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THE CROSSING NUMBERS OF JOIN OF SOME GRAPHS WITH n ISOLATED VERTICES

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Abstract

There are only few results concerning crossing numbers of join of some graphs. In this paper, for some graphs on five vertices, we give the crossing numbers of its join with n isolated vertices.

Keywords: disconnected graph, crossing number, join product.

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1. Introduction

For graph theory terminology not defined here, we direct the reader to [2]. A drawing of a graph $G = (V, E)$ is a mapping ϕ that assigns to each vertex in V a distinct point in the plane and to each edge uv in E a continuous arc (i.e., a homeomorphic image of a closed interval) connecting $\phi(u)$ and $\phi(v)$, not passing through the image of any other vertex. For simplicity, we impose the following conditions on a drawing: (a) no three edges have an interior point in common, (b) if two edges share an interior point p, then they cross at p, and (c) any two edges of a drawing have only a finite number of crossings (common interior points). The crossing number, $cr(G)$, of a graph G is the minimum number of edge crossings in any drawing of G. Let D be a drawing of the graph G, we denote the number of crossings in D by $cr_D(G)$.

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For a graph G, let $A, B \subseteq E(G)$, then, for a drawing D of G, let

$$cr_D(A,B) = \sum_{a\in A, b\in B} |D(a) \cap D(b)|.$$

Additionally, let $cr_D(A,A) = cr_D(A)$. Informally, $cr_D(A,B)$ denotes the number of crossings between every pair of edges where one edge is in A and the other in B. For three mutually disjoint subsets $A, B, C \subseteq E(G)$, the following equations hold:

$$cr_D(A \cup B) = cr_D(A) + cr_D(B) + cr_D(A,B),$$
$$cr_D(A, B \cup C) = cr_D(A,B) + cr_D(A,C).$$

For more about crossing number, we refer the reader to [3]. The investigation on the crossing number of graphs is a classical but very difficult problem. It is well known that there are only few results concerning crossing numbers of join of some graphs. The join product of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is obtained from vertex-disjoint copies of G_1 and G_2 by adding all edges between $V(G_1)$ and $V(G_2)$. For $|V(G_1)| = m$ and $|V(G_2)| = n$, the edge set of $G_1 + G_2$ is the union of disjoint edge sets of the graphs G_1, G_2, and the complete bipartite graph $K_{m,n}$. Let nK_1 denote the graph on n isolated vertices (i.e., the complement of the complete graph K_n).

It has been long conjectured in [8] that the crossing number of the complete bipartite graph $K_{m,n}$ equals $\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$. This conjecture was verified by Kleitman for $\min\{m,n\} \leqslant 6$ in [6]. Similarly, most results about $G_1 + G_2$ deal with the case $|V(G_1)| \leqslant 6$. Moreover, usually G_1 is connected and G_2 is some special graph, such as the graph nK_1, the path P_n on n vertices or the cycle C_n. The k-spoke wheel, denoted by W_k, has vertices $v_0, v_1, \ldots, v_k$, where $v_1, v_2, \ldots, v_k$ form a cycle, and v_0 is adjacent to all of $v_1, v_2, \ldots, v_k$.

Using Kleitman's result [6], the crossing numbers for join of two paths, join of two cycles, or for join of a path and a cycle were studied in [7]. Moreover, the exact values for crossing numbers of $G_1 + nK_1$ for all graphs G_1 of order at most four are given. In 1986, Asano started to study crossing numbers of multipartite complete graphs. In [1], he established the crossing numbers of the tripartite graphs $K_{1,3,n}$ and $K_{2,3,n}$ (namely $K_{1,3} + nK_1$ and $K_{2,3} + nK_1$). For the graph $K_{1,4,n}$ (namely $K_{1,4} + nK_1$), the crossing number was given independently in [4] and [5].

