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FACIAL RAINBOW COLORING OF PLANE GRAPHS

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Abstract

A vertex coloring of a plane graph G is a *facial rainbow coloring* if any two vertices of G connected by a facial path have distinct colors. The *facial rainbow number* of a plane graph G, denoted by $rb(G)$, is the minimum number of colors that are necessary in any facial rainbow coloring of G. Let $L(G)$ denote the order of a longest facial path in G. In the present note we prove that $rb(T) \leq \left\lfloor \frac{3}{2} L(T) \right\rfloor$ for any tree T and $rb(G) \leq \left\lceil \frac{5}{3} L(G) \right\rceil$ for arbitrary simple graph G. The upper bound for trees is tight. For any simple 3-connected plane graph G we have $rb(G) \leq L(G) + 5$.

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1. Introduction

All graphs considered in this note are simple connected plane graphs. We use a standard graph theory terminology according to West [30]. However, we recall some important notions.

A *plane* graph is a particular drawing of a planar graph in the Euclidean plane. Let G be a connected plane graph with vertex set $V(G)$, edge set $E(G)$, and face set $F(G)$. Faces of G are open 2-cells. The *boundary* of a face α is the boundary in the usual topological sense. It is a collection of all edges and vertices contained in the closure of α that can be organized into a closed walk in G traversing along a simple closed curve lying just inside the face α. This closed walk is unique up to the choice of initial vertex and direction, and is called the *boundary walk* of the face α (see [16], p. 101).

The *size* of a face $\alpha \in F(G)$ is the length of its boundary walk.

The *boundary cycle* of a bounded (respectively, unbounded) face α of a connected plane graph G, not being a tree, denoted by $C(\alpha)$, is a cycle which is a subgraph of the boundary walk of α and the whole α is in the interior (respectively, in the exterior) of this cycle. In the case when G is a tree, we define $C(\alpha)$ to be empty.

Let α be a face of G having size k and the boundary walk $v_0v_1 \cdots v_{k-1}v_k$, where $v_k = v_0$ with $v_i \in V(G)$ and $v_iv_{i+1} \in E(G)$ for every $i = 0, 1, \ldots, k-1$. A *facial path* of α is any path of the form $v_mv_{m+1} \cdots v_{n-1}v_n$ (subscripts modulo k) which is a contiguous subsequence of the boundary walk of α.

Let $\Delta^*(G)$ and $L(G)$ denote the maximum face size and the order of a longest facial path of G, respectively.

Two vertices (two edges or two faces) are *adjacent* if they are connected by an edge (have a common end-vertex or their boundaries have a common edge, respectively). A vertex and an edge are *incident* if the vertex is an end-vertex of the edge. A vertex (or an edge) and a face are *incident* if the vertex (or the edge) lies on the boundary of the face.

A *block* of a plane graph G is a maximal connected subgraph of G that has no cut-vertex. If G itself is connected and has no cut-vertex then G is a block. An edge is a block if and only if it is a cut-edge. A block consisting of a cut-edge is called *trivial*. Note that any non-trivial block is 2-connected. In any plane graph each non-trivial block B is bounded by a cycle belonging to this block, which is called the *block-cycle* and is denoted by $C(B)$. A cycle C is called *separating* in G if there are edges of G in the interior and also in the exterior of C.

Let α be a face of G. A non-trivial block B is an *interior block* of α, if all edges of its block-cycle $C(B)$ are incident with α and edge sets of $C(B)$ and $C(\alpha)$ are disjoint. A trivial block of G, which is an edge e, is an *interior block* of α if e is incident with α. An interior block B of α is an *end-block* of α if there is, on the block-cycle $C(B)$ of B, exactly one (unique) cut-vertex with the property that no other vertex of $C(B)$ is a cut-vertex of any other interior block of α.

