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TURÁN FUNCTION AND H-DECOMPOSITION PROBLEM FOR GEM GRAPHS

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Abstract

Given a graph H, the *Turán function* $\mathrm{ex}(n,H)$ is the maximum number of edges in a graph on n vertices not containing H as a subgraph. For two graphs G and H, an *H-decomposition* of G is a partition of the edge set of G such that each part is either a single edge or forms a graph isomorphic to H. Let $\phi(n,H)$ be the smallest number ϕ such that any graph G of order n admits an H-decomposition with at most ϕ parts. Pikhurko and Sousa conjectured that $\phi(n,H)=\mathrm{ex}(n,H)$ for $\chi(H)\geq 3$ and all sufficiently large n. Their conjecture has been verified by Özkahya and Person for all edge-critical graphs H. In this article, we consider the *gem graphs* gem_4 and gem_5. The graph gem_4 consists of the path P_4 with four vertices a,b,c,d and edges ab,bc,cd plus a universal vertex u adjacent to a,b,c,d, and the graph gem_5 is similarly defined with the path P_5 on five vertices. We determine the Turán functions $\mathrm{ex}(n, \mathrm{gem}_4)$ and $\mathrm{ex}(n, \mathrm{gem}_5)$, and verify the conjecture of Pikhurko and Sousa when H is the graph gem_4 and gem_5.

Keywords: gem graph, Turán function, extremal graph, graph decomposition.

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1. Introduction

Given a graph H, the *Turán function* $\mathrm{ex}(n, H)$ is the maximum number of edges in a graph on n vertices, and not containing a copy of H as a subgraph. The important result of Turán [13] states that when $H = K_r$ is the complete graph on $r \geq 3$ vertices, we have $\mathrm{ex}(n, K_r) = t_{r-1}(n)$. Here $t_{r-1}(n)$ denotes the number of edges in the *Turán graph* of order n, $T_{r-1}(n)$, which is the unique complete $(r-1)$-partite graph on n vertices where every partition class has either $\left\lfloor \frac{n}{r-1} \right\rfloor$ or $\left\lceil \frac{n}{r-1} \right\rceil$ vertices. Moreover, $T_{r-1}(n)$ is the unique extremal graph on n vertices that has the maximum number of edges not containing K_r as a subgraph. For general graphs H, the Turán function $\mathrm{ex}(n, H)$ has been well studied by numerous researchers, which led to many important results and open problems in extremal graph theory. For example, when $H = C_{2k}$ is the even cycle of length $2k$, where $k \geq 2$, the exact determination of the function $\mathrm{ex}(n, C_{2k})$ is still a wide open problem. It has been conjectured that $\mathrm{ex}(n, C_{2k}) = (c_k + o(1))n^{1+1/k}$ for some constant $c_k > 0$, and this conjecture is only known to be true for $k = 2, 3, 5$. See for example [8] and the references therein. When $H = P_k$ is the path of order $k \geq 3$, Faudree and Schelp [5] have determined the function $\mathrm{ex}(n, P_k)$ exactly. In order to obtain $\mathrm{ex}(n, P_k)$, we can take the graph on n vertices containing as many disjoint copies of K_{k-1} as possible, and a smaller complete graph on the remaining vertices. For odd k, this graph is the unique P_k-free extremal graph attaining $\mathrm{ex}(n, P_k)$, and for even k and certain values of n, there are other such extremal graphs. Here we state the result of Faudree and Schelp as follows, which will be useful in this paper.

Theorem 1.1 [5]**.** *Let $k \geq 3$ and $n = a(k-1)+b$, where $a \geq 0$ and $0 \leq b < k-1$. Then $\mathrm{ex}(n, P_k) = a\binom{k-1}{2} + \binom{b}{2}$. Moreover, a P_k-free graph on n vertices attaining $\mathrm{ex}(n, P_k)$ is $aK_{k-1} \dot\cup K_b$, the disjoint union of a copies of K_{k-1} and one copy of K_b.*

For two graphs G and H, an *H-decomposition* of G is a partition of the edge set of G such that each part is either a single edge or forms a graph isomorphic to H. Let $\phi(G, H)$ be the smallest possible number of parts in an H-decomposition of G. It is easy to see that, for non-empty H, we have $\phi(G, H) = e(G) -$

$p_H(G)(e(H)-1)$, where $p_H(G)$ is the maximum number of pairwise edge-disjoint copies of H that can be packed into G and $e(G)$ denotes the number of edges in G. Dor and Tarsi [3] showed that if H has a component with at least three edges, then the problem of checking whether a graph G admits a partition into H-subgraphs is NP-complete. Thus, it is NP-hard to compute the function $\phi(G, H)$ for such H. Here we study the function

$$\phi(n, H) = \max\{\phi(G, H) \mid v(G) = n\},$$

which is the smallest number ϕ such that any graph G of order n admits an H-decomposition with at most ϕ parts.

This function was first studied, in 1966, by Erdős, Goodman and Pósa [4], who were motivated by the problem of representing graphs by set intersections. They proved that $\phi(n, K_3) = t_2(n)$. A decade later, this result was extended by Bollobás [2], who proved that $\phi(n, K_r) = t_{r-1}(n)$, for all $n \geq r \geq 3$.

General graphs H were only considered recently by Pikhurko and Sousa [9]. They proved the following result.

Theorem 1.2 (See Theorem 1.1 from [9]). *Let H be any fixed graph of chromatic number $r \geq 3$. Then,*

$$\phi(n, H) = \mathrm{ex}(n, H) + o(n^2).$$

Pikhurko and Sousa also made the following conjecture.

Conjecture 1.3 [9]. *For any graph H of chromatic number $r \geq 3$, there exists $n_0 = n_0(H)$ such that $\phi(n, H) = \mathrm{ex}(n, H)$ for all $n \geq n_0$.*

A graph H is *edge-critical* if there exists an edge $e \in E(H)$ such that $\chi(H) > \chi(H - e)$, where $\chi(H)$ denotes the *chromatic number* of H. For $r \geq 4$ a *clique-extension of order* r is a connected graph that consists of a K_{r-1} plus another vertex, say v, adjacent to at most $r-2$ vertices of K_{r-1}. Conjecture 1.3 has been verified by Sousa for some edge-critical graphs, namely, clique-extensions of order $r \geq 4$ ($n \geq r$) [11] and the cycles of length 5 ($n \geq 6$) and 7 ($n \geq 10$) [10, 12]. Later, Özkahya and Person [7] verified the conjecture for all edge-critical graphs with chromatic number $r \geq 3$. Their result is the following.

Theorem 1.4 (See Theorem 3 from [7]). *For any edge-critical graph H with chromatic number $r \geq 3$, there exists $n_0 = n_0(H)$ such that $\phi(n, H) = \mathrm{ex}(n, H)$, for all $n \geq n_0$. Moreover, the only graph attaining* $\mathrm{ex}(n, H)$ *is the Turán graph $T_{r-1}(n)$.*

Recently, as an extension of Özkahya and Person's work, Allen, Böttcher, and Person [1] improved the error term obtained by Pikhurko and Sousa in Theorem 1.2. In fact, they proved that the error term $o(n^2)$ can be replaced by $O(n^{2-\alpha})$

for some $\alpha > 0$. Furthermore, they also showed that this error term has the correct order of magnitude. Their result is indeed an extension of Theorem 1.4 since the error term $O(n^{2-\alpha})$ that they obtained vanishes for every edge-critical graph H.

Conjecture 1.3 has also been verified by Liu and Sousa [6] for the *k-fan graph* F_k, which is the graph on $2k+1$ vertices consisting of k triangles intersecting in exactly one common vertex. Observe that $\chi(F_k) = 3$ and for $k \geq 2$ the graph F_k is not edge-critical. Thus, the result of Liu and Sousa is not a particular case of Theorem 1.4 by Özkahya and Person.

In this article, we consider the *gem graphs* gem_4 and gem_5, defined as follows. For the graph gem_4, we take the path P_4 with vertices a, b, c, d and edges ab, bc, cd and add a universal vertex u adjacent to a, b, c, d. Similarly for the graph gem_5, we take the path P_5 with vertices a, b, c, d, e and edges ab, bc, cd, de and add a universal vertex u adjacent to a, b, c, d, e. See Figure 1 below. For convenience, we write $abcd + u$ and $abcde + u$ for these two graphs.

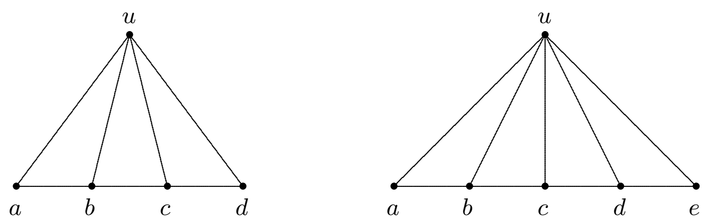

Figure 1. The graphs gem_4 and gem_5.

In Section 2, we will determine the Turán functions $\mathrm{ex}(n, \mathrm{gem}_4)$ for $n \geq 6$, and $\mathrm{ex}(n, \mathrm{gem}_5)$ for $n \geq 8$. Then, in Section 3, we will prove Pikhurko and Sousa conjecture for these two gem graphs. That is, we will show that $\phi(n, \mathrm{gem}_4) = \mathrm{ex}(n, \mathrm{gem}_4)$ for $n \geq 6$, and $\phi(n, \mathrm{gem}_5) = \mathrm{ex}(n, \mathrm{gem}_5)$ for $n \geq 8$. Note that $\chi(\mathrm{gem}_4) = \chi(\mathrm{gem}_5) = 3$, and that gem_4 and gem_5 are not edge-critical graphs. Thus, our results are again not implied by Theorem 1.4.

Our notations throughout the paper are fairly standard. For a vertex v in a graph G, the *neighbourhood* of v, denoted by $N(v)$, is the set of vertices in G that are adjacent to v. The *degree* of v is $\deg(v) = |N(v)|$, and the *minimum degree* and *maximum degree* of G are $\delta(G)$ and $\Delta(G)$, respectively. For a set $U \subset V(G)$, let $\deg(v, U)$ denote the number of vertices in U that are adjacent to v, and let $G[U]$ denote the subgraph of G induced by U.

2. Turán Function for the Gem Graphs

In this section, we will determine the Turán functions $\mathrm{ex}(n, \mathrm{gem}_4)$ for $n \geq 6$, and $\mathrm{ex}(n, \mathrm{gem}_5)$ for $n \geq 8$. Furthermore, we will determine the extremal graphs in each case. That is, we will determine all gem_4-free graphs on $n \geq 6$ vertices with $\text{ex}(n, \text{gem}_4)$ edges, and all gem_5-free graphs on $n \geq 8$ vertices with $\text{ex}(n, \text{gem}_5)$ edges.