As it is difficult to determine the crossing number of join of the disconnected graph with nK_1, there are only few results concerning that. In this paper, we determine the crossing number for the join of nK_1 with a disconnected graph G on five vertices, as shown in Figure 1. The approach is seemingly new. As the disconnected graph may be a subgraph of many connected graphs, we can gain

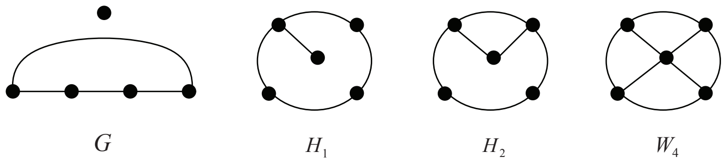

Figure 1. Some graphs on five vertices.

crossing numbers of join of some graphs H_1 and H_2 (see Figure 1) on five vertices with nK_1 directly. The crossing number for the graph $G + nK_1$ enables us, in Section 3, to give the crossing number of $W_4 + nK_1$ cleverly and simply.

In the paper, some proofs are based on Kleitman's result on crossing numbers of complete bipartite graphs. More precisely, he [6] proved that

$$cr(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \ m \leqslant 6.$$

For convenience, the number $\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ is often denoted by $Z(m, n)$ in our paper. In the proofs of the paper, we will often use the term "region" also in nonplanar drawings. In this case, crossings are considered to be vertices of the "map".

2. THE CROSSING NUMBER OF $G + nK_1$

The graph G in Figure 1 is isomorphic to the graph $C_4 \cup K_1$. The graph $G + nK_1$ consists of one copy of the graph G and n vertices $t_1, t_2, \ldots, t_n$, where every vertex $t_i, i = 1, 2, \ldots, n$, is adjacent to every vertex of G. For $i = 1, 2, \ldots, n$, let T_i denote the subgraph induced by five edges incident with the vertex t_i and let $F_i = G \cup T_i$. For the simpler labelling, let G_n denote the graph $G + nK_1$ in this paper. In Figure 2, we have $G + nK_1 = G_n = G \cup (\bigcup_{i=1}^n T_i)$, and we also have $\bigcup_{i=1}^n T_i = K_{5,n}$.

Lemma 1. *$cr(G + nK_1) \leqslant Z(5, n) + \lfloor \frac{n}{2} \rfloor$ for $n \geqslant 1$.*

Proof. We will display a drawing ϕ of G_n in the plane such that $cr_\phi(G_n) = Z(5, n) + \lfloor \frac{n}{2} \rfloor$. The desired drawing ϕ is constructed as follows (see Figure 2).

(i) Set all vertices of G on y-axis, $\lceil \frac{n}{2} \rceil$ isolated vertices on the negative x-axis and $\lfloor \frac{n}{2} \rfloor$ isolated vertices on the positive x-axis.

(ii) The image of each edge for G is a thick line segment.

Then it is not difficult to see that $cr_\phi(G_n) = Z(5, n) + \lfloor \frac{n}{2} \rfloor$, and so $cr(G + nK_1) \leqslant Z(5, n) + \lfloor \frac{n}{2} \rfloor$. ∎

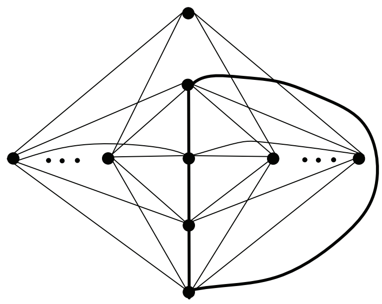

Figure 2. A drawing ϕ of G_n.

Lemma 2. *$cr(G+K_1)=0$, $cr(G+2K_1)=1$.*

Proof. The graph $G+K_1$ is planar, so $cr(G+K_1)=0$. As the graph $G+2K_1$ contains a subgraph which is homeomorphic to $K_{3,3}$, then $cr(G+2K_1) \geqslant 1$. On the other hand, by Lemma 1, we have $cr(G+2K_1) \leqslant 1$. This completes the proof. ■

Lemma 3. *Let D be an optimal drawing of G_n and let C_4 be the 4-cycle of G. Then we have $cr_D(C_4)=0$.*

Proof. We assume that there exists an optimal drawing D of G_n such that $cr_D(C_4) \neq 0$. There exist two crossed edges $e, f \in E(C_4)$. We assume that $e = y_iy_j$, $f = y_ky_l$, where i, j, k, l are distinct and the 4-cycle of G is $y_iey_jy_lfy_ky_i$. For convenience, we denote the crossing between e and f by v. In the following, we shall produce a new good drawing D^* of G_n as shown in Figure 3.