Consider now the boundary cycle of $C(\alpha)$. It contains all former cut-vertices which are, in α, the cut-vertices of some interior blocks of α. Let us denote these vertices $c_1, \ldots, c_d$ in order following an orientation of $C(\alpha)$ and call them the *vertices of attachment* of α.

Let G be a vertex colored graph. A subgraph H of G is called *rainbow* if distinct vertices of H receive different colors.

2. Cyclic and Rainbow Colorings

There are two main motivations for this paper. First one comes from an intensive research of various types of rainbow vertex colorings of graphs, see e.g.

[12, 15, 23] and [24]. The second motivation is from different investigations of colorings of plane graphs, where restrictions on the properties of colorings are given by properties of color sequences of facial paths, facial walks, and facial cycles. For more information in this directions see a recent survey [10].

Many questions considered in facial colorings have their origin in the famous Four Color Conjecture (4CC). In spite of the fact that the 4CC has become the Four Color Theorem (see [2, 3]) one can still find some motivations for a new research. One of the equivalent formulations of the 4CC is: Vertices of any plane triangulation T can be colored with four colors so that two distinct vertices incident with the same face of T receive different colors.

This formulation of the 4CC led Ore and Plummer [25] to introduce the cyclic coloring of embedded graphs. A *cyclic coloring* of a connected plane graph G is a coloring of its vertices such that two distinct vertices incident with the same face of G receive different colors. The *cyclic chromatic number* of a connected plane graph G, denoted by $\chi_c(G)$, is the smallest number of colors used in a cyclic coloring of G.

It is an intensively studied parameter of plane graphs. If G is a 2-connected plane graph, then $\chi_c(G)$ is trivially bounded from below by the size $\Delta^*(G)$ of a largest face of G.

Ore and Plummer [25] proved the first upper bound $2\Delta^*$ for $\chi_c(G)$. Borodin [6] slightly improved this bound to $2\Delta^* - 3$ for $\Delta^* \geq 8$. Significant progress has been made by Borodin, Sanders and Zhao [9]. They managed to prove the upper bound of $\lceil \frac{9}{5}\Delta^* \rceil$. The following, currently best known, general upper bound is due to Sanders and Zhao [29].

Theorem 1 [29]**.** *If G is a connected plane graph, then* $\chi_c(G) \leq \lceil \frac{5}{3}\Delta^*(G) \rceil$.

Better results are known for graphs with small maximum face sizes, i.e., for small values of Δ^*. The case of cyclic coloring of plane triangulations, i.e., $\Delta^* = 3$, is equivalent to the famous Four Color Theorem which was proved by Appel and Haken in [2] and [3] (see also [28] for a refinement of its proof). Hence $\chi_c(G) \leq 4$ for $\Delta^* = 3$. The case of $\Delta^* = 4$ is Ringel's problem [27]. The problem was solved and it was shown that $\chi_c(G) \leq 6$ by Borodin [5, 7]. The case $\chi_c(G) \leq 9$ for $\Delta^* = 6$ was proved by Hebdige and Kráľ [18]. The bounds of $\chi_c(G)$ for $\Delta^* = 3$, $\Delta^* = 4$, and $\Delta^* = 6$ are the only ones which are currently known to be tight. The upper bound 8 for $\Delta^* = 5$ is proved in [6], 11 for $\Delta^* = 7$ in [18] and 14 for $\Delta^* = 8$ in [29].

The bound $\lfloor \frac{3}{2}\Delta^* \rfloor$ for cyclic coloring of connected plane graphs is conjectured to be the best possible upper bound by Borodin [5], see also the well-known monograph [22] on graph coloring problems by Jensen and Toft.

Amini, Esperet and van den Heuvel [1] proved the following theorem.