2.1. Turán function for gem_4

We will now determine the function $\text{ex}(n, \text{gem}_4)$. In order to state our result, we first define the family of graphs $\mathcal{F}_{n,4}$, which will consist of all the extremal graphs. Let $n \geq 6$ and $\mathcal{F}_{n,4}$ be the family of graphs on n vertices as follows. For $n \equiv 0 \pmod 4$, let G_n^0 be the graph obtained by taking the Turán graph $T_2(n)$ and embedding a maximum matching into a class of $T_2(n)$. For $n \equiv 1 \pmod 4$, let G_n^{11} and G_n^{12} be the graphs obtained by embedding a maximum matching into the smaller class and the larger class of $T_2(n)$, respectively. For $n \equiv 2 \pmod 4$, let G_n^{21} and G_n^{22} be the graphs obtained by embedding a maximum matching into a class of $T_2(n)$, and into the larger class of the complete bipartite graph $K_{n/2-1,n/2+1}$, respectively. For $n \equiv 3 \pmod 4$, let G_n^3 be the graph obtained by embedding a maximum matching into the larger class of $T_2(n)$. Let the vertex classes of G_n^0 be A_n^0 and B_n^0, with similar notations for the other graphs. Let $\mathcal{F}_{n,4} = \{G_n^0\}$, $\mathcal{F}_{n,4} = \{G_n^{11}, G_n^{12}\}$, $\mathcal{F}_{n,4} = \{G_n^{21}, G_n^{22}\}$ and $\mathcal{F}_{n,4} = \{G_n^3\}$ for $n \equiv 0, 1, 2, 3 \pmod 4$, respectively. Figure 2 below shows the graphs of $\mathcal{F}_{n,4}$. Note that in G_n^{12}, we have an unmatched vertex in the class B_n^{12}, and similarly for G_n^{21} with the class B_n^{21}.

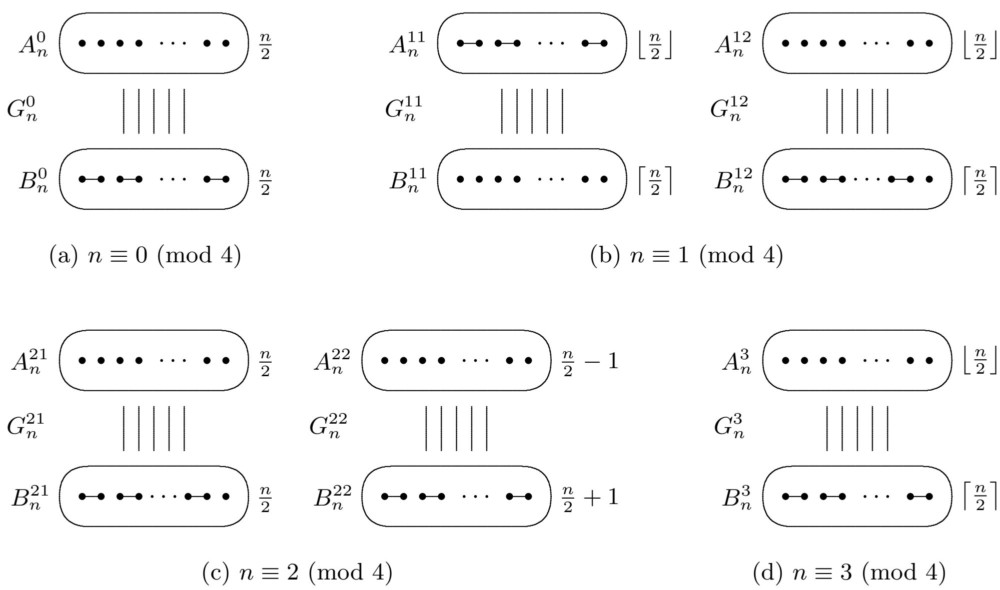

Figure 2. The graphs of $\mathcal{F}_{n,4}$.

It is easy to see that every graph of $\mathcal{F}_{n,4}$ is gem_4-free. Let $G \in \mathcal{F}_{n,4}$, and suppose that there exists a copy of gem_4 in G, say $abcd + u$. We may consider in turn whether u is in the independent class of G, or in the class containing the maximum matching. In each case, we can easily verify that no four neighbours of u form a path P_4 in G, which is a contradiction. Also, for any graph of $\mathcal{F}_{n,4}$, by adding an edge, we obtain a graph that contains a copy of gem_4. Indeed, let $G \in \mathcal{F}_{n,4}$. Since $n \geq 6$, if an edge cu is added to the independent class of G, then we may find an edge ab and another vertex d in the other class. If an edge bu is added to the class of G containing the maximum matching, then we may assume that du is an edge in the matching, and choose vertices a, c in the other class. In both cases, we have $abcd + u$ is a copy of gem_4.

We can easily check that for $n \geq 6$, all graphs of $\mathcal{F}_{n,4}$ have the same number of edges. Thus for $G \in \mathcal{F}_{n,4}$, we let e_n denote the number of edges in the graph G. Then, we can easily check that the number of edges of G is

$$e(G) = e_n = \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \begin{cases} 0 & \text{if } n \equiv 0, 1, 2 \pmod{4}, \\ 1 & \text{if } n \equiv 3 \pmod{4}. \end{cases} \tag{1}$$

Moreover, for $n \geq 7$, $G \in \mathcal{F}_{n,4}$ and $G' \in \mathcal{F}_{n-1,4}$, we have

$$e(G) - e(G') = e_n - e_{n-1} = \left\lfloor \frac{n}{2} \right\rfloor + \begin{cases} 0 & \text{if } n \equiv 0, 1, 2 \pmod{4}, \\ 1 & \text{if } n \equiv 3 \pmod{4}. \end{cases} \tag{2}$$

We have the following result for the Turán function $\mathrm{ex}(n, \mathrm{gem}_4)$.

Theorem 2.1. *For $n \geq 6$, we have*

$$\mathrm{ex}(n, \mathrm{gem}_4) = e_n = \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \begin{cases} 0 & \textit{if } n \equiv 0, 1, 2 \pmod{4}, \\ 1 & \textit{if } n \equiv 3 \pmod{4}. \end{cases}$$

Moreover, the only gem_4-free graphs with n vertices and $\mathrm{ex}(n, \mathrm{gem}_4)$ edges are the members of $\mathcal{F}_{n,4}$.

We will prove Theorem 2.1 by induction on n. We first prove the base case as follows.

Lemma 2.2. $\mathrm{ex}(6, \mathrm{gem}_4) = e_6 = 10$ *and the only gem_4-free graphs with six vertices and* 10 *edges are G_6^{21} and G_6^{22}.*

Proof. It suffices to prove that, for any graph G with six vertices and $e_6 = 10$ edges, either G contains a copy of the graph gem_4, or $G \in \mathcal{F}_{6,4} = \{G_6^{21}, G_6^{22}\}$. Then for any graph G' with six vertices and $e(G') \geq 11$, we can take a spanning subgraph $G \subset G'$ with $e(G) = e_6 = 10$, so that either G contains a copy of gem_4, or $G \in \mathcal{F}_{6,4}$. In either case, G' contains a copy of gem_4 and we are done.

Let G be a graph with six vertices and $e_6 = 10$ edges. Note that G has either a vertex of degree 5, or two vertices of degree 4. Otherwise, we have $e(G) \leq \left\lfloor \frac{1}{2}(4 + 5 \cdot 3) \right\rfloor = 9 < 10 = e_6$, a contradiction.

Suppose first that G has a vertex u with $\deg(u) = 5$. By Theorem 1.1, we have $\mathrm{ex}(5, P_4) = \binom{3}{2} + \binom{2}{2} = 4$. We have $e(G - u) = 10 - 5 = 5 > 4 = \mathrm{ex}(5, P_4)$, and thus $G - u$ contains a copy of the path P_4, which together with u, form a copy of gem_4 in G.

Now, suppose that G has two vertices of degree 4, say u and v. Let x_1, x_2, x_3, x_4 be the remaining four vertices, and assume that G does not contain a copy of gem_4. Suppose first that $uv \in E(G)$. If u and v have three common neighbours, say x_1, x_2, x_3, then we must have $x_i x_4 \in E(G)$ for $i = 1, 2, 3$, so that $G = G_6^{21}$. If u and v have two common neighbours, say x_1, x_2, then let $ux_3, vx_4 \in E(G)$ and $ux_4, vx_3 \notin E(G)$. We see that only the edges x_1x_2, x_3x_4 can be added to avoid creating a copy of gem_4, so that G can only have at most nine edges, a contradiction. Now, suppose that $uv \notin E(G)$. Then G contains all edges between $\{u, v\}$ and $\{x_1, x_2, x_3, x_4\}$. If G does not contain a copy of gem_4, then the remaining two edges must be independent within $\{x_1, x_2, x_3, x_4\}$, so that $G = G_6^{22}$.

We conclude that either G contains a copy of gem_4, or $G \in \mathcal{F}_{6,4}$, as required. ■

We are now able to prove Theorem 2.1.

***Proof of Theorem* 2.1.** Let $n \geq 6$. The lower bound $\mathrm{ex}(n, \mathrm{gem}_4) \geq e_n$ follows instantly by considering any graph of $\mathcal{F}_{n,4}$. We prove the upper bound $\mathrm{ex}(n, \mathrm{gem}_4) \leq e_n$ by induction on n. Lemma 2.2 proves the result for $n = 6$. Now suppose that $n \geq 7$, and the theorem holds for $n-1$. We will prove that if G is a graph on n vertices and $e(G) = e_n$, then either G contains a copy of gem_4, or G is one of the graphs of $\mathcal{F}_{n,4}$. This clearly implies the upper bound $\mathrm{ex}(n, \mathrm{gem}_4) \leq e_n$, and thus the theorem for n. Indeed, if we have a graph G' with n vertices and $e(G') > e_n$, then by taking a spanning subgraph $G \subset G'$ with $e(G) = e_n$, we see that either G contains a copy of gem_4, or $G \in \mathcal{F}_{n,4}$. In either case, G' contains a copy of gem_4.

First, suppose that $\delta(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$ and let $v \in V(G)$ be a vertex of minimum degree. Then by (2), we have

$$e(G - v) = e(G) - \deg(v) \geq e_n - \left\lfloor \frac{n}{2} \right\rfloor \geq e_{n-1}. \tag{3}$$

If $e(G - v) > e_{n-1}$, then by induction, $G - v$, and thus G, contains a copy of gem_4. Next, $e(G - v) = e_{n-1}$ holds if and only if $\deg(v) = \left\lfloor \frac{n}{2} \right\rfloor$ and $e_n - e_{n-1} = \left\lfloor \frac{n}{2} \right\rfloor$. The latter condition holds for $n \not\equiv 3 \pmod 4$. By induction, either $G - v$, and thus G, contains a copy of gem_4 and we are done, or $G - v \in \mathcal{F}_{n-1,4}$, and we must consider the following cases.

Case 1. $n \equiv 0 \pmod 4$. We have $G - v = G^3_{n-1}$ with classes A^3_{n-1} and B^3_{n-1}, where $|A^3_{n-1}| = \frac{n}{2} - 1$ and $|B^3_{n-1}| = \frac{n}{2}$, and B^3_{n-1} containing a perfect matching. Since $\deg(v) = \frac{n}{2}$, if $N(v) = B^3_{n-1}$, then $G = G^0_n$. Otherwise, if v has neighbours $c \in A^3_{n-1}$ and $u \in B^3_{n-1}$, then $abcv + u$ is a copy of gem$_4$ in G, where $a \in A^3_{n-1} \setminus \{c\}$ and $b \in B^3_{n-1}$ is the vertex adjacent to u.

Case 2. $n \equiv 1 \pmod 4$. We have $G - v = G^0_{n-1}$ with classes A^0_{n-1} and B^0_{n-1}, where $|A^0_{n-1}| = |B^0_{n-1}| = \frac{n-1}{2}$, with B^0_{n-1} containing a perfect matching. Since $\deg(v) = \frac{n-1}{2}$, it follows that if $N(v) = B^0_{n-1}$ then $G = G^{11}_n$, and if $N(v) = A^0_{n-1}$ then $G = G^{12}_n$. Otherwise, v has a neighbour in both A^0_{n-1} and B^0_{n-1}, so that as in Case 1, G contains a copy of gem$_4$.