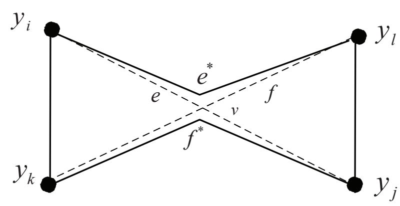

Figure 3. A 4-cycle of G.

First, we connect y_i to y_l sufficiently close to the section between y_i and v of e and the section between y_l and v of f, then we get a new edge $e^* = y_iy_l$. Analogously, we can get another new edge $f^* = y_jy_k$. Second, we delete two original edges e and f. In this way, we produce a new 4-cycle $(y_ie^*y_ly_jf^*y_ky_i)$ and a new good drawing D^* of G_n, such that the crossing v in D is deleted in D^*, the other crossings are unchanged from D to D^*, and there is no new crossing occurring in D^*. Now $cr_{D^*}(G_n) = cr_D(G_n) - 1$ contradicts to D being an optimal drawing. ■

Theorem 4. *$cr(G+nK_1) = Z(5,n) + \lfloor \frac{n}{2} \rfloor$ for $n \geqslant 1$.*

Proof. Lemma 1 shows that $cr(G+nK_1) \leqslant Z(5,n) + \lfloor \frac{n}{2} \rfloor$ and that the theorem is true if the equality holds. We prove the reverse inequality by induction on n. Lemma 2 implies that the result is true for the case $n = 1, 2$.

Suppose now that for $3 \leqslant k \leqslant n-1$

$$cr(G+kK_1) \geqslant Z(5,k) + \left\lfloor \frac{k}{2} \right\rfloor,$$

and assume there exists such an optimal drawing D of G_n that

$$cr_D(G+nK_1) < Z(5,n) + \left\lfloor \frac{n}{2} \right\rfloor. \tag{1}$$

Since

$$\begin{aligned} cr_D(G+nK_1) &= cr_D\left(G \cup \bigcup_{i=1}^{n} T_i\right) = cr_D\left(\bigcup_{i=1}^{n} T_i\right) + cr_D(G) + cr_D\left(G, \bigcup_{i=1}^{n} T_i\right) \\ &= cr_D(K_{5,n}) + cr_D(G) + cr_D\left(G, \bigcup_{i=1}^{n} T_i\right) \\ &\geqslant Z(5,n) + cr_D(G) + cr_D\left(G, \bigcup_{i=1}^{n} T_i\right), \end{aligned}$$

this implies that

$$cr_D(G) + cr_D\left(G, \bigcup_{i=1}^{n} T_i\right) < \left\lfloor \frac{n}{2} \right\rfloor. \tag{2}$$

The following claims hold for the drawing D.

Claim 5. *$cr_D(T_i, T_j) \geqslant 1$ for all $i, j = 1, 2, \dots, n, i \neq j$.*

Proof. Assume that for some $i \neq j, cr_D(T_i, T_j) = 0$, implying that $cr_D(T_i \cup T_j) = 0$ as $cr_D(T_i) = cr_D(T_j) = 0$ due to D being a good drawing. The subgraph $G \cup T_i \cup T_j$ is isomorphic to the graph G_2. Since $cr(G_2) \leqslant cr_D(G \cup T_i \cup T_j) = cr_D(G, T_i \cup T_j) + cr_D(T_i \cup T_j) + cr_D(G)$ and $cr_D(G) = cr_D(C_4) = 0$ (by Lemma 3), we have $cr_D(G, T_i \cup T_j) \geqslant cr(G_2) - cr_D(T_i \cup T_j) - cr_D(G) = 1$. For every subgraph $T_k, k = 1, 2, \dots, n, k \neq i, j$, $T_k \cup T_i \cup T_j$ is isomorphic to the graph $K_{3,5}$. As $cr(K_{3,5}) \leqslant cr_D(T_i \cup T_j \cup T_k) = cr_D(T_k, T_i \cup T_j) + cr_D(T_i \cup T_j) + cr_D(T_k)$, we have $cr_D(T_k, T_i \cup T_j) \geqslant cr(K_{3,5}) - cr_D(T_i \cup T_j) - cr_D(T_k) = 4$. Thus,

$$\begin{aligned}
cr_D(G_n) &= cr_D\left(G_{n-2} \cup T_i \cup T_j\right) \\
&= cr_D(G_{n-2}) + cr_D(G_{n-2}, T_i \cup T_j) + cr_D(T_i \cup T_j) \\
&= cr_D(G_{n-2}) + cr_D(G, T_i \cup T_j) + cr_D(K_{5,n-2}, T_i \cup T_j) + 0 \\
&\geqslant Z(5, n-2) + \left\lfloor \frac{n-2}{2} \right\rfloor + 1 + 4(n-2) = Z(5,n) + \left\lfloor \frac{n}{2} \right\rfloor .
\end{aligned}$$

