# TREES WITH UNIQUE LEAST CENTRAL SUBTREES 

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#### Abstract

A subtree $S$ of a tree $T$ is a central subtree of $T$ if $S$ has the minimum eccentricity in the join-semilattice of all subtrees of $T$. Among all subtrees lying in the join-semilattice center, the subtree with minimal size is called the least central subtree. Hamina and Peltola asked what is the characterization of trees with unique least central subtree? In general, it is difficult to characterize completely the trees with unique least central subtree. Nieminen and Peltola [The subtree center of a tree, Networks 34 (1999) 272-278] characterized the trees with the least central subtree consisting just of a single vertex. This paper characterizes the trees having two adjacent vertices as a unique least central subtree.


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## 1. Introduction

The "central part" of a graph has many important applications in the facility location, and it has been well studied in the literature (see, for example, $[2,4,5,7,11-15])$. Applications of the center problem include the location of industrial plants, warehouses, distribution centers, and public service facilities in transportation networks, as well as the location of various facilities in telecommunication networks.

The concepts of central subtrees and least central subtrees were introduced in [10]. For every tree $T$, a join-semilattice $L(T)$ of subtrees of $T$ is defined in [10] as follows. The meet $S_{1} \wedge S_{2}$ of subtrees $S_{1}$ and $S_{2}$ equals the subtree induced by the intersection of the vertex sets of $S_{1}$ and $S_{2}$ whenever the intersection is nonempty, while the join $S_{1} \vee S_{2}$ is the least subtree of $T$ containing the subtrees $S_{1}$ and $S_{2}$. In other words, $S_{1} \vee S_{2}$ is the subtree induced by the union of vertices of $S_{1}$ and $S_{2}$ whenever the intersection of the vertex sets of $S_{1}$ and $S_{2}$ is nonempty. In the case of nonintersection subtrees, $S_{1} \vee S_{2}$ is the subtree induced by the union of vertices of $S_{1}$ and $S_{2}$ together with the vertices of the path from $S_{1}$ to $S_{2}$. A subtree $S$ of a tree $T$ is a central subtree of $T$ if $S$ has the minimum eccentricity in the join-semilattice of all subtrees of $T$. It can be a vertex, or a path, or some other kind of subtrees such that the subtree is the most central when compared with all subtrees of the tree. The Hasse diagram graph $G_{L}$ of $L(T)$ is a median graph $[1,8,9]$. The graph center of the median graph $G_{L}$ is closely related to central subtrees. The set of all central subtrees is, in fact, the set of central vertices of the graph $G_{L}$. Among all subtrees lying in the join-semilattice center, the best is the one with minimal size. That is the least central subtree. For paths and stars, the least central subtree is unique and coincides with the center of the tree.

Nieminen and Peltola [10] described the general properties of a least central subtree of a tree, they gave some connections between the least central subtree and the center/centroid of a tree, and proved that the intersection of two least central subtrees is nonempty. Hamina and Peltola [6] proved that every least central subtree of a tree contains the center and at least one vertex of the centroid of the tree.

Hamina and Peltola posed the following problem in [6].
Problem. What is the characterization of trees with unique least central subtree?

Motivated by this problem, Nieminen and Peltola [10] describes the structure of the trees with the least central subtree consisting just of a single vertex.

Theorem 1 [10]. The least central subtree of a tree $T$ is a single vertex if and only if $T$ is either a path of odd order or a star $K_{1, p}(p \geq 2)$.

In general, characterizing completely the trees with unique least central subtrees seems to be difficult. In this paper we characterize the trees having two adjacent vertices as a unique least central subtree. Our main result is the following.

Theorem 2. The unique least central subtree of a tree $T$ is $P_{2}$ if and only if $T$ is one of trees $T_{4}, T_{6}, T_{8}, T_{9}$, a double star and a path of even order.

The trees $T_{4}, T_{6}, T_{8}, T_{9}$ in Theorem 2 are defined in Section 3. In the next section, we give some basic notation and terminology. In Section 3, we give the proof of Theorem 2.

## 2. Notation and Preliminaries

The vertex set of a graph $G$ is referred to as $V(G)$, its edge set as $E(G)$. The number of vertices of $G$ is its order, written as $|G|$. If $U \subseteq V(G), G[U]$ is the subgraph of $G$ induced by $U$ and we write $G-U$ for $G[V(G)-U]$. In other words, $G-U$ is obtained from $G$ by deleting all the vertices in $U \cap V(G)$ and their incident edges. If $U=\{v\}$ is a singleton, we write $G-v$ rather than $G-\{v\}$. As usual, $P_{n}$ denotes the path of order $n$ and the complete bipartite graph $K_{1, p}(p \geq 1)$ is called a star. A double star is the tree obtained from two vertex disjoint stars by connecting their centers. The subdivision of a star $K_{1, p}$ is the tree obtained from $K_{1, p}$ by subdividing each edge of $K_{1, p}$ exactly once. The distance $d_{G}(x, y)$ in $G$ of two vertices $x, y$ is the length of a shortest $x-y$ path in $G$; if no such path exists, we set $d_{G}(x, y)=\infty$. The eccentricity $e(v)$ of a vertex $v$ in a connected graph $G$ is the distance to a vertex farthest from $v$, i.e., $e(v)=$ $\max \left\{d_{G}(u, v) \mid u \in V(G)\right\}$. The number of components of a graph $G$ is denoted by $\omega(G)$. The center $C(G)$ of $G$ consists of vertices with minimum eccentricity, i.e., $C(G)=\{v \mid e(v)=\min \{e(u) \mid u \in V(G)\}\}$ and the radius $\operatorname{Rad}(G)$ of $G$ is its minimum eccentricity.