Case 3. $n \equiv 2 \pmod 4$. We have $G - v \in \{G^{11}_{n-1}, G^{12}_{n-1}\}$. Suppose first that $G - v = G^{11}_{n-1}$. Then the classes of $G - v$ are A^{11}_{n-1} and B^{11}_{n-1}, where $|A^{11}_{n-1}| = \frac{n}{2} - 1$ and $|B^{11}_{n-1}| = \frac{n}{2}$, with A^{11}_{n-1} containing a perfect matching. Since $\deg(v) = \frac{n}{2}$, it follows that if $N(v) = B^{11}_{n-1}$, then $G = G^{21}_n$. Otherwise, v has a neighbour in both A^{11}_{n-1} and B^{11}_{n-1}, and G contains a copy of gem$_4$ as in Case 1. Now suppose that $G - v = G^{12}_{n-1}$. Then the classes are A^{12}_{n-1} and B^{12}_{n-1}, where $|A^{12}_{n-1}| = \frac{n}{2} - 1$ and $|B^{12}_{n-1}| = \frac{n}{2}$, with B^{12}_{n-1} containing a maximum matching with one unmatched vertex, say w. Since $\deg(v) = \frac{n}{2}$, it follows that if $N(v) = B^{12}_{n-1}$ then again $G = G^{21}_n$, and if $N(v) = A^{12}_{n-1} \cup \{w\}$ then $G = G^{22}_n$. Otherwise, v has a neighbour in both A^{12}_{n-1} and $B^{12}_{n-1} \setminus \{w\}$, and again as in Case 1, G contains a copy of gem$_4$.

Next, suppose that $\delta(G) \geq \lfloor \frac{n}{2} \rfloor + 1$. In view of (1), if n is even, then we have $e(G) \geq \frac{n}{2}(\frac{n}{2} + 1) > e_n$. If $n \equiv 1 \pmod 4$, then $e(G) \geq \lceil \frac{n}{2} \left(\lfloor \frac{n}{2} \rfloor + 1 \right) \rceil = \left\lfloor \frac{n^2}{4} \right\rfloor + \lfloor \frac{n}{4} \rfloor + 1 > e_n$. We have a contradiction in these cases. Now let $n \equiv 3 \pmod 4$. We have $e(G) \geq \lceil \frac{n}{2} \left(\lfloor \frac{n}{2} \rfloor + 1 \right) \rceil = \left\lfloor \frac{n^2}{4} \right\rfloor + \lfloor \frac{n}{4} \rfloor + 1 = e_n$. We must have equality, and thus G is a $\left(\lfloor \frac{n}{2} \rfloor + 1 \right)$-regular graph. Let $v \in V(G)$, so that by (2)

$$e(G - v) = e(G) - \deg(v) = e_n - \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) = e_{n-1}. \tag{4}$$

By induction, either $G - v$, and thus G, contains a copy of gem$_4$, or $G - v \in \mathcal{F}_{n-1,4}$. If the latter holds, then $G - v \in \{G^{21}_{n-1}, G^{22}_{n-1}\}$. Suppose first that $G - v = G^{21}_{n-1}$. The classes are A^{21}_{n-1} and B^{21}_{n-1}, where $|A^{21}_{n-1}| = |B^{21}_{n-1}| = \frac{n-1}{2}$, with B^{21}_{n-1} containing a maximum matching with one unmatched vertex, say w. Since $\deg(v) = \frac{n-1}{2} + 1$, in order for G to be $\left(\lfloor \frac{n}{2} \rfloor + 1 \right)$-regular, we must have $N(v) = A^{21}_{n-1} \cup \{w\}$. This gives $G = G^3_n$. Now, suppose that $G - v = G^{22}_{n-1}$. The classes are A^{22}_{n-1} and B^{22}_{n-1}, where $|A^{22}_{n-1}| = \frac{n-1}{2} - 1$ and $|B^{22}_{n-1}| = \frac{n-1}{2} + 1$, with B^{22}_{n-1} containing a perfect matching. Again since G is $\left(\lfloor \frac{n}{2} \rfloor + 1 \right)$-regular, we must have $N(v) = B^{22}_{n-1}$, and this also implies $G = G^3_n$.

This completes the proof of Theorem 2.1. ■

2.2. Turán function for gem$_5$

We will next determine the function $\mathrm{ex}(n, \mathrm{gem}_5)$. Analogously, we first define the family of graphs $\mathcal{F}_{n,5}$, which will consist of all the extremal graphs. Let $n \geq 8$ and $\mathcal{F}_{n,5}$ be the family of graphs on n vertices as follows. For $n \geq 11$, we let $\mathcal{F}_{n,5} = \mathcal{F}_{n,4}$. For $n = 8, 9, 10$, the family $\mathcal{F}_{n,5}$ will consist of all graphs of $\mathcal{F}_{n,4}$ and some additional graphs. Let G'_n be the graph obtained by adding one edge into each class of $T_2(n)$. Also for $n = 8$, let G''_8 be the graph obtained by embedding two vertex-disjoint triangles into the larger class of the complete bipartite graph $K_{2,6}$. For $n = 9$, let G''_9 be the graph obtained by taking G'_8 and joining another vertex to the four unmatched vertices within the classes of G'_8. As before, let A'_8 and B'_8 be the classes of G'_8, with similar notations for the other graphs. Figure 3 below shows these additional graphs. Let $\mathcal{F}_{8,5} = \{G^0_8, G'_8, G''_8\}$, $\mathcal{F}_{9,5} = \{G^{11}_9, G^{12}_9, G'_9, G''_9\}$, and $\mathcal{F}_{10,5} = \{G^{21}_{10}, G^{22}_{10}, G'_{10}\}$.

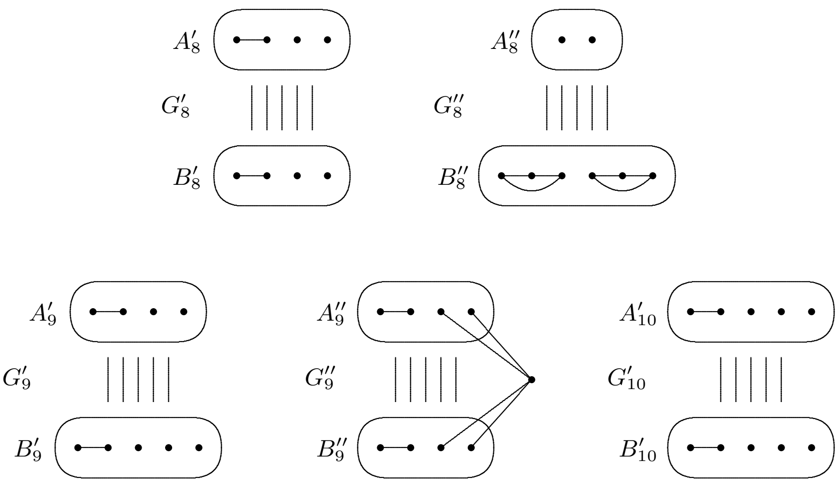

Figure 3. The additional graphs in $\mathcal{F}_{n,5}$ for $n = 8, 9, 10$.

Note that every graph of $\mathcal{F}_{n,5}$ is gem$_5$-free. Indeed, let $G \in \mathcal{F}_{n,5}$. If $G \notin \{G'_8, G''_8, G'_9, G''_9, G'_{10}\}$, then G is gem$_4$-free as before, so that G is gem$_5$-free. Suppose that $G \in \{G'_8, G''_8, G'_9, G''_9, G'_{10}\}$ and G contains a copy of gem$_5$, say $abcde + u$. It is easy to check that in each choice for G, whichever vertex of G is chosen for u, we have that u does not have five neighbours that form a path P_5 in G. This is a contradiction.

Also, by adding an edge to any graph of $\mathcal{F}_{n,5}$, we obtain a graph that contains a copy of gem$_5$. To see this, let $G \in \mathcal{F}_{n,5}$. Suppose first that $G \notin \{G'_8, G''_8, G'_9, G''_9, G'_{10}\}$. Then similar to before, since $n \geq 8$, it follows that if an edge cu is added to the independent class of G, then we can find two independent edges ab, de in the other class. If an edge bu is added to the class of G containing the maximum matching, then we may assume that du is an edge in the matching, and choose vertices a, c, e in the other class. In both cases, we have $abcde + u$ is a copy of gem_5. Next, the case $G \in \{G'_8, G'_9, G'_{10}\}$ can be considered similarly, according to whether or not the added edge is incident with an edge within a class of G. Now, consider $G = G''_8$. If the edge bu is added into A''_8, then let cde be a triangle and a be another vertex in B''_8. If an edge is added into B''_8, then there exists a path $abcde$ of order 5 in B''_8, and we let $u \in A''_8$. In both cases, $abcde + u$ is a copy of gem_5. Finally, consider $G = G''_9$. Since G''_9 contains G'_8 as a subgraph on $A''_9 \cup B''_9$, it follows that if an edge is added into A''_9 or B''_9, then we have a copy of gem_5. Thus, we may assume that the edge au is added to G''_9, where a is an end-vertex of the edge in A''_9, and u is the vertex outside of $A''_9 \cup B''_9$. Then if $c, e \in A''_9$ and $b, d \in B''_9$ are the neighbours of u in G''_9, we have $abcde + u$ is a copy of gem_5.

We can easily check that for $n \geq 8$, all graphs of $\mathcal{F}_{n,5}$ have the same number of edges, which is also the same as the number of edges in any graph of $\mathcal{F}_{n,4}$. Thus, we may also let e_n denote the number of edges in any graph of $\mathcal{F}_{n,5}$. Then, equations (1) and (2) remain true. That is, for $G \in \mathcal{F}_{n,5}$, we have

$$(5) \qquad e(G) = e_n = \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \begin{cases} 0 & \text{if } n \equiv 0, 1, 2 \pmod 4, \\ 1 & \text{if } n \equiv 3 \pmod 4, \end{cases}$$

and for $n \geq 9$, $G \in \mathcal{F}_{n,5}$ and $G' \in \mathcal{F}_{n-1,5}$, we have

$$(6) \qquad e(G) - e(G') = e_n - e_{n-1} = \left\lfloor \frac{n}{2} \right\rfloor + \begin{cases} 0 & \text{if } n \equiv 0, 1, 2 \pmod 4, \\ 1 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

We have the following result for the Turán function $\mathrm{ex}(n, \mathrm{gem}_5)$.

Theorem 2.3. *For $n \geq 8$, we have*

$$\mathrm{ex}(n, \mathrm{gem}_5) = e_n = \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \begin{cases} 0 & \text{if } n \equiv 0, 1, 2 \pmod 4, \\ 1 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Moreover, the only gem_5-free graphs with n vertices and $\mathrm{ex}(n, \mathrm{gem}_5)$ edges are the members of $\mathcal{F}_{n,5}$.

As before, Theorem 2.3 will be proved by induction on n. We first prove the base case, which will involve a bit more of case analysis than in Lemma 2.2.

Lemma 2.4. $\mathrm{ex}(8, \mathrm{gem}_5) = e_8 = 18$ *and the only gem_5-free graphs with eight vertices and* 18 *edges are G^0_8, G'_8 and G''_8.*

To prove Lemma 2.4, the following lemma will be useful.

Lemma 2.5. *Let H be a graph with vertex set $A \cup B$, where $A = \{x, y\}$ and $B = \{z_1, z_2, z_3, z_4\}$. Suppose that $xy, xz_4 \in E(H)$, and H also contains all edges between $\{x, y\}$ and $\{z_1, z_2, z_3\}$. Suppose that $H[B]$ contains two edges f_1, f_2, and either z_4 belongs to at least one of f_1, f_2, or $yz_4 \in E(H)$. Then H contains a copy of* gem_5.