This contradicts (1), and therefore $cr_D(T_i, T_j) \geqslant 1$ for all $i, j = 1, 2, \ldots, n, i \neq j$. □

The inequality (2) immediately implies Claim 6.

Claim 6. *In D, there are at least $\left(\lceil \frac{n}{2} \rceil + 1\right)$ subgraphs T_i, neither of which crosses G.*

Assume, without loss of generality, that $cr_D(G, T_n) = 0$. According to Lemma 3 ($cr_D(G) = cr_D(C_4) = 0$), the subgraph $F_n = G \cup T_n$ of the graph G_n has a unique drawing as shown in Figure 4(1).

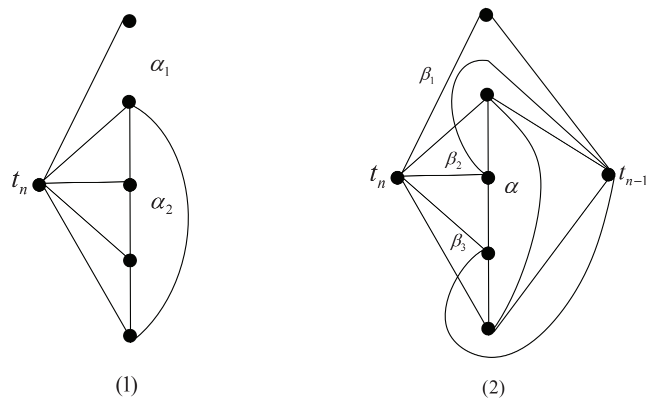

Figure 4. F_n and G^*.

Claim 7. *If for any j with $1 \leqslant j \leqslant n-1$ we have $cr_D(T_j, F_n) = 2$, then $cr_D(T_j, G) \geqslant 1$.*

Proof. Without loss of generality, let $j = n-1$. If $cr_D(T_{n-1}, F_n) = 2$ and $cr_D(T_{n-1}, G) = 0$, then the subgraph $G^* = T_{n-1} \cup F_n$ of the graph G_n has a unique drawing (see Figure 4(2)). We claim that no matter where t_i ($1 \leqslant i \leqslant n-2$) is placed, we have either $cr_D(T_i, G^*) \geqslant 6$, or both $cr_D(T_i, G) \geqslant 1$ and $cr_D(T_i, G^*) \geqslant 3$. It is also worth mentioning that it is a straightforward case analysis and some details are left to the reader.

It is easy to verify that, for every subgraph $T_i, 1 \leqslant i \leqslant n-2$, if t_i is placed in the region α, we have $cr_D(T_i, G) \geqslant 1$. According to Claim 5 ($cr_D(T_i, T_j) \geqslant 1, i \neq j$), we have $cr_D(T_i, G^*) = cr_D(T_i, G) + cr_D(T_i, T_{n-1}) + cr_D(T_i, T_n) \geqslant 3$.

If t_i is placed in the region β_1, β_2 or β_3, then $cr_D(T_i, G^*) \geqslant 5$. Especially, if $cr_D(T_i, G^*) = 5$, we have $cr_D(T_i, G) \geqslant 1$.