Theorem 2 [1]**.** *For every* $\varepsilon > 0$*, there exists* Δ_ε *such that every connected plane graph of maximum face size* $\Delta^* \geq \Delta_\varepsilon$ *admits a cyclic coloring with at most* $\left(\frac{3}{2} + \varepsilon\right) \Delta^*$ *colors.*

Restricting attention to 3-connected plane graphs, Plummer and Toft [26] proved that $\chi_c(G) \leq \Delta^* + 9$ and proposed the conjecture that every 3-connected plane graph has a cyclic coloring with at most $\Delta^* + 2$ colours. This conjecture is true for 3-connected plane graphs with $\Delta^* \geq 16$, see Horňák and Jendroľ [19] for $\Delta^* \geq 24$, Horňák and Zlámalová [20] for $\Delta^* \geq 18$ and Dvořák *et al.* [11] for the remaining cases. Enomoto, Horňák, and Jendroľ [14] obtained for $\Delta^* \geq 60$ even stronger results, namely $\chi_c \leq \Delta^* + 1$. Azarija *et al.* [4] proved the same bound for plane graphs in which all faces of size four or more are vertex-disjoint.

The best known general upper bound is due to Enomoto and Horňák [13].

Theorem 3 [13]**.** *If* G *is a* 3*-connected plane graph, then*

$$\chi_c(G) \leq \Delta^*(G) + 5.$$

The above mentioned formulation of the 4CC has led us to introduce the following new type of a coloring of plane graphs. A *facial rainbow coloring* of a connected plane graph G is a coloring of its vertices such that two distinct vertices connected by a facial path receive different colors. The *facial rainbow number* of G, denoted by $rb(G)$, is the smallest number of colors used in a facial rainbow coloring of G. Observe that if G is 2-connected, then $\chi_c(G) = rb(G)$. In general these two types of colorings differ. For example, for the star $K_{1,r}$, $r \geq 3$, we have $rb(K_{1,r}) = 3$ if r is even and $rb(K_{1,r}) = 4$ if r is odd. It is easy to see that $\chi_c(T) = n$ for any tree on n vertices. Evidently, $rb(G) \geq L(G)$ for any plane graph G.

The following three theorems are our main results.

Theorem 4. *If* T *is a tree, then* $rb(T) \leq \left\lfloor \frac{3}{2} L(T) \right\rfloor$*. Moreover, the bound is tight.*

Theorem 5. *If* G *is a connected plane graph, then* $rb(G) \leq \left\lceil \frac{5}{3} L(G) \right\rceil$*.*

Theorem 6. *For every* $\varepsilon > 0$*, there exists an* n_ε *such that every connected plane graph with* $L(G) \geq n_\varepsilon$ *admits a rainbow coloring with* $\left(\frac{3}{2} + \varepsilon\right) L(G)$ *colors.*

3. Proofs of Theorems

***Proof of Theorem* 4.** The theorem is evidently true for paths and stars. Suppose that the theorem is not true. Let T be a counterexample on minimum number n of vertices. Then $\Delta(T) \geq 3$ and $L = L(T) \geq 4$. Let S be a tree obtained from T by replacing any maximal $u - v$-path P with all internal degree-2 vertices (in

T) by an edge uv. Clearly, $\deg(u) \neq 2 \neq \deg(v)$. Let x be an internal vertex of S adjacent to exactly one other internal vertex of S. Clearly, $\deg_S(x) \geq 3$.

Let v_1, v_2, and v_3 be consecutive neighbors of x in S in an order around x such that v_1 and v_2 are leaves of S. Let $x - v_1$-path P_1 and $x - v_2$-path P_2 be corresponding paths, in T, to the facially adjacent edges xv_1 and xv_2 of S, respectively, having lengths a and b. We can suppose, without loss of generality, that $b \geq a$. Let Q_1 be the (unique) maximal facial $v_1 - v_2$-path in T, which is a concatenation of P_1 and P_2. (Recall that a path is a maximal facial path, if it is not a proper subgraph of any longer facial path.) Let Q_2 be the unique maximal facial $v_2 - y$-path containing P_2 and passing the vertex v_3 (or terminating in v_3 if it is a leaf). Here the vertex y is a leaf of T (and also of S). Let P_3 be the facial $x - y$-path which is a subpath of Q_2. Let the length of P_3 be denoted by c. From the above considerations we have

$$1 + a + b \leq L, \ 1 + b + c \leq L, \text{ and } 1 + a + c \leq L,$$

this gives $a + b + c \leq \frac{3}{2}(L - 1)$.