Proof. First, if z_4 belongs to one of f_1, f_2, then we may assume that either $f_1 = z_1z_2, f_2 = z_3z_4$ or $f_1 = z_1z_2, f_2 = z_2z_4$ or $f_1 = z_1z_4, f_2 = z_2z_4$. Then $z_1z_2yz_3z_4+x$ or $z_3yz_1z_2z_4+x$ or $z_3yz_1z_4z_2+x$ is a copy of gem_5 in H, respectively.

Next, if $yz_4 \in E(H)$ and z_4 does not belong to f_1 and f_2, then we may assume that $f_1 = z_1z_2$ and $f_2 = z_2z_3$. Then $z_1z_2z_3yz_4 + x$ is a copy of gem_5 in H. ■

***Proof of Lemma* 2.4.** Let G be a graph with eight vertices and $e_8 = 18$ edges. As in Lemma 2.2, it suffices to prove that either G contains a copy of gem_5, or $G \in \mathcal{F}_{8,5} = \{G_8^0, G_8', G_8''\}$. Let $\Delta = \Delta(G)$ be the maximum degree of G. Note that $5 \le \Delta \le 7$, otherwise if $\Delta \le 4$, then $e(G) \le \lfloor \frac{1}{2} \cdot 8 \cdot 4 \rfloor = 16 < 18 = e_8$, a contradiction. Let $d_1 \ge d_2 \ge \cdots \ge d_8$ be the degree sequence of G. Let $u \in V(G)$ be a vertex of maximum degree, so that $\deg(u) = \Delta = d_1$. We consider three cases according to the value of Δ.

Case 1. $\Delta = 7$. By Theorem 1.1, we have $\text{ex}(7, P_5) = \binom{4}{2} + \binom{3}{2} = 9$. Thus $e(G - u) = 18 - 7 = 11 > 9 = \text{ex}(7, P_5)$, and there exists a copy of the path P_5 in $G - u$, which together with u, form a copy of gem_5 in G.

Case 2. $\Delta = 6$. Let $v \in V(G) \setminus \{u\}$ be a vertex with $\deg(v) = d_2$. Note that $\deg(v) = 6$ or $\deg(v) = 5$, otherwise $e(G) \le \lfloor \frac{1}{2}(6 + 7 \cdot 4) \rfloor = 17 < 18 = e_8$, a contradiction.

Subcase 2.1. $\deg(v) = 6$. Suppose first that $uv \notin E(G)$. We have $e(G - \{u, v\}) = 18 - 2 \cdot 6 = 6$. If there exists $x \in V(G) \setminus \{u, v\}$ with at least three neighbours in $V(G) \setminus \{u, v, x\}$, say x_1, x_2, x_3, then $x_1ux_2vx_3 + x$ is a copy of gem_5 in G. Otherwise, since $e(G - \{u, v\}) = 6$, we see that every vertex of $V(G) \setminus \{u, v\}$ must have exactly two neighbours in $V(G) \setminus \{u, v\}$, and thus, the subgraph $G - \{u, v\}$ must be either C_6 or two vertex-disjoint copies of C_3. If the former, then there is a copy of P_5 in $G - \{u, v\}$, which together with u, form a copy of gem_5. If the latter, then $G = G_8''$.

Now, suppose that $uv \in E(G)$. Observe first that u and v have at least four common neighbours in $V(G) \setminus \{u, v\}$. If $G[N(u) \setminus \{v\}]$ contains two edges, then Lemma 2.5 implies that G contains a copy of gem_5. Otherwise, we may assume that $G[N(u) \setminus \{v\}]$ contains at most one edge. If y is the vertex not adjacent to u in G, then y has at most five neighbours in $N(u) \setminus \{v\}$. Therefore, we have $e(G - \{u, v\}) \le 1 + 5 = 6$. This is a contradiction, since we have $e(G - \{u, v\}) = 18 - 1 - 2 \cdot 5 = 7$.

Subcase 2.2. $\deg(v) = 5$. Let $w \in V(G)\setminus\{u, v\}$ be a vertex with $\deg(w) = d_3$. Note that $\deg(w) = 5$, otherwise, $e(G) \le \lfloor \frac{1}{2}(6 + 5 + 6 \cdot 4)\rfloor = 17 < 18 = e_8$. Thus, without loss of generality, we may assume $uv \in E(G)$, so that $e(G - \{u, v\}) = 18 - 1 - 5 - 4 = 8$. Let y be the vertex not adjacent to u. Suppose that G does not contain a copy of gem_5.

Let $vy \notin E(G)$. Then v has exactly four neighbours in $N(u) \setminus \{v\}$, and by Lemma 2.5, $G[N(u) \setminus \{v\}]$ contains at most one edge, so that $e(G - \{u, v\}) \le 6$, a contradiction.

Now let $vy \in E(G)$. Let x_1, x_2, x_3 be the common neighbours of u and v, and z_1, z_2 be the remaining two vertices, so that $uz_1, uz_2 \in E(G)$ and $vz_1, vz_2 \notin E(G)$. Again by Lemma 2.5, each of y, z_1, z_2 has at most one neighbour in $\{x_1, x_2, x_3\}$. If there are no edges between $\{y, z_1, z_2\}$ and $\{x_1, x_2, x_3\}$, then $e(G - \{u, v\}) \le 6$, a contradiction. Otherwise, if there exists an edge between $\{y, z_1, z_2\}$ and $\{x_1, x_2, x_3\}$, then by Lemma 2.5, there are no edges in $G[\{x_1, x_2, x_3\}]$. Since there are at most three edges in $G[\{y, z_1, z_2\}]$ and at most three edges between $\{y, z_1, z_2\}$ and $\{x_1, x_2, x_3\}$, we have $e(G - \{u, v\}) \le 6$, another contradiction.

Case 3. $\Delta = 5$. We have $d_1 = d_2 = d_3 = d_4 = \Delta = 5$, otherwise, $e(G) \le \lfloor \frac{1}{2}(3 \cdot 5 + 5 \cdot 4)\rfloor = 17 < 18 = e_8$. This means that, we may assume there exists $v \in V(G) \setminus \{u\}$ with $\deg(v) = 5$ and $uv \in E(G)$, so that $e(G - \{u, v\}) = 18 - 1 - 2 \cdot 4 = 9$. If G contains a copy of gem_5, then we are done, so assume otherwise.

Suppose first that u and v have four common neighbours, say x_1, x_2, x_3, x_4. Let y_1, y_2 be the remaining two vertices. By Lemma 2.5, $G[\{x_1, x_2, x_3, x_4\}]$ contains at most one edge. If there is exactly one edge, say $x_1x_2 \in E(G)$, then there are 10 edges already in G. The edges between $\{y_1, y_2\}$ and $\{x_1, x_2, x_3, x_4\}$, as well as y_1y_2, may possibly be present, and since $e(G) = 18$, exactly one of these nine edges is not present. Suppose first that $y_1y_2 \in E(G)$. We may assume that $y_1x_1, y_1x_2, y_2x_1 \in E(G)$, but then $uvx_2y_1y_2 + x_1$ is a copy of gem_5. Otherwise, if $y_1y_2 \notin E(G)$, then we have $G = G'_8$. Finally, if there does not exist an edge in $G[\{x_1, x_2, x_3, x_4\}]$, then a similar edge count shows that G contains all edges between $\{y_1, y_2\}$ and $\{x_1, x_2, x_3, x_4\}$, as well as y_1y_2. This gives $G = G^0_8$.

Next, suppose that u and v have three common neighbours, say x_1, x_2, x_3. Let y, z_1, z_2 be the remaining vertices, where $uz_1, vz_2 \in E(G)$ and $uy, vy, uz_2, vz_1 \notin E(G)$. By Lemma 2.5, each of z_1, z_2 has at most one neighbour in $\{x_1, x_2, x_3\}$. If there exists an edge between $\{z_1, z_2\}$ and $\{x_1, x_2, x_3\}$, then again by Lemma 2.5, there are no edges in $G[\{x_1, x_2, x_3\}]$. Since there are at most three edges in $G[\{y, z_1, z_2\}]$, and at most five edges between $\{y, z_1, z_2\}$ and $\{x_1, x_2, x_3\}$, we have $e(G - \{u, v\}) \le 8$, a contradiction. Otherwise, suppose that there are no edges between $\{z_1, z_2\}$ and $\{x_1, x_2, x_3\}$. Then we have $\deg(z_i) \le 3$ for $i = 1, 2$. This implies that the remaining six vertices must each have degree 5, otherwise $e(G) \le \lfloor \frac{1}{2}(5 \cdot 5 + 4 + 2 \cdot 3)\rfloor = 17 < 18 = e_8$. In particular, we have $x_ix_j \in E(G)$

for $1 \leq i \neq j \leq 3$ and $yx_i \in E(G)$ for $i = 1, 2, 3$. But then $uvx_2x_3y + x_1$ is a copy of gem$_5$.

Finally, suppose that u and v have two common neighbours, say x_1, x_2. Let y_1, y_2, z_1, z_2 be the remaining vertices, where $uy_1, uy_2, vz_1, vz_2 \in E(G)$ and $uz_1, uz_2, vy_1, vy_2 \notin E(G)$. Suppose first that there are at most two edges in $G[\{x_1, x_2, y_1, y_2\}]$, and at most two edges in $G[\{x_1, x_2, z_1, z_2\}]$. Since there are at most four edges between $\{y_1, y_2\}$ and $\{z_1, z_2\}$, we have $e(G - \{u, v\}) \leq 2 \cdot 2 + 4 = 8$, a contradiction. Now, suppose that there are at least three edges in $G[\{x_1, x_2, y_1, y_2\}]$. If $x_1y_1, y_1y_2 \in E(G)$ or $x_1y_1, x_2y_2 \in E(G)$, then $x_2vx_1y_1y_2 + u$ or $y_1x_1vx_2y_2 + u$ is a copy of gem$_5$. Thus, we may assume that $x_1x_2, x_1y_1, x_2y_1 \in E(G)$ and $x_1y_2, x_2y_2, y_1y_2 \notin E(G)$. If there are at most two edges in $G[\{x_1, x_2, z_1, z_2\}]$, including x_1x_2, then since there are at most four edges between $\{y_1, y_2\}$ and $\{z_1, z_2\}$, we have $e(G - \{u, v\}) \leq 3 + 1 + 4 = 8$, a contradiction. Thus, there are at least three edges in $G[\{x_1, x_2, z_1, z_2\}]$, and by similarly considering the edges in $G[\{x_1, x_2, z_1, z_2\}]$, we may assume that $x_1z_1, x_2z_1 \in E(G)$ and $x_1z_2, x_2z_2, z_1z_2 \notin E(G)$. But now, $y_1ux_2vz_1 + x_1$ is a copy of gem$_5$.

Therefore, we conclude that either G contains a copy of gem$_5$, or $G \in \mathcal{F}_{8,5}$. This completes the proof of Lemma 2.4. ∎

We are now able to prove Theorem 2.3. The proof is generally similar to that of Theorem 2.1 but with a little more case analysis.

***Proof of Theorem* 2.3.** Let $n \geq 8$. Again, the lower bound $\text{ex}(n, \text{gem}_5) \geq e_n$ follows by considering any graph of $\mathcal{F}_{n,5}$. We prove the upper bound $\text{ex}(n, \text{gem}_5) \leq e_n$ by induction on n. Lemma 2.4 proves the result for $n = 8$. Now suppose that $n \geq 9$, and the theorem holds for $n - 1$. As before, it suffices to prove that if G is a graph on n vertices and $e(G) = e_n$, then either G contains a copy of gem$_5$, or $G \in \mathcal{F}_{n,5}$.