If t_i is placed in other regions, we have $cr_D(T_i, G^*) \geqslant 4$. Especially, if $cr_D(T_i, G^*) = 4$, then $cr_D(T_i, G) \geqslant 1$. If $cr_D(T_i, G^*) = 5$, we have $cr_D(T_i, G) \geqslant 1$. Thus, for all $T_i, 1 \leqslant i \leqslant n-2$, let

$$A_1 = \{T_i | 3 \leqslant cr_D(T_i, G^*) \leqslant 5, cr_D(T_i, G) \geqslant 1\},$$
$$A_2 = \{T_i | cr_D(T_i, G^*) \geqslant 6\}.$$

The inequality (2) implies $|A_1| \leqslant \lfloor \frac{n}{2} \rfloor - 1$. Therefore, using $|A_1| + |A_2| = n-2$, we have

$$\begin{aligned}
cr_D(G_n) &= cr_D\left(G \cup \bigcup_{i=1}^{n} T_i\right) = cr_D\left(\bigcup_{i=1}^{n-2} T_i\right) + cr_D\left(G^*, \bigcup_{i=1}^{n-2} T_i\right) + cr_D(G^*) \\
&= cr_D(K_{5,n-2}) + cr_D\left(G^*, \bigcup_{i=1}^{n-2} T_i\right) + cr_D(G^*) \\
&\geqslant Z(5, n-2) + 3|A_1| + 6|A_2| + 2 \\
&= Z(5, n) - (4n-8) + 3|A_1| + 6|A_2| + 2 \\
&= Z(5, n) - (4n-8) + 6(n-2) - 3|A_1| + 2 = Z(5, n) + 2n - 2 - 3|A_1| \\
&\geqslant Z(5, n) + 2n - 2 - 3\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) \geqslant Z(5, n) + \left\lfloor \frac{n}{2} \right\rfloor.
\end{aligned}$$

This contradicts (1). □

Now, we consider Figure 4(1). It is easy to verify that, for every subgraph $T_i, 1 \leqslant i \leqslant n-1$, if t_i is placed in the region α_1, we have $cr_D(T_i, F_n) \geqslant 2$. Especially, according to Claim 7, if $cr_D(T_i, F_n) = 2$, then $cr_D(T_i, G) \geqslant 1$.

If t_i is placed in the region α_2, then $cr_D(T_i, G) \geqslant 1$ and $cr_D(T_i, F_n) = cr_D(T_i, T_n) + cr_D(T_i, G) \geqslant 2$.

If t_i is placed in other regions, we have $cr_D(T_i, F_n) \geqslant 3$. Thus, for all $T_i, 1 \leqslant i \leqslant n-1$, let

$$B_1 = \{T_i | cr_D(T_i, F_n) = 2,\ cr_D(T_i, G) \geqslant 1\},$$
$$B_2 = \{T_i | cr_D(T_i, F_n) \geqslant 3\}.$$

The inequality (2) implies $|B_1| \leqslant \lfloor \frac{n}{2} \rfloor - 1$.

Hence, using $|B_1| + |B_2| = n - 1$, we have

$$\begin{aligned} cr_D(G_n) &= cr_D\left(G \cup \bigcup_{i=1}^{n} T_i\right) = cr_D\left(\bigcup_{i=1}^{n-1} T_i\right) + cr_D\left(F_n, \bigcup_{i=1}^{n-1} T_i\right) + cr_D(F_n) \\ &= cr_D(K_{5,n-1}) + cr_D\left(F_n, \bigcup_{i=1}^{n-1} T_i\right) + 0 \\ &\geqslant Z(5, n-1) + 2|B_1| + 3|B_2| = Z(5,n) - 4\left\lfloor \frac{n-1}{2} \right\rfloor + 3(n-1) - |B_1| \\ &\geqslant Z(5,n) - 4\left\lfloor \frac{n-1}{2} \right\rfloor + 3(n-1) - \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) \geqslant Z(5,n) + \left\lfloor \frac{n}{2} \right\rfloor . \end{aligned}$$

This contradicts our assumption (1) and completes the proof. ■

Corollary 8. *For $n \geqslant 1$, $cr(H_1 + nK_1) = cr(H_2 + nK_1) = Z(5,n) + \lfloor \frac{n}{2} \rfloor$.*

Proof. The graphs H_1 and H_2 are shown in Figure 1. Figure 2 shows the drawing of the graph $G + nK_1$ with $Z(5,n) + \lfloor \frac{n}{2} \rfloor$ crossings. One can easily see that in this drawing it is possible to add one edge which form the graph H_1 or add two edges which form the graph H_2 on the vertices of G in such a way that there is no new crossing occurring. Hence, we have $cr(H_1 + nK_1) \leqslant Z(5,n) + \lfloor \frac{n}{2} \rfloor$ and $cr(H_2 + nK_1) \leqslant Z(5,n) + \lfloor \frac{n}{2} \rfloor$. On the other hand, as G is a subgraph of H_1 and is a subgraph of H_2, clearly, we have $cr(H_2 + nK_1) \geqslant cr(H_1 + nK_1) \geqslant cr(G + nK_1)$. Thus, this completes the proof. ■