Let T' be the tree obtained from T by deleting all vertices of P_2 except of x. The resulting tree T' has less vertices than T and, by the above, $L(T') \leq L$. Hence, T' is no counterexample to our theorem and so it has a facial rainbow coloring with at most $\left\lfloor \frac{3}{2}L \right\rfloor$ colors. This coloring induces a partial rainbow coloring of T with only b vertices of the path P_2 not being colored. Because of the inequality in the previous paragraph we have enough colors to our disposal for coloring the uncolored vertices.

To see that the bound $\left\lfloor \frac{3}{2}L \right\rfloor$ is tight, consider a generalized star $S(3, r)$ consisting of a central vertex x of degree-3 from which three paths emanate, each of length r. For any two vertices u and v of this graph there exists a facial $u-v$-path, so every facial rainbow coloring of this graph requires exactly $3r + 1$ colors. As the order of a longest facial path of this graph is $2r + 1$, we are done. ■

***Proof of Theorem* 5.** Let G be a counterexample with minimum number b of non-trivial blocks and with minimum number of trivial blocks among all counterexamples having b non-trivial blocks. If G is a tree or G is 2-connected, then from Theorem 3 or Theorem 1, respectively, it follows that $rb(G) \leq \left\lceil \frac{5}{3}L(G) \right\rceil$. So we can suppose that $b \geq 1$ and G has at least two blocks.

Observation 1. *No face of G has an interior non-trivial end-block.*

Proof. Suppose that there is a face α having an interior non-trivial end-block B. Let x be the (unique) common cut-vertex of B and α. We distinguish two cases.

Case 1. $C(B)$ is not a separating cycle in G. Let e be an edge of $C(B)$ incident with the vertex x. Put $G' = G - \{e\}$. Observe that G' has $b - 1$ non-trivial blocks, $L(G') = L(G)$, and that $rb(G) = rb(G') \leq \lceil \frac{5}{3}L(G) \rceil$, which is a contradiction.

Case 2. $C(B)$ is a separating cycle in G. In this case we consider two subgraphs G_1 and G_2 of G, where G_1 consists of the cycle $C(B)$ and the interior of $C(B)$, and G_2 is isomorphic to G with the interior of $C(B)$ deleted. Clearly, G_1 has less than b non-trivial blocks or has exactly b non-trivial blocks but less trivial blocks. Hence, G_1 is not a counterexample. Because $L(G_1) \leq L(G)$, G_1 has a facial rainbow coloring with at most $\lceil \frac{5}{3}L(G) \rceil$.

The graph G_2 has also $L(G_2) \leq L(G)$. Now, as $C(B)$ is not a separating cycle in G_2, we continue as in Case 1. The result is that $rb(G_2) \leq \lceil \frac{5}{3}L(G) \rceil$.

In any facial rainbow colorings of all three graphs G, G_1, and G_2 the vertices of the cycle $C(B)$ must be colored with different colors. Hence we can suppose, without loss of generality, that the corresponding vertices of $C(B)$ receive the same colors in the corresponding facial rainbow colorings of G_1 and G_2. These colorings provide a facial rainbow coloring of G with at most $\lceil \frac{5}{3}L(G) \rceil$ colors, which is a contradiction. □

Observation 2. *No face α of G has a vertex of attachment.*

Proof. Suppose first that there is a face α with $d \geq 2$ vertices of attachment. All of the interior end-blocks of α are trivial by Observation 1. Denote their degree-1 vertices by $v_1, v_2, \ldots, v_l$ in an order given by the unique boundary walk of α. Denote by $P_{i,i+1}$ the unique facial $v_i - v_{i+1}$-path in α, subscripts modulo l. Because $d \geq 2$, we have $l \geq 2$. It is easy to see that the set $\{P_{i,i+1} : i = 1, \ldots, l,$ subscripts modulo $l\}$ is the set of (all) maximal facial paths in α.