First, suppose that $\delta(G) \leq \lfloor \frac{n}{2} \rfloor$ and let $v \in V(G)$ be a vertex of minimum degree. Then exactly as in (3), we have $e(G - v) \geq e_{n-1}$. Again we are done unless $e(G - v) = e_{n-1}$, whence $\deg(v) = \lfloor \frac{n}{2} \rfloor$ and $e_n - e_{n-1} = \lfloor \frac{n}{2} \rfloor$, and $n \not\equiv 3 \pmod 4$. By induction, either $G - v$, and thus G, contains a copy of gem$_5$ and we are done, or $G - v \in \mathcal{F}_{n-1,5}$, and we must consider the following cases.

Case 1. $n \equiv 0 \pmod 4$. We have $G - v = G^3_{n-1}$ with classes A^3_{n-1} and B^3_{n-1}, where $|A^3_{n-1}| = \frac{n}{2} - 1$ and $|B^3_{n-1}| = \frac{n}{2}$, and B^3_{n-1} containing a perfect matching. We have $\deg(v) = \frac{n}{2}$. If $N(v) = B^3_{n-1}$, then $G = G^0_n$. Otherwise, if v has neighbours $c, d \in A^3_{n-1}$ and $u \in B^3_{n-1}$, then $abcvd + u$ is a copy of gem$_5$ in G, where $a \in A^3_{n-1} \setminus \{c, d\}$ and $b \in B^3_{n-1}$ is the vertex adjacent to u. If v has exactly one neighbour $u \in A^3_{n-1}$, then since $|B^3_{n-1}| = \frac{n}{2} > 4$, we can find $a, b, c, d \in B^3_{n-1}$ such that $ab, cd, bv, cv \in E(G)$. We have $abvcd + u$ is a copy of gem$_5$ in G.

Case 2. $n \equiv 1 \pmod 4$. If $n \geq 13$, we have $G - v = G^0_{n-1}$. If $n = 9$, we have $G - v \in \{G^0_8, G'_8, G''_8\}$.

Subcase 2.1. $n \geq 9$ and $G - v = G^0_{n-1}$. The classes of $G - v$ are A^0_{n-1} and B^0_{n-1}. Since $|B^0_{n-1}| = \frac{n-1}{2} \geq 4$, this subcase can be considered by combining the arguments used in Case 2 of Theorem 2.1 and in Case 1 above. We find that either G contains a copy of gem_5, or $G \in \{G^{11}_n, G^{12}_n\}$.

Subcase 2.2. $n = 9$ and $G - v \in \{G'_8, G''_8\}$. Suppose first that $G - v = G'_8$, so that the classes of $G - v$ are A'_8 and B'_8 with $|A'_8| = |B'_8| = 4$, and each class containing one edge, say cu and ab are the edges in A'_8 and B'_8. We have $\deg(v) = 4$. If $N(v) = A'_8$ or $N(v) = B'_8$, then $G = G'_9$, and if $N(v) = (A'_8 \cup B'_8) \setminus \{a, b, c, u\}$, then $G = G''_9$. Otherwise, let $d \in B'_8 \setminus \{a, b\}$. We may assume that $uv \in E(G)$, and either $av \in E(G)$ or $dv \in E(G)$. Then $vabcd + u$ or $abcdv + u$ is a copy of gem_5.

Now, suppose that $G - v = G''_8$. The classes of $G - v$ are A''_8 and B''_8 with $|A''_8| = 2$, $|B''_8| = 6$, and there are two vertex-disjoint triangles embedded into B''_8. Let $A''_8 = \{b, d\}$ and acu be one of the triangles in B''_8. We have $\deg(v) = 4$. If $bv, dv \in E(G)$, then we may assume that $uv \in E(G)$. We have $abcdv + u$ is a copy of gem_5. Otherwise, v has at least three neighbours in B''_8, and we may assume that $av, uv \in E(G)$. Then $vabcd + u$ is a copy of gem_5.

Case 3. $n \equiv 2 \pmod 4$. If $n \geq 14$, then we have $G - v \in \{G^{11}_{n-1}, G^{12}_{n-1}\}$. If $n = 10$, then we have $G - v \in \{G^{11}_9, G^{12}_9, G'_9, G''_9\}$.

Subcase 3.1. $n \geq 10$ and $G - v \in \{G^{11}_{n-1}, G^{12}_{n-1}\}$. If $G - v = G^{11}_{n-1}$, then $|A^{11}_{n-1}| = \frac{n}{2} - 1 \geq 4$. If $G - v = G^{12}_{n-1}$, then $G - v$ has the class B^{12}_{n-1} which contains a maximum matching with an unmatched vertex, say w. We have $|B^{12}_{n-1} \setminus \{w\}| = \frac{n}{2} - 1 \geq 4$. Since $\deg(v) = \frac{n}{2}$, this subcase can be considered by combining the arguments used in Case 3 of Theorem 2.1 and in Case 1 above. We find that either G contains a copy of gem_5, or $G \in \{G^{21}_n, G^{22}_n\}$.

Subcase 3.2. $n = 10$ and $G - v \in \{G'_9, G''_9\}$. Suppose first that $G - v = G'_9$, so that the classes of $G - v$ are A'_9 and B'_9 with $|A'_9| = 4$, $|B'_9| = 5$, and each class containing one edge. We have $\deg(v) = 5$. If $N(v) = B'_9$, then $G = G'_{10}$. If v has a neighbour which is incident with the edge in A'_9 or the edge in B'_9, then as in the argument in the first part of Subcase 2.2, G contains a copy of gem_5. Otherwise, $N(v)$ consists of the five vertices not incident with the two edges within A'_9 and B'_9. Therefore, if $b, d \in A'_9$ and $a, c, e \in B'_9$ are these five neighbours of v, then $abcde + v$ is a copy of gem_5.

Now, suppose that $G - v = G''_9$. The graph $G - v$ consists of two sets A''_9 and B''_9 where $|A''_9| = |B''_9| = 4$, with one edge in each set, say f_1 in A''_9 and f_2 in B''_9, and another vertex, say z, joined to the four vertices not incident with f_1, f_2. Let $b, d \in A''_9$ and $a, c \in B''_9$ be the neighbours of z in $G - v$. We have $\deg(v) = 5$.

Again, if v has a neighbour in each of A''_9 and B''_9 where at least one is incident with f_1 or f_2, then by the argument in Subcase 2.2, G contains a copy of gem_5. Otherwise, we may assume that $N(v) = A''_9 \cup \{z\}$ or $N(v) = \{a, b, c, d, z\}$, and $abcdv + z$ is a copy of gem_5.

This concludes the case when $\delta(G) \le \lfloor \frac{n}{2} \rfloor$.

Next, suppose that $\delta(G) \ge \lfloor \frac{n}{2} \rfloor + 1$. Then exactly as in the proof of Theorem 2.1, we must have $n \equiv 3 \pmod 4$, and that G is a $(\lfloor \frac{n}{2} \rfloor + 1)$-regular graph. Again for $v \in V(G)$, we have $e(G-v) = e_{n-1}$, using exactly the same argument as in (4). By induction, either $G-v$, and thus G, contains a copy of gem_5, or $G-v \in \mathcal{F}_{n-1,5}$. If the latter holds, then for $n \ge 15$ we have $G-v \in \{G^{21}_{n-1}, G^{22}_{n-1}\}$, and for $n = 11$ we have $G - v \in \{G^{21}_{10}, G^{22}_{10}, G'_{10}\}$. If $n \ge 11$ and $G - v \in \{G^{21}_{n-1}, G^{22}_{n-1}\}$, then as in the proof of Theorem 2.1, the fact that G is a $(\lfloor \frac{n}{2} \rfloor + 1)$-regular graph implies that $G = G^3_n$. Otherwise, we have $n = 11$ and $G-v = G'_{10}$. Then G is a 6-regular graph, which means that $N(v)$ consists of the six vertices not incident with the two edges within A'_{10} and B'_{10}. Therefore, if $a, c, e \in A'_{10}$ and $b, d \in B'_{10}$ are neighbours of v, then $abcde + v$ is a copy of gem_5.

This completes the proof Theorem 2.3. ■

3. Decompositions of Graphs Into Gem Graphs and Single Edges

Recall that for a fixed graph H, $\phi(n, H)$ denotes the smallest integer ϕ such that any graph on n vertices admits an H-decomposition with at most ϕ parts. In this section we will verify Pikhurko and Sousa conjecture (Conjecture 1.3) for the gem graphs gem_4 and gem_5. That is, we will show that $\phi(n, \text{gem}_4) = \text{ex}(n, \text{gem}_4)$ for $n \ge 6$, and $\phi(n, \text{gem}_5) = \text{ex}(n, \text{gem}_5)$ for $n \ge 8$.

3.1. gem_4-decompositions

We begin by considering gem_4-decompositions, and prove the following result.

Theorem 3.1. *For $n \ge 6$ we have*

$$\phi(n, \text{gem}_4) = \text{ex}(n, \text{gem}_4).$$

Moreover, the only graphs attaining $\text{ex}(n, \text{gem}_4)$ are the members of $\mathcal{F}_{n,4}$.

Proof. Let $n \ge 6$. The lower bound $\phi(n, \text{gem}_4) \ge \text{ex}(n, \text{gem}_4)$ holds by considering any graph of $\mathcal{F}_{n,4}$. We prove the matching upper bound. By Theorem 2.1, we know that $\text{ex}(n, \text{gem}_4) = e_n$ for $n \ge 6$. Let G be a graph on $n \ge 6$ vertices. We must prove that $\phi(G, \text{gem}_4) \le \text{ex}(n, \text{gem}_4) = e_n$, with equality if and only if $G \in \mathcal{F}_{n,4}$.

We proceed by induction on n. For $n = 6$, if $e(G) < e_6 = 10$, then we can simply decompose G into single edges to obtain $\phi(G, \text{gem}_4) < e_6$. Otherwise, let

$10 = e_6 \le e(G) \le 15$. By Theorem 2.1, we either have $G \in \mathcal{F}_{6,4}$, or G contains a copy of gem_4. If $G \in \mathcal{F}_{6,4}$, then $e(G) = e_6 = 10$ and we must decompose G into single edges, thus, $\phi(G, \text{gem}_4) = e_6$ as required. If G contains a copy of gem_4, then $\phi(G, \text{gem}_4) \le 1 + e(G) - e(\text{gem}_4) \le 9 < 10 = e_6$. Thus, the theorem holds for $n = 6$.

Now, let $n \ge 7$, and suppose that the theorem holds for $n-1$. Let G be a graph on n vertices. As before, if $e(G) < e_n$, then $\phi(G, \text{gem}_4) < e_n$, simply by decomposing G into single edges. If $e(G) = e_n$, then by Theorem 2.1, either G contains a copy of gem_4, in which case $\phi(G, \text{gem}_4) \le 1 + e(G) - e(\text{gem}_4) = e_n - 6 < e_n$, or $G \in \mathcal{F}_{n,4}$, in which case we can only decompose G into e_n single edges for a gem_4-decomposition, and $\phi(G, \text{gem}_4) = e_n$ as required.