3. The Crossing Number of $W_4 + nK_1$

Lemma 9. $cr(W_4 + nK_1) \leqslant Z(5,n) + n + \lfloor \frac{n}{2} \rfloor$ *for $n \geqslant 1$.*

Proof. We will display a drawing φ of $W_4 + nK_1$ in the plane such that $cr_\varphi(W_4 + nK_1) = Z(5,n) + n + \lfloor \frac{n}{2} \rfloor$. The desired drawing φ is constructed as follows (see Figure 5).

(i) Set all vertices of W_4 on y-axis, $\lceil \frac{n}{2} \rceil$ isolated vertices on the negative x-axis and $\lfloor \frac{n}{2} \rfloor$ isolated vertices on the positive x-axis.

(ii) The image of each edge for W_4 is a thick line segment.

Then it is not difficult to see that $cr_\varphi(W_4 + nK_1) = Z(5,n) + n + \lfloor \frac{n}{2} \rfloor$, and so $cr(W_4 + nK_1) \leqslant Z(5,n) + n + \lfloor \frac{n}{2} \rfloor$. ■

Theorem 10. $cr(W_4 + nK_1) = Z(5,n) + n + \lfloor \frac{n}{2} \rfloor$ *for $n \geqslant 1$.*

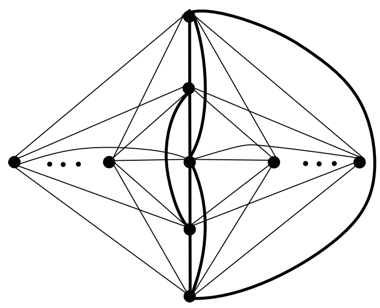

Figure 5. A drawing φ of $W_4 + nK_1$.

Proof. Lemma 9 shows that $cr(W_4+nK_1) \leqslant Z(5,n)+n+\left\lfloor \frac{n}{2} \right\rfloor$. Thus, in order to prove the theorem, we need only to prove that $cr_\phi(W_4+nK_1) \geqslant Z(5,n)+n+\left\lfloor \frac{n}{2} \right\rfloor$ for any drawing ϕ of $W_4 + nK_1$. Let e_i denote the edge xy_i of $W_4, i = 1, 2, 3, 4$, wherein x is a vertex of degree 4 and y_i are vertices of degree 3 in W_4. Without loss of generality, assume that under any drawing ϕ, the clockwise order of these four images $\phi(e_i)$ around $\phi(x)$ is $\phi(e_1) \to \phi(e_2) \to \phi(e_3) \to \phi(e_4)$. The graph W_4+nK_1 has additional n edges $f_j = t_j x$ incident with x $(1 \leqslant j \leqslant n)$. Let A_i denote the set of all those images f_j, each of which lies in the angle α_i formed between $\phi(e_i)$ and $\phi(e_{i+1})$, where the indices are read modulo 4 (see Figure 6(1)). We note that $|A_1| + |A_2| + |A_3| + |A_4| = n$. Again, we see that in the plane $\mathbb{R}^2$, there exists a circular neighborhood around $\phi(x)$, $N(\phi(x), \varepsilon) = \{s \in \mathbb{R}^2 : ||s - \phi(x)|| < \varepsilon\}$, where ε is a sufficiently small positive number, such that for any other edge e of W_4+nK_1 not incident with x, $\phi(e) \cap N(\phi(x), \varepsilon) = \emptyset$. We now consider two cases.

