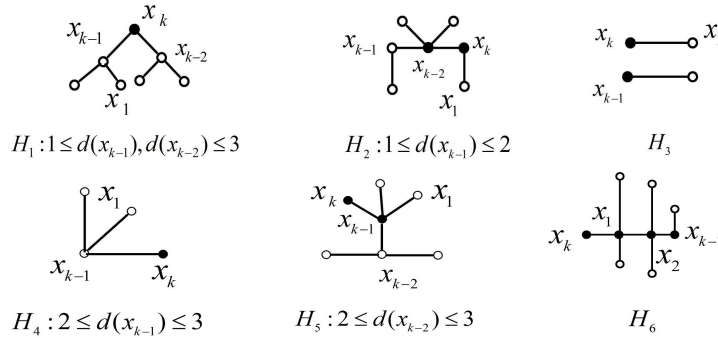


Figure 1

Each configuration depicted in Figure 1 is such that: (1) hollow vertices may be not distinct while solid vertices are distinct, (2) the degree of the solid vertices is fixed, and (3) except for specially pointed, the degree of a hollow vertices may be any integer from  $[d, \Delta(G)]$ , where  $d$  is the number of edges incident with the hollow vertex in the configuration.

**Observation 3.2.**  $n_4(G) \geq n_2(G) + r$ .

By Lemma 2.5, we have  $\delta(G) \leq 2$ .

Suppose  $\delta(G) = 2$ . By Observation 3.2, we have  $ad(G) = \frac{2n_2(G)+3n_3(G)+4n_4(G)}{n_2(G)+n_3(G)+n_4(G)} \geq \frac{2n_2(G)+3n_3(G)+4(n_2(G)+r)}{n_2(G)+n_3(G)+n_2(G)+r} = \frac{6n_2(G)+3n_3(G)+4r}{2n_2(G)+n_3(G)+r} = \frac{3[2n_2(G)+n_3(G)+r]+r}{2n_2(G)+n_3(G)+r} \geq 3$ , a contradiction to  $mad(G) < 3$ .

Suppose  $\delta(G) = 1$ . Since  $G$  contains no structure  $H_3$ , there is only one 1-vertex  $v$  in  $G$ . Furthermore, the vertex  $v$  must be adjacent to a 4-vertex  $u$  for the reason that  $G$  contains no structure  $H_4$ . Since  $G$  contains no structure  $H_5$ , the other adjacent vertices of  $u$  must be 4-vertices. For convenience, we use  $u_i$  ( $1 \leq i \leq 3$ ) to denote the 4-vertices which are adjacent to  $u$ . Since  $G$  contains no structure  $H_6$ ,  $u_i$  ( $1 \leq i \leq 3$ ) is not adjacent to any 2-vertex. From the above discussion, we have  $r \geq 4$ . Obviously, we have  $ad(G) = \frac{n_1(G)+2n_2(G)+3n_3(G)+4n_4(G)}{n_1(G)+n_2(G)+n_3(G)+n_4(G)} = \frac{1+2n_2(G)+3n_3(G)+4(n_2(G)+r)}{1+n_2(G)+n_3(G)+n_2(G)+r} = \frac{1+6n_2(G)+3n_3(G)+4r}{1+2n_2(G)+n_3(G)+r} = \frac{1+6n_2(G)+3n_3(G)+3r+4}{1+2n_2(G)+n_3(G)+r} = \frac{3[1+2n_2(G)+n_3(G)+r]+2}{1+2n_2(G)+n_3(G)+r} \geq 3$ , a contradiction to  $mad(G) < 3$ . ■

In the following, let us give the proof of the main theorems.

**Theorem 3.3.** *If  $G$  is a graph such that  $mad(G) < 3$ , then  $G$  is equitably  $k$ -colorable where  $k \geq \max\{\Delta(G), 4\}$ .*

**Proof.** By Lemma 2.6, we only need to focus on the situation where  $\Delta(G) \leq 4$ . Let  $G$  be a counterexample with the smallest number of vertices. Clearly,  $\delta(G) \geq 1$ . If each component of  $G$  has at most four vertices, then  $\Delta(G) \leq 3$ . So  $G$  is equitably  $k$ -colorable by Lemma 2.3. Otherwise, there is at least one component with at least four vertices. By Lemma 3.1,  $G$  has one of the structures  $H_1 \sim H_6$ , taking it and the vertices are labelled as they are in Figure 1. If there are vertices labelled repeatedly, then we take the larger ( $x_i$  is larger than  $x_{i-1}$ ). In the following, we show how to find  $S$  in Lemma 2.2. If  $G$  has  $H_1$ ,  $H_2$  or  $H_5$ , then let  $S' = \{x_k, x_{k-1}, x_{k-2}, x_1\}$ . If  $G$  has  $H_3$  or  $H_4$ , then let  $S' = \{x_k, x_{k-1}, x_1\}$ . If  $G$  has  $H_6$ , then let  $S' = \{x_k, x_{k-1}, x_2, x_1\}$ . By Lemma 2.5,  $G$  is 2-degenerate, thus we can find the remaining unspecified positions in  $S$  from highest to lowest indices by choosing a vertex with minimum degree in the graph obtained from  $G$  by deleting the vertices already being chosen for  $S$  at each step. By the minimality of  $|V(G)|$  and since  $k \geq \Delta(G) \geq \Delta(G - S)$ ,  $G - S$  is equitably  $k$ -colorable. So  $G$  is also equitably  $k$ -colorable by Lemma 2.2. ■