Now, suppose that $e(G) > e_n$, and let $v \in V(G)$ be a vertex of minimum degree. If $\deg(v) \le \lfloor \frac{n}{2} \rfloor$, then by equation (2) we have $e(G-v) = e(G) - \deg(v) > e_n - \lfloor \frac{n}{2} \rfloor \ge e_{n-1}$, that is, $G - v \notin \mathcal{F}_{n-1,4}$ and by the induction hypothesis we have

$$\phi(G - v, \text{gem}_4) < \text{ex}(n-1, \text{gem}_4) = e_{n-1}.$$

Therefore, when going from $G - v$ to G we only need to use the edges joining v to the other vertices of G, and there are at most $\lfloor \frac{n}{2} \rfloor$ of these edges at v. We have

$$\phi(G, \text{gem}_4) \le \phi(G - v, \text{gem}_4) + \deg(v) < e_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor \le e_n,$$

as required.

Therefore, we may assume that $\deg(v) \ge \lfloor \frac{n}{2} \rfloor + 1$ and let $\deg(v) = \lfloor \frac{n}{2} \rfloor + m$ for some integer $m \ge 1$. For every $x \in N(v)$, we have

$$\begin{aligned} \deg(x, N(v)) &\ge \left\lfloor \frac{n}{2} \right\rfloor + m - \left(n - \left\lfloor \frac{n}{2} \right\rfloor - m\right) \\ &= 2\left\lfloor \frac{n}{2} \right\rfloor + 2m - n \ge 2m - 1. \end{aligned} \tag{7}$$

This means that $G[N(v)]$ must contain a path P_{2m} on $2m$ vertices. Otherwise, if the longest path in $G[N(v)]$ has at most $2m - 1$ vertices, say with an end-vertex y, then all neighbours of y in $N(v)$ must lie in the path, so that $\deg(y, N(v)) \le 2m - 2$, contradicting (7).

If $m \ge 2$, then the path P_{2m} contains $\lfloor \frac{2m}{4} \rfloor = \lfloor \frac{m}{2} \rfloor$ vertex-disjoint paths of order 4. Thus, we have $\lfloor \frac{m}{2} \rfloor$ edge-disjoint copies of gem_4, where each copy is formed by a path of order 4, together with v. Let $F \subset G - v$ be the subgraph of order $n - 1$ obtained by deleting the edges of the paths of order 4 from $G - v$. By induction and (2), and since $m \ge 2$, we have

$$\begin{aligned} \phi(G, \text{gem}_4) &\le \phi(F, \text{gem}_4) + \left\lfloor \frac{m}{2} \right\rfloor + \deg(v) - 4\left\lfloor \frac{m}{2} \right\rfloor \\ &\le e_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor + m - 3\left\lfloor \frac{m}{2} \right\rfloor < e_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor \le e_n. \end{aligned}$$

To complete the proof it remains to consider the case $m = 1$. For this case, we will repeatedly use the following claim.

Claim 3.2. *Suppose that there exists a vertex $z \in V(G)$ with $\deg(z) = \lfloor \frac{n}{2} \rfloor + 1$, and G has a copy of gem_4 with at least three edges incident to z. Then $\phi(G, \mathrm{gem}_4) < e_n$.*

Proof. Let $F \subset G - z$ be the subgraph on $n - 1$ vertices obtained from $G - z$ by deleting the edges of the copy of gem_4. By induction and (2), we have

$$\phi(G, \mathrm{gem}_4) \leq \phi(F, \mathrm{gem}_4) + 1 + \deg(z) - 3 \leq e_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor - 1 < e_n. \qquad \square$$

We now consider three cases. Let $\overline{N}(v) = V(G) \setminus (N(v) \cup \{v\})$, and note that

$$|N(v)| = \left\lfloor \frac{n}{2} \right\rfloor + 1 \geq 4 \quad \text{and} \quad |\overline{N}(v)| = \left\lceil \frac{n}{2} \right\rceil - 2 \geq 2.$$

Case 1. $G[N(v)]$ contains a path P of order 4. Then P and v form a copy of gem_4, and we have $\phi(G, \mathrm{gem}_4) < e_n$ by Claim 3.2.

Case 2. The order of the longest path in $G[N(v)]$ is 3. Let $x_1 x x_2$ be a path of order 3 in $G[N(v)]$.

Subcase 2.1. $x_1 x_2 \in E(G)$. We have $\deg(x, N(v)) = 2$, for otherwise $G[N(v)]$ would contain a P_4. We must have $\deg(x, \overline{N}(v)) \geq \lfloor \frac{n}{2} \rfloor + 1 - 3 \geq |\overline{N}(v)| - 1$. Similarly for x_1, x_2. This implies that two of x, x_1, x_2 have a common neighbour in $\overline{N}(v)$, say $y \in \overline{N}(v)$ is a common neighbour of x, x_1. Then $x_2 v x_1 y + x$ is a copy of gem_4, and by Claim 3.2 with $z = v$, we have $\phi(G, \mathrm{gem}_4) < e_n$.

Subcase 2.2. $x_1 x_2 \notin E(G)$. Let $N(v) = \{x, x_1, x_2, \ldots, x_{\lfloor n/2 \rfloor}\}$. For $i = 1, 2$, we have $\deg(x_i, N(v)) = 1$, and

$$\text{(8)} \qquad \deg(x_i, \overline{N}(v)) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 - 2 \geq \left\lceil \frac{n}{2} \right\rceil - 2 = |\overline{N}(v)|.$$

We must have equality to hold throughout, whence n is odd, $\deg(x_1) = \deg(x_2) = \lfloor \frac{n}{2} \rfloor + 1$, and both x_1, x_2 are adjacent to all vertices of $\overline{N}(v)$. If x has a neighbour $y \in \overline{N}(v)$, then $x_1 v x_2 y + x$ is a copy of gem_4, and again $\phi(G, \mathrm{gem}) < e_n$ by Claim 3.2 with $z = v$.

Otherwise, suppose that x does not have a neighbour in $\overline{N}(v)$. Then $\deg(x) \leq |N(v) \cup \{v\}| - 1 = \lfloor \frac{n}{2} \rfloor + 1$, so that $\deg(x) = \lfloor \frac{n}{2} \rfloor + 1$ and $x x_i \in E(G)$ for all $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. Moreover, we have $x_i x_j \notin E(G)$ for all $i \neq j$, otherwise there would exist a copy of P_4 in $G[N(v)]$. By a similar argument as in (8), we have $\deg(x_i) = \lfloor \frac{n}{2} \rfloor + 1$, and x_i is adjacent to all vertices of $\overline{N}(v)$ for all $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. In order to get a contradiction, suppose that there does not exist a path of order 3 in $G[\overline{N}(v)]$. Then the maximum number of edges in $G[\overline{N}(v)]$ is $\lfloor \frac{1}{2}|\overline{N}(v)| \rfloor$. Recall that n is odd. We have

$$\begin{aligned} e(G) &\leq 2|N(v)| - 1 + (|N(v)| - 1)|\overline{N}(v)| + \left\lfloor \frac{1}{2}|\overline{N}(v)| \right\rfloor \\ &= 2\left\lfloor \frac{n}{2} \right\rfloor + 1 + \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lceil \frac{n}{2} \right\rceil - 2 \right) + \left\lfloor \frac{1}{2}\left(\left\lceil \frac{n}{2} \right\rceil - 2 \right) \right\rfloor \\ &= \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor = e_n, \end{aligned}$$

by (1), which contradicts the assumption $e(G) > e_n$. Therefore, $G[\overline{N}(v)]$ must have a path of order 3, say $y_1y_2y_3$. Note that $|\overline{N}(v)| = \lceil \frac{n}{2} \rceil - 2 \geq 3$ and thus we must have n odd and $n \geq 9$. Then, $x_1y_1x_2y_3 + y_2$ is a copy of gem_4, and by Claim 3.2 with $z = x_1$, we have $\phi(G, \text{gem}) < e_n$.

Case 3. The longest path in $G[N(v)]$ has order 2. Note that this is indeed the remaining case, since $\deg(x, N(v)) \geq 2m - 1 = 1$ for all $x \in N(v)$ by (7). Moreover, $N(v)$ induces a perfect matching in G. Now by a similar argument as in (8), we must have n odd, and for every $x \in N(v)$, we have $\deg(x) = \lfloor \frac{n}{2} \rfloor + 1$ and x is adjacent to all vertices of $\overline{N}(v)$. Thus, we can find an edge x_1x_2 in $G[N(v)]$ and a common neighbour $y \in \overline{N}(v)$ of x_1, x_2. Now, since vx_2y is a path of order 3 in $G[N(x_1)]$, we are done by applying Case 1 or Case 2 with x_1 in place of v.

The induction step is complete, and this completes the proof of Theorem 3.1. ∎

3.2. gem_5-decompositions

By using the same ideas as in the proof of Theorem 3.1, but with more case analysis, we will be able to prove a similar result for gem_5-decompositions. That is, we will prove the following theorem.

Theorem 3.3. *For $n \geq 8$ we have*

$$\phi(n, \text{gem}_5) = \text{ex}(n, \text{gem}_5).$$

Moreover, the only graphs attaining $\text{ex}(n, \text{gem}_5)$ are the members of $\mathcal{F}_{n,5}$.

Proof. Let $n \geq 8$. As before, we have $\phi(n, \text{gem}_5) \geq \text{ex}(n, \text{gem}_5)$ by considering any graph of $\mathcal{F}_{n,5}$. By Theorem 2.3, to prove the matching upper bound, we must prove that if G is a graph on $n \geq 8$ vertices, then $\phi(G, \text{gem}_5) \leq \text{ex}(n, \text{gem}_5) = e_n$, with equality if and only if $G \in \mathcal{F}_{n,5}$.

We proceed by induction on n. For $n = 8$, if $e(G) < e_8 = 18$, then we can simply decompose G into single edges to obtain $\phi(G, \text{gem}_4) < e_8$. Next, suppose that $18 = e_8 \leq e(G) \leq 25$. By Theorem 2.3, we either have $G \in \mathcal{F}_{8,5}$, or G contains a copy of gem_5. If $G \in \mathcal{F}_{8,5}$, then $e(G) = e_8 = 18$ and we must decompose G into single edges, and $\phi(G, \text{gem}_5) = e_8$. If G contains a copy of gem_5, then $\phi(G, \text{gem}_5) \leq 1 + e(G) - e(\text{gem}_5) \leq 17 < 18 = e_8$. Finally, suppose that $26 \leq e(G) \leq 28$. Clearly, there exist two vertices $x, y \in V(G)$ of degree 7, so that $e(G - \{x, y\}) \geq 26 - 1 - 2 \cdot 6 = 13$. Since $\text{ex}(6, P_5) = \binom{4}{2} + \binom{2}{2} = 7$ by Theorem 1.1, this means that we can find two edge-disjoint copies of P_5 in $G - \{x, y\}$. These two copies of P_5, together with x and y, form two edge-disjoint copies of gem_5 in G. Thus, $\phi(G, \text{gem}_5) \leq 2 + e(G) - 2e(\text{gem}_5) \leq 12 < 18 = e_8$. The theorem holds for $n = 8$.

Now, let $n \geq 9$, and suppose that the theorem holds for $n - 1$. Let G be a graph on n vertices. As before, if $e(G) < e_n$, then $\phi(G, \text{gem}_5) < e_n$, simply by decomposing G into single edges. If $e(G) = e_n$, then by Theorem 2.3, either G contains a copy of gem_5, in which case $\phi(G, \text{gem}_5) \leq 1 + e(G) - e(\text{gem}_5) = e_n - 8 < e_n$, or $G \in \mathcal{F}_{n,5}$, in which case we can only decompose G into e_n single edges for a gem_5-decomposition, and $\phi(G, \text{gem}_5) = e_n$ as required.