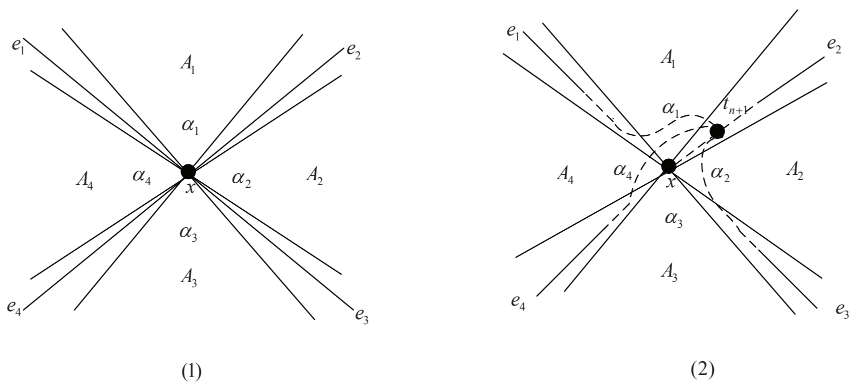

Figure 6. A drawing ϕ^* of G_{n+1} obtained from $\phi(W_4 + nK_1)$.

Case 1. Assume n is even. We consider arbitrarily a pair A_1 and A_3 or A_2 and A_4, say A_1 and A_3. Without loss of generality, assume $|A_1| \leqslant |A_3|$. In the following, we produce the graph G_{n+1} ($G_{n+1} = G + (n+1)K_1$ and $G = C_4 \cup K_1$ in Figure 1) together with its drawing ϕ^*.

Step 1. Add a new vertex t_{n+1} in some location of $\phi(e_2) \cap N(\phi(x), \varepsilon)$.

Step 2. For all $1 \leqslant i \leqslant 4$, remove the part of $\phi(e_i)$ lying in $N(\phi(x), \varepsilon)$(do not remove the vertex t_{n+1}).

Step 3. Connect t_{n+1} to each vertex in $\{\phi(x), \phi(y_1), \phi(y_2), \phi(y_3), \phi(y_4)\}$ in such a way as described in Figure 6(2).

For example, connect t_{n+1} to $\phi(x)$ along the section of $\phi(e_2) \cap N(\phi(x), \varepsilon)$, connect t_{n+1} to $\phi(y_2)$ first along the section of $\phi(e_2) \cap N(\phi(x), \varepsilon)$ and then along the original section $\phi(e_2)$ outside $N(\phi(x), \varepsilon)$. Again, the way of connecting t_{n+1} to $\phi(y_4)$ is first by successively traversing through the angles α_1 and α_4 (near to $\phi(x)$), and then along the original section of $\phi(e_4)$ outside $N(\phi(x), \varepsilon)$. Thus, we obtain a drawing ϕ^* of the graph G_{n+1}. It is easy to see that

$$\begin{aligned} cr_{\phi^*}(G_{n+1}) &= cr_\phi(W_4 + nK_1) + 2|A_1| + |A_2| + |A_4| \\ &\leqslant cr_\phi(W_4 + nK_1) + n. \end{aligned}$$

Under any drawing ϕ of $W_4 + nK_1$, using Theorem 4,

$$\begin{aligned} cr_\phi(W_4 + nK_1) &\geqslant cr_{\phi^*}(G_{n+1}) - n \geqslant Z(5, n+1) + \left\lfloor \frac{n+1}{2} \right\rfloor - n \\ &\geqslant Z(5, n) + n + \left\lfloor \frac{n}{2} \right\rfloor. \end{aligned}$$

Case 2. Assume n is odd. First, if $|A_1| = |A_3|$ and $|A_2| = |A_4|$, then it follows that $n = \sum_{i=1}^4 |A_i| = 2(|A_1| + |A_2|)$. This contradicts that n is odd. Therefore, either $|A_1| \neq |A_3|$ or $|A_2| \neq |A_4|$. Without loss of generality, let $|A_1| \neq |A_3|$, and moreover let $|A_3| \geqslant |A_1| + 1$. Completely analogously to Case 1 above,

$$\begin{aligned} cr_{\phi^*}(G_{n+1}) &= cr_\phi(W_4 + nK_1) + 2|A_1| + |A_2| + |A_4| \\ &\leqslant cr_\phi(W_4 + nK_1) + n - 1. \end{aligned}$$

Thus, we have

$$\begin{aligned} cr_\phi(W_4 + nK_1) &\geqslant cr_{\phi^*}(G_{n+1}) - n + 1 \geqslant Z(5, n+1) + \left\lfloor \frac{n+1}{2} \right\rfloor - n + 1 \\ &\geqslant Z(5, n) + n + \left\lfloor \frac{n}{2} \right\rfloor. \end{aligned}$$

Thereby, from the arguments above we finish the proof of the theorem. ■

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