**Corollary 3.4.** *Let  $G$  be a graph such that  $\text{mad}(G) < 3$ . If  $\Delta(G) \geq 4$ , then  $\chi_e(G) \leq \Delta(G)$ .*

**Corollary 3.5.** *Let  $G$  be a graph such that  $\text{mad}(G) < 3$ . If  $\Delta(G) \geq 4$ , then  $\chi_e^*(G) \leq \Delta(G)$ .*

**Theorem 3.6.** *If  $G$  is a graph such that  $\text{mad}(G) < 3$  and  $k \geq \max\{4, \Delta(G)\}$ , then  $G$  is equitably  $k$ -choosable.*

**Proof.** Let  $G$  be a counterexample with the smallest number of vertices. If each component of  $G$  has at most 4 vertices, then  $\Delta(G) \leq 3$ . So  $G$  is equitably  $k$ -choosable by Lemma 2.4. Otherwise, the statement is similar to that in the corresponding cases of Theorem 3.3. By Lemma 2.1 and Lemma 2.4, we have this theorem. ■

**Corollary 3.7.** *Let  $G$  be a graph such that  $\text{mad}(G) < 3$ . If  $\Delta(G) \geq 4$ , then  $G$  is equitably  $\Delta(G)$ -choosable.*

For a planar graph with girth  $g$ , by  $\text{mad}(G) < \frac{2g}{g-2}$ , we have the following corollary.

**Corollary 3.8.** *Let  $G$  be a planar graph with girth  $g \geq 6$ . If  $\Delta(G) \geq 4$ , then  $G$  is equitably  $\Delta(G)$ -colorable and equitably  $\Delta(G)$ -choosable.*

#### 4. GRAPHS WITH $\text{mad}(G) < \frac{12}{5}$

**Lemma 4.1.** *Let  $G$  be a connected graph with order at least 4 and  $\text{mad}(G) < \frac{12}{5}$ . Then  $G$  has at least one of the structures in Figure 2.*

**Proof.** Let  $G$  be a counterexample. Then  $G$  does not contain any configuration  $F_1 \sim F_4$  presented in Figure 2.

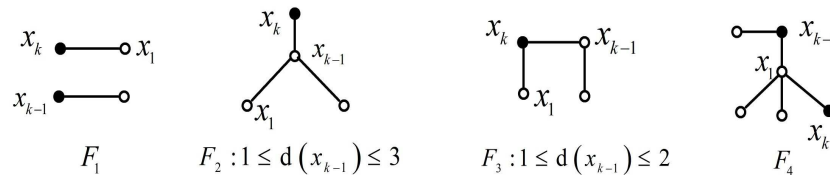


Figure 2

Each configuration depicted in Figure 2 is such that: (1) hollow vertices may be not distinct while solid vertices are distinct, (2) the degree of the solid vertices is fixed, and (3) except for specially pointed, the degree of a hollow vertices may be any integer from  $[d, \Delta(G)]$ , where  $d$  is the number of edges incident with the hollow vertex in the configuration.

In the following, we use the discharging method to get a contradiction. For every  $v \in V(G)$ , we define the original charge of  $v$  to be  $w(v) = d(v) - \frac{12}{5}$ . The total charge of the vertices of  $G$  is equal to

$$\sum_{v \in V(G)} \left( d(v) - \frac{12}{5} \right) = |V(G)| \times \left( ad(G) - \frac{12}{5} \right) \leq |V(G)| \times \left( mad(G) - \frac{12}{5} \right) < 0.$$

In the following, we redistribute the charge according to the given discharging rules and let  $w'(v)$  be the new charge of a vertex  $v \in V(G)$ , for convenience. If  $\sum_{v \in V(G)} w'(v) > 0$  can be deduced, we can show that the assumption is wrong.

Define discharging rules as the following statements.

**D1** Transfer charge  $\frac{7}{5}$  from each  $4^+$ -vertex to every adjacent 1-vertex.

**D2** Transfer charge  $\frac{1}{5}$  from each  $3^+$ -vertex to every adjacent 2-vertex.

In the following, let us check the charge of each element  $v$  for  $v \in V(G)$ . For each  $v \in V(G)$ , if  $d(v) = 1$ , then  $w(v) = -\frac{7}{5}$ . Since  $G$  contains no structure  $F_1$ , there is at most one 1-vertex in  $G$ . Furthermore, the 1-vertex must be adjacent to a  $4^+$ -vertex for the reason that  $G$  contains no structure  $F_2$ . So  $w'(v) \geq -\frac{7}{5} + \frac{7}{5} = 0$  by D1.