Now, suppose that $e(G) > e_n$, and let $v \in V(G)$ be a vertex of minimum degree. If $\deg(v) \leq \lfloor \frac{n}{2} \rfloor$, then by equation (6), we have $e(G - v) = e(G) - \deg(v) > e_n - \lfloor \frac{n}{2} \rfloor \geq e_{n-1}$, that is, $G - v \notin \mathcal{F}_{n-1,5}$. By induction, we have $\phi(G - v, \text{gem}_5) < \text{ex}(n - 1, \text{gem}_5) = e_{n-1}$. Thus, when going from $G - v$ to G we only need to use the edges joining v to the other vertices of G. We have

$$\phi(G, \text{gem}_5) \leq \phi(G - v, \text{gem}_5) + \deg(v) < e_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor \leq e_n.$$

Therefore, we may assume that $\deg(v) \geq \lfloor \frac{n}{2} \rfloor + 1$ and let $\deg(v) = \lfloor \frac{n}{2} \rfloor + m$ for some integer $m \geq 1$. As in (7), for every $x \in N(v)$, we have $\deg(x, N(v)) \geq 2m - 1$, and that $G[N(v)]$ must contain a path P_{2m} on $2m$ vertices.

If $m \geq 3$, then the path P_{2m} contains $\lfloor \frac{2m}{5} \rfloor$ vertex-disjoint paths of order 5. Thus, we have $\lfloor \frac{2m}{5} \rfloor$ edge-disjoint copies of gem_5, where each copy is formed by a path of order 5, together with v. Let $F \subset G - v$ be the subgraph of order $n - 1$ obtained by deleting the edges of the paths of order 5 from $G - v$. By induction and (6), and since $m \geq 3$, we have

$$\begin{aligned} \phi(G, \text{gem}_5) &\leq \phi(F, \text{gem}_5) + \left\lfloor \frac{2m}{5} \right\rfloor + \deg(v) - 5 \left\lfloor \frac{2m}{5} \right\rfloor \\ &\leq e_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor + m - 4 \left\lfloor \frac{2m}{5} \right\rfloor < e_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor \leq e_n. \end{aligned}$$

For the rest of the proof, let $\overline{N}(v) = V(G) \setminus (N(v) \cup \{v\})$. Next, suppose that $m = 2$, so that $|N(v)| = \lfloor \frac{n}{2} \rfloor + 2 \geq 6$ and $|\overline{N}(v)| = \lceil \frac{n}{2} \rceil - 3 \geq 2$. If $G[N(v)]$ contains a path P_5 of order 5, then this path together with v form a copy of gem_5.

Let $F \subset G - v$ be the subgraph of order $n-1$, obtained by deleting the edges of the P_5. Then,

$$\phi(G, \text{gem}_5) \leq \phi(F, \text{gem}_5) + 1 + \deg(v) - 5 \leq e_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor + 2 - 4 < e_n.$$

Therefore, we may assume that the longest path in $G[N(v)]$ has order 4. Let $x_1x_2x_3x_4$ be such a path in $G[N(v)]$. Since $\deg(x_1, N(v)) \geq 2 \cdot 2 - 1 = 3$, we must have $x_1x_3, x_1x_4 \in E(G)$. Moreover, the only neighbours of x_1 in $N(v)$ are x_2, x_3, x_4, so that

$$\deg(x_1, \overline{N}(v)) \geq \left\lfloor \frac{n}{2} \right\rfloor + 2 - 4 \geq \left\lceil \frac{n}{2} \right\rceil - 3 = |\overline{N}(v)|.$$

We must have equality, so that n is odd, $\deg(x_1) = \left\lfloor \frac{n}{2} \right\rfloor + 2$, and x_1 is adjacent to every vertex of $\overline{N}(v)$. The same argument holds for x_4, so that x_1, x_4 have a common neighbour $y \in \overline{N}(v)$. Now, since $vx_2x_3x_4y$ is a path of order 5 in $G[N(x_1)]$, we are done by applying the previous argument with x_1 in place of v.

To complete the proof it remains to consider the case $m = 1$. As before, we will repeatedly use the following claim which is analogous to Claim 3.2.

Claim 3.4. *Suppose that there exists a vertex $z \in V(G)$ with $\deg(z) = \left\lfloor \frac{n}{2} \right\rfloor + 1$, and G has a copy of gem_5 with at least three edges incident to z. Then $\phi(G, \text{gem}_5) < e_n$.*

Proof. Exactly the same as the proof of Claim 3.2. □

We now consider four cases. Note that we have

$$|N(v)| = \left\lfloor \frac{n}{2} \right\rfloor + 1 \geq 5 \quad \text{and} \quad |\overline{N}(v)| = \left\lceil \frac{n}{2} \right\rceil - 2 \geq 3.$$

Case 1. $G[N(v)]$ contains a path P of order 5. Then P and v form a copy of gem_5, and we have $\phi(G, \text{gem}_5) < e_n$ by Claim 3.4.

Case 2. The order of the longest path in $G[N(v)]$ is 4. Let $x_1x_2x_3x_4$ be such a path in $G[N(v)]$. It suffices to consider the following subcases.

Subcase 2.1. $x_1x_3, x_1x_4 \in E(G)$. For $i = 1, 2, 3, 4$, x_i does not have a neighbour in $N(v) \setminus \{x_1, x_2, x_3, x_4\}$, so that $\deg(x_i, N(v)) \leq 3$. Thus,

$$\deg(x_i, \overline{N}(v)) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 - 4 \geq \left\lceil \frac{n}{2} \right\rceil - 4 = |\overline{N}(v)| - 2. \tag{9}$$

If $x_2x_4 \notin E(G)$, then we have $\deg(x_j, N(v)) = 2$, and $\deg(x_j, \overline{N}(v)) \geq |\overline{N}(v)| - 1$ for $j = 2, 4$. With (9), this implies that either x_1, x_2 or x_2, x_3 or x_1, x_3, have a common neighbour $y \in \overline{N}(v)$. Then, either $x_4vx_3x_2y + x_1$; or $x_4vx_1x_2y + x_3$; or $x_4vx_2x_3y + x_1$, is a copy of gem_5, respectively. By Claim 3.4 with $z = v$, we have $\phi(G, \text{gem}_5) < e_n$. Now, if $x_2x_4 \in E(G)$, then by (9), two of x_1, x_2, x_3, x_4 have a common neighbour in $\overline{N}(v)$. We may assume that x_1, x_2 have a common neighbour $y \in \overline{N}(v)$. Then we have $\phi(G, \text{gem}_5) < e_n$ by the same argument.

Subcase 2.2. $x_1x_3 \in E(G)$ and $x_1x_4, x_2x_4 \notin E(G)$. We see that x_3 is the only neighbour of x_4 in $N(v)$, so that

$$\deg(x_4, \overline{N}(v)) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 - 2 \geq \left\lceil \frac{n}{2} \right\rceil - 2 = |\overline{N}(v)|.$$

We must have equality throughout, so that $\deg(x_4) = \left\lfloor \frac{n}{2} \right\rfloor + 1$ and n is odd. Moreover, x_4 is adjacent to every vertex of $\overline{N}(v)$. If x_3 has a neighbour $y \in \overline{N}(v)$, then $x_1x_2vx_4y + x_3$ is a copy of gem_5, and we have $\phi(G, \text{gem}_5) < e_n$ by Claim 3.4 with $z = v$. Now suppose that x_3 does not have a neighbour in $\overline{N}(v)$. Let $x_5, x_6, \ldots, x_{\lfloor n/2 \rfloor + 1}$ be the remaining vertices of $N(v)$. Then $\deg(x_3) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$ implies that $x_3x_i \in E(G)$ for every $i \geq 5$. Moreover, we have $x_1x_i, x_2x_i \notin E(G)$ for all $i \geq 5$, otherwise we are in Subcase 2.1. This means that $\deg(x_i) = \left\lfloor \frac{n}{2} \right\rfloor + 1$ and x_i is adjacent to every vertex of $\overline{N}(v)$ for all $i \geq 4$. Also, note that for $i = 1, 2$,

$$\deg(x_i, \overline{N}(v)) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 - 3 = \left\lceil \frac{n}{2} \right\rceil - 3 = |\overline{N}(v)| - 1.$$

Suppose first that $G[\overline{N}(v)]$ contains a path of order 3, say $y_1y_2y_3$. If $n \geq 11$ so that $|N(v)| = \left\lfloor \frac{n}{2} \right\rfloor + 1 \geq 6$, then $x_4y_1x_5y_3x_6 + y_2$ is a copy of gem_5, and we have $\phi(G, \text{gem}_5) < e_n$ by Claim 3.4 with $z = x_5$. Now let $n = 9$, and suppose that $x_1y_1, x_1y_2 \in E(G)$. Then $x_1y_1x_4y_3x_5 + y_2$ is a copy of gem_5, and we have $\phi(G, \text{gem}_5) < e_n$ by Claim 3.4 with $z = x_4$. Thus, we may assume that $x_1y_1, x_1y_3, x_2y_1, x_2y_3 \in E(G)$ and $x_1y_2, x_2y_2 \notin E(G)$. It is easy to check that G is the graph G''_9 with $A''_9 = \{x_1, x_2, x_4, x_5\}$, $B''_9 = \{v, x_3, y_1, y_3\}$, and y_2 is the remaining vertex, so that $\phi(G, \text{gem}_5) = e_9 = \text{ex}(9, \text{gem}_5)$.

Now, suppose that $G[\overline{N}(v)]$ contains an edge, say y_1y_2. If x_1 is adjacent to every vertex in $\overline{N}(v)$, then we may assume that $x_2y_1 \in E(G)$. Then $x_3vx_2y_1y_2 + x_1$ is a copy of gem_5, and we have $\phi(G, \text{gem}_5) < e_n$ by Claim 3.4 with $z = v$. Thus we may assume that x_1 and x_2 are not adjacent to exactly one vertex in $\overline{N}(v)$. Since there are at most $|N(v)|$ edges in $G[N(v)]$ and at most $\left\lfloor \frac{1}{2}|\overline{N}(v)| \right\rfloor$ edges in $G[\overline{N}(v)]$, we have

$$\begin{aligned} e(G) &\leq 2|N(v)| + 2(|\overline{N}(v)| - 1) + (|N(v)| - 3)|\overline{N}(v)| + \left\lfloor \frac{1}{2}|\overline{N}(v)| \right\rfloor \\ &= 2n - 4 + \Big(\left\lfloor \frac{n}{2} \right\rfloor - 2\Big)\Big(\left\lceil \frac{n}{2} \right\rceil - 2\Big) + \left\lfloor \frac{1}{2}\Big(\left\lceil \frac{n}{2} \right\rceil - 2\Big) \right\rfloor \\ &= \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor = e_n, \end{aligned}$$

by (5) and since n is odd, which contradicts the assumption $e(G) > e_n$. Finally, if $G[\overline{N}(v)]$ does not contain an edge, then

$$\begin{aligned} e(G) &\le 2|N(v)| + (|N(v)| - 1)|\overline{N}(v)| \\ &= 2\Big(\Big\lfloor \frac{n}{2} \Big\rfloor + 1\Big) + \Big\lfloor \frac{n}{2} \Big\rfloor \Big(\Big\lceil \frac{n}{2} \Big\rceil - 2\Big) = \Big\lfloor \frac{n^2}{4} \Big\rfloor + 2 \le e_n, \end{aligned}$$

another contradiction.