Next we extend the graph G to a graph G_1. Let $l \geq 3$. In the first step, we insert new edges v_iv_{i+1} for every $i = 1, \ldots, l$; subscripts modulo l. The face α is replaced by the faces $\alpha_1, \ldots, \alpha_l$, and β, where α_i is bounded by the path $P_{i,i+1}$ and the edge v_iv_{i+1}. The remaining l-gonal face β is bounded by the inserted edges v_iv_{i+1}. Next we insert $l-3$ diagonals into β to get $l-2$ triangular faces instead of β. If $l = 2$, we only add the edge v_1v_2.

In both cases the result is graph G_1. Because all interior blocks of α and the cycle $C(\alpha)$ are subgraphs of the same block of G_1, the number b_1 of non-trivial blocks of G_1 is at most b. However, the number of trivial blocks has been reduced. Because of our construction, $L(G_1) \leq L(G)$, so the graph G_1 has a facial rainbow coloring with $rb(G_1) \leq \lceil \frac{5}{3}L(G) \rceil$. This facial coloring of G_1 induces a facial rainbow coloring of G with at most $\lceil \frac{5}{3}L(G) \rceil$ colors. This is because outside of α the rainbow colorings of G_1 and G are identical and, in α, each facial path $P_{i,i+1}$ is colored rainbowly. Again we have a contradiction.

Suppose now that there is a face α with exactly one vertex, x, of attachment. Consider the cycle $C(\alpha)$. We define two graphs G_1 and G_2 as follows. The graph

G_1 is a plane graph consisting of $C(\alpha)$ and its interior in G. The graph G_2 is obtained from G by deleting the interior of $C(\alpha)$. Obviously, $L(G_i) \leq L(G)$ for $i = 1, 2$. The graph G_2 has less nontrivial blocks than G, so it has a required coloring with all vertices of $C(\alpha)$ colored distinctly. To see that G_1 has a required coloring with all vertices on $C(\alpha)$ colored with different colors we first delete an edge xy from $C(\alpha)$, where y is a neighbor of x and then continue as in the proof of Observation 1. □

From Observations 1 and 2 it follows that G has to be a 2-connected plane graph. By Theorem 1 and the fact that in this case $\chi_c(G) = rb(G)$, the graph G has a facial rainbow coloring with $\left\lceil \frac{5}{3} L(G) \right\rceil$ colors; a contradiction. ■

***Proof of Theorem* 6.** This proof is analogous to the proof of Theorem 5. The only difference is that we apply Theorem 2 instead of Theorem 1. Also we let $n_\varepsilon = \Delta_\varepsilon$ and $L(G) = \Delta^*$, and use the bound $\left(\frac{3}{2} + \varepsilon\right) L(G)$ instead of $\left\lceil \frac{5}{3} L(G) \right\rceil$. ■

4. Concluding Remarks

We believe that the following analogue of the above mentioned conjecture of Borodin [6] holds.

Conjecture 7. *Let G be a connected plane graph. Then $rb(G) \leq \left\lfloor \frac{3}{2} L(G) \right\rfloor$.*

A weaker result for trees than Theorem 4 (namely with the upper bound $\left\lceil \frac{5}{3} L(T) \right\rceil$) is proved in our paper [21]. For trees having no degree-2 vertices there are proved stronger results, e.g. the following.

Theorem 8 [21]**.** *If T is a plane tree having no degree-2 vertices, then $rb(G) \leq L(T) + 5$.*

It would be interesting to study an extension of the problem studied in this paper for cellular embeddings of connected graphs into compact surfaces different from the sphere.

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