If  $d(v) = 2$ , then  $w(v) = -\frac{2}{5}$ . Since  $G$  contains no structure  $F_3$ ,  $v$  is not adjacent to any  $2^-$ -vertex. We have  $w'(v) \geq -\frac{2}{5} + \frac{1}{5} \times 2 = 0$  by D2.

If  $d(v) = 3$ , then  $w(v) = \frac{3}{5}$ . Since  $G$  contains no structure  $F_2$ ,  $v$  is not adjacent to any 1-vertex. Then we have  $w'(v) \geq \frac{3}{5} - \frac{1}{5} \times 3 = 0$  by D2.

Suppose  $d(v) \geq 4$ . Then  $w(v) = d(v) - \frac{12}{5}$ . Since  $G$  contains no structure  $F_4$ , the vertex  $v$  is adjacent to at most one 1-vertex. If  $v$  is adjacent to a 1-vertex,

then  $v$  is not adjacent to any  $2^-$ -vertex for the reason that  $G$  contains no structure  $F_4$ . We have  $w'(v) \geq d(v) - \frac{12}{5} - \frac{7}{5} \geq 4 - \frac{12}{5} - \frac{7}{5} = \frac{1}{5} > 0$  by  $D1$ . Otherwise, we have  $w'(v) \geq d(v) - \frac{12}{5} - \frac{1}{5} \times d(v) = \frac{4}{5}d(v) - \frac{12}{5} \geq \frac{4}{5} \times 4 - \frac{12}{5} = \frac{4}{5} > 0$  by  $D2$ .

From the above discussion, we have  $\sum_{v \in V(G)} w'(v) \geq 0$ , a contradiction. ■

In the following, let us give the proof of the main theorem.

**Theorem 4.2.** *If  $G$  is a graph such that  $\text{mad}(G) < \frac{12}{5}$ , then  $G$  is equitably  $k$ -colorable where  $k \geq \max\{\Delta(G), 3\}$ .*

**Proof.** Let  $G$  be a counterexample with smallest number of vertices. If each component of  $G$  has at most 3 vertices, then  $\Delta(G) \leq 2$ . So  $G$  is equitably  $k$ -colorable by Lemma 2.3. Otherwise, there is at least one component with at least four vertices. By Lemma 4.1,  $G$  has one of the structures  $F_1 \sim F_4$ , taking it and the vertices are labelled as they are in Figure 1. If there are vertices labelled repeatedly, then we take the larger ( $x_i$  is larger than  $x_{i-1}$ ). In the following, we show how to find  $S$  in Lemma 2.2. Let  $S' = \{x_k, x_{k-1}, x_1\}$ . By Lemma 2.5,  $G$  is 2-degenerate, hence we can find the remaining unspecified positions in  $S$  from highest to lowest indices by choosing a vertex with minimum degree in the graph obtained from  $G$  by deleting the vertices already being chosen for  $S$  at each step. By the minimality of  $|V(G)|$  and since  $k \geq \Delta(G) \geq \Delta(G - S)$ ,  $G - S$  is equitably  $k$ -colorable. So  $G$  is also equitably  $k$ -colorable by Lemma 2.2. ■

**Corollary 4.3.** *Let  $G$  be a graph such that  $\text{mad}(G) < \frac{12}{5}$ . If  $\Delta(G) \geq 3$ , then  $\chi_e(G) \leq \Delta(G)$ .*

**Corollary 4.4.** *Let  $G$  be a graph such that  $\text{mad}(G) < \frac{12}{5}$ . If  $\Delta(G) \geq 3$ , then  $\chi_e^*(G) \leq \Delta(G)$ .*

**Theorem 4.5.** *If  $G$  is a graph such that  $\text{mad}(G) < \frac{12}{5}$  and  $k \geq \max\{3, \Delta(G)\}$ , then  $G$  is equitably  $k$ -choosable.*

**Proof.** Let  $G$  be a counterexample with the smallest number of vertices. If each component of  $G$  has at most 3 vertices, then  $\Delta(G) \leq 2$ . So  $G$  is equitably  $k$ -choosable by Lemma 2.4. Otherwise, the statement is similar to that in the corresponding cases of Theorem 4.2. By Lemma 2.1 and Lemma 2.4, we have this theorem. ■

**Corollary 4.6.** *Let  $G$  be a graph such that  $\text{mad}(G) < \frac{12}{5}$ . If  $\Delta(G) \geq 3$ , then  $G$  is equitably  $\Delta(G)$ -choosable.*

For a planar graph with girth  $g$ , we have the following corollary.

**Corollary 4.7.** *Let  $G$  be a planar graph with girth  $g \geq 12$ . If  $\Delta(G) \geq 3$ , then  $G$  is equitably  $\Delta(G)$ -colorable and equitably  $\Delta(G)$ -choosable.*



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