Subcase 2.3. $x_1x_4 \in E(G)$ and $x_1x_3, x_2x_4 \notin E(G)$. For $i = 1, 2, 3, 4$, x_i does not have a neighbour in $N(v) \setminus \{x_1, x_2, x_3, x_4\}$, so that $\deg(x_i, N(v)) = 2$. Thus,

$$\deg(x_i, \overline{N}(v)) \ge \Big\lfloor \frac{n}{2} \Big\rfloor + 1 - 3 \ge \Big\lceil \frac{n}{2} \Big\rceil - 3 = |\overline{N}(v)| - 1. \tag{10}$$

If $\deg(x_1, \overline{N}(v)) = |\overline{N}(v)|$, then we can find $y_1, y_2 \in \overline{N}(v)$ such that y_1 is a common neighbour of x_1, x_2, and y_2 is a common neighbour of x_2, x_3. Then $y_1x_1vx_3y_2 + x_2$ is a copy of gem_5, and we have $\phi(G, \mathrm{gem}_5) < e_n$ by Claim 3.4 with $z = v$. Otherwise, we must have equality in (10) for $i = 1, 2, 3, 4$, so that n is odd, and for $i = 1, 2, 3, 4$, we have $\deg(x_i) = \lfloor \frac{n}{2} \rfloor + 1$, and x_i is not adjacent to exactly one vertex in $\overline{N}(v)$. If $n \ge 11$ so that $|\overline{N}(v)| = \lceil \frac{n}{2} \rceil - 2 \ge 4$, then we can again find the vertices $y_1, y_2 \in \overline{N}(v)$ and we are done as before. Now let $n = 9$, so that $|N(v)| = 5$, $|\overline{N}(v)| = 3$, and each x_i has exactly two neighbours in $\overline{N}(v)$. If x_1 and x_2 have two common neighbours in $\overline{N}(v)$, then we can again find $y_1, y_2 \in \overline{N}(v)$ as before and we are done. Otherwise, we may assume that $\overline{N}(v) = \{z_1, z_2, z_3\}$ with $x_1z_1, x_1z_2, x_2z_1, x_2z_3 \in E(G)$. If $z_1z_2 \in E(G)$, then $x_4vx_2z_1z_2 + x_1$ is a copy of gem_5, and again $\phi(G, \mathrm{gem}_5) < e_n$ by Claim 3.4 with $z = v$. A similar argument holds if $z_1z_3 \in E(G)$. Otherwise, we have at most one edge in $G[\overline{N}(v)]$, and since there are exactly nine edges in $G[N(v) \cup \{v\}]$ and at most $4 \cdot 2 + 3 = 11$ edges between $N(v)$ and $\overline{N}(v)$, we have $e(G) \le 1 + 9 + 11 = 21 < 22 = e_9$, which is a contradiction.

Subcase 2.4. $x_1x_3, x_1x_4, x_2x_4 \notin E(G)$. We first note that x_2 is the only neighbour of x_1 in $N(v)$, so that

$$\deg(x_1, \overline{N}(v)) \ge \Big\lfloor \frac{n}{2} \Big\rfloor + 1 - 2 \ge \Big\lceil \frac{n}{2} \Big\rceil - 2 = |\overline{N}(v)|.$$

We must have equality throughout, so that n is odd, $\deg(x_1) = \lfloor \frac{n}{2} \rfloor + 1$, and x_1 is adjacent to all vertices of $\overline{N}(v)$. The exact same properties hold for x_4. Next, suppose that x_2 has p neighbours in $N(v) \setminus \{x_1, x_2, x_3, x_4\}$, where $0 \le p \le \lfloor \frac{n}{2} \rfloor - 3$. Let S_2 be the set of these p neighbours. We have

$$\deg(x_2, \overline{N}(v)) \ge \Big\lfloor \frac{n}{2} \Big\rfloor + 1 - 3 - p = \Big\lceil \frac{n}{2} \Big\rceil - 3 - p. \tag{11}$$

Now, x_3 does not have a neighbour in S_2, otherwise there would exist a path of order 5 in $G[N(v)]$. Thus, x_3 has at most $|N(v)|-4-p=\lfloor\frac{n}{2}\rfloor-3-p$ neighbours in $N(v)\setminus\{x_1,x_2,x_3,x_4\}$. Let S_3 be these neighbours of x_3, so that $S_2\cap S_3=\emptyset$. We have

$$\deg(x_3,\overline{N}(v))\geq\left\lfloor\frac{n}{2}\right\rfloor+1-3-\left(\left\lfloor\frac{n}{2}\right\rfloor-3-p\right)=p+1. \tag{12}$$

Suppose that x_2,x_3 have a common neighbour $y_1\in\overline{N}(v)$. Clearly, from (11) and (12), at least one of x_2,x_3 has at least two neighbours in $\overline{N}(v)$. If x_2 has this property, then x_1,x_2 have a common neighbour $y_2\in\overline{N}(v)\setminus\{y_1\}$. Thus, $y_1x_3vx_1y_2+x_2$ is a copy of gem$_5$, and by Claim 3.4 with $z=v$, we have $\phi(G,\text{gem}_5)<e_n$. A similar argument holds if x_3 has at least two neighbours in $\overline{N}(v)$, with x_4 in place of x_1.

Thus, if $T_2,T_3\subset\overline{N}(v)$ are the sets of neighbours of x_2,x_3 in $\overline{N}(v)$, respectively, then we may assume that $T_2\cap T_3=\emptyset$. Note that from (11) and (12), we have

$$\deg(x_2,\overline{N}(v))+\deg(x_3,\overline{N}(v))\geq\left\lceil\frac{n}{2}\right\rceil-2=|\overline{N}(v)|.$$

Thus, we must have equality above, as well as in (11) and (12). This means that $\deg(x_2)=\deg(x_3)=\lfloor\frac{n}{2}\rfloor+1$, and we have the partitions $N(v)\setminus\{x_1,x_2,x_3,x_4\}=S_2\,\dot\cup\,S_3$ and $\overline{N}(v)=T_2\,\dot\cup\,T_3$. Clearly, there are no edges in $G[S_2\cup S_3]$, otherwise there would exist a path of order 5 in $G[N(v)]$. Next, suppose that there is a path of order 3 in $G[\overline{N}(v)]$, say $y_1y_2y_3$. Suppose that $y_2\in T_2$. Then $x_2x_1y_1x_4y_3+y_2$ is a copy of gem$_5$, so that by Claim 3.4 with $z=x_1$, we have $\phi(G,\text{gem}_5)<e_n$. A similar argument holds if $y_2\in T_3$. Otherwise, we have $|N(v)|-1$ edges in $G[N(v)]$, $|\overline{N}(v)|$ edges between $\{x_2,x_3\}$ and $\overline{N}(v)$, and at most $\lfloor\frac{1}{2}|\overline{N}(v)|\rfloor$ edges in $G[\overline{N}(v)]$. By (5) and since n is odd,

$$\begin{aligned}e(G)&\leq 2|N(v)|-1+|\overline{N}(v)|+(|N(v)|-2)|\overline{N}(v)|+\left\lfloor\frac{1}{2}|\overline{N}(v)|\right\rfloor\\&=2\left\lfloor\frac{n}{2}\right\rfloor+\left\lceil\frac{n}{2}\right\rceil-1+\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\left(\left\lceil\frac{n}{2}\right\rceil-2\right)+\left\lfloor\frac{1}{2}\left(\left\lceil\frac{n}{2}\right\rceil-2\right)\right\rfloor\\&=\left\lfloor\frac{n^2}{4}\right\rfloor+\left\lfloor\frac{n+1}{4}\right\rfloor=e_n,\end{aligned}$$

which contradicts the assumption $e(G)>e_n$.

Case 3. The order of the longest path in $G[N(v)]$ is 3. Let x_1xx_2 be such a path in $G[N(v)]$. We consider the following subcases.

Subcase 3.1. $x_1x_2\in E(G)$. We have $\deg(x,N(v))=2$, for otherwise $G[N(v)]$ would contain a P_4. Thus

$$\deg(x,\overline{N}(v))\geq\left\lfloor\frac{n}{2}\right\rfloor+1-3\geq\left\lceil\frac{n}{2}\right\rceil-3=|\overline{N}(v)|-1.$$

Similar inequalities hold for x_1, x_2. If $\deg(x, \overline{N}(v)) = |\overline{N}(v)|$, then there exist $y_1, y_2 \in \overline{N}(v)$ such that y_i is a common neighbour of x, x_i for $i = 1, 2$. Then $y_1x_1vx_2y_2 + x$ is a copy of gem_5, and by Claim 3.4 with $z = v$, we have $\phi(G, \text{gem}_5) < e_n$. Otherwise, we have $\deg(x, \overline{N}(v)) = |\overline{N}(v)| - 1$, whence n is odd and $\deg(x) = \lfloor \frac{n}{2} \rfloor + 1$. We may assume that x, x_1 have a common neighbour $y \in \overline{N}(v)$. Now, vx_2x_1y is a path of order 4 in $G[N(x)]$, and we are done by applying Case 1 or Case 2 with x in place of v.

Subcase 3.2. $x_1x_2 \notin E(G)$. Let $N(v) = \{x, x_1, x_2, \ldots, x_{\lfloor n/2 \rfloor}\}$. For $i = 1, 2$, we have

$$\deg(x_i, \overline{N}(v)) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 - 2 \geq \left\lceil \frac{n}{2} \right\rceil - 2 = |\overline{N}(v)|. \tag{13}$$

We must have equality to hold throughout, whence n is odd, $\deg(x_1) = \deg(x_2) = \lfloor \frac{n}{2} \rfloor + 1$, and both x_1, x_2 are adjacent to all vertices of $\overline{N}(v)$. If x has neighbours $y_1, y_2 \in \overline{N}(v)$, then we are done as in Subcase 3.1. If x has exactly one neighbour $y \in \overline{N}(v)$, then we have

$$\deg(x, N(v) \setminus \{x, x_1, x_2\}) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 - 4 \geq 1,$$

and we may assume that $xx_3 \in E(G)$. Then $x_1yx_2vx_3 + x$ is a copy of gem_5, and we have $\phi(G, \text{gem}_5) < e_n$ by Claim 3.4 with $z = v$. Otherwise, suppose that x does not have a neighbour in $\overline{N}(v)$. We may apply the exact same argument as in Subcase 2.2 of Theorem 3.1 to deduce that x_i is adjacent to all vertices of $\overline{N}(v)$ for all $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, and $G[\overline{N}(v)]$ must contain a path of order 3, say $y_1y_2y_3$. Then $x_1y_1x_2y_3x_3 + y_2$ is a copy of gem_5, and by Claim 3.4 with $z = x_2$, we have $\phi(G, \text{gem}_5) < e_n$.

Case 4. The longest path in $G[N(v)]$ has order 2. Note that this is indeed the remaining case, since $\deg(x, N(v)) \geq 2m - 1 = 1$ for all $x \in N(v)$. Moreover, $N(v)$ induces a perfect matching in G. By a similar argument as in (13), we must have n odd, and for every $x \in N(v)$, we have $\deg(x) = \lfloor \frac{n}{2} \rfloor + 1$ and x is adjacent to all vertices of $\overline{N}(v)$. Thus, we can find an edge x_1x_2 in $G[N(v)]$ and a common neighbour $y \in \overline{N}(v)$ of x_1, x_2. Now, since vx_2y is a path of order 3 in $G[N(x_1)]$, we are done by applying Case 1, Case 2 or Case 3 with x_1 in place of v.

The induction step is complete, and this completes the proof of Theorem 3.3. ∎

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