In a tree $T$, a vertex of degree one is referred to as a leaf and a vertex which is adjacent to a leaf is a support vertex. An edge incident to a leaf is a pendant edge. For subtrees $S_{1}$ and $S_{2}$ of a tree $T$, the distance $d_{T}\left(S_{1}, S_{2}\right)$ between $S_{1}$ and $S_{2}$ in $T$ is the length of the shortest path joining two vertices of $S_{1}$ and $S_{2}$ in $T$. The median graph $G_{L}$ of $T$ is the graph on the set of all subtrees of $T$ in which two subtrees $S_{1}$ and $S_{2}$ are adjacent as vertices of $G_{L}$ if and only if $V\left(S_{1}\right) \supseteq V\left(S_{2}\right)$ and $\left|V\left(S_{1}\right)-V\left(S_{2}\right)\right|=1$ or $V\left(S_{2}\right) \supseteq V\left(S_{1}\right)$ and $\left|V\left(S_{2}\right)-V\left(S_{1}\right)\right|=1$. In $G_{L}$, we simply write $d_{L}\left(S_{1}, S_{2}\right)$ for the distance between two vertices $S_{1}$ and $S_{2}$ instead of $d_{G_{L}}\left(S_{1}, S_{2}\right)$. For a subtree $S$ of $T$, the $L$ eccentricity $e_{L}(S)$ of $S$ is the distance from $S$ to a vertex most remote from it in $G_{L}$, that is, $e_{L}(S)=\max \left\{d_{L}\left(S, S^{\prime}\right) \mid S^{\prime}\right.$ is a subtree of $\left.T\right\}$. A subtree $S$ is a
central subtree of $T$ if it has the minimum eccentricity $e_{L}(S)$ in the graph $G_{L}$. A tree may contain several central subtrees [10]. A central subtree of $T$ is called a least central subtree if it has the minimum number of vertices.

The following are some basic results on the least central subtrees of a tree in $[6,10]$ that will be useful in the next section.

Lemma 3 [10]. Let $G_{L}$ be the semilattice graph of all subtrees of a tree $T$, and $S_{1}$ and $S_{2}$ be two subtrees of $T$. Then, the distance $d_{L}\left(S_{1}, S_{2}\right)$ between $S_{1}$ and $S_{2}$ in $G_{L}$ is $\left|S_{1}\right|+\left|S_{2}\right|+2\left(d_{T}\left(S_{1}, S_{2}\right)-1\right)$ if $d_{T}\left(S_{1}, S_{2}\right) \geq 1$, and $\left|S_{1} \vee S_{2}\right|-\left|S_{1} \wedge S_{2}\right|$ if $d_{T}\left(S_{1}, S_{2}\right)=0$.

Let $C_{L}$ be a least central subtree of a tree $T$ and $v$ a vertex adjacent to $C_{L}$. Let $S_{L}$ be the component of $T-v$ containing $C_{L}$ and $C_{v}$ the subtree of $T$ induced by $V\left(C_{L}\right) \cup\{v\}$. Furthermore, we set $S_{v}=T \backslash V\left(S_{L}\right)$ and let $S_{v}^{*}$ be the subtree of $T$ satisfying $e_{L}\left(C_{v}\right)=d_{L}\left(C_{v}, S_{v}^{*}\right)$.
Lemma 4 [6]. Let $C_{L}$ be a least central subtree of a tree $T$ and $v$ be a vertex adjacent to $C_{L}$. If $e_{L}\left(C_{v}\right)=d_{L}\left(C_{v}, S_{v}^{*}\right)$, then $S_{v}^{*} \neq T$ and
(1) if $V\left(C_{L}\right) \cap V\left(S_{v}^{*}\right) \neq \emptyset$, then $v \notin V\left(S_{v}^{*}\right)$,
(2) if $V\left(C_{L}\right) \cap V\left(S_{v}^{*}\right)=\emptyset$, then $v$ does not lie in the $C_{L}-S_{v}^{*}$ path in $T$.

Lemma 5 [6]. $\left|S_{v}\right| \leq d_{T}\left(S_{v}, S_{v}^{*}\right)$.
Theorem 6 [6]. The center of a tree is a subtree of every least central subtree.
The following theorem reveals a close connection between least central subtrees and leaves of a tree.

Theorem 7 [10]. If $C_{L}$ is a least central subtree of a tree $T$ with at least three vertices, then the subtree $C_{L}$ contains no leaf of $T$.

## 3. Proof of Theorem 2

In this section we give the proof of our main result. For this purpose, we first give some special trees as follows.
$T_{1}$ : the subdivision of $K_{1,3}$.
$T_{2}$ : the tree obtained from $K_{1,3}$ by subdividing exactly two edges of $K_{1,3}$ once.
$T_{3}$ : the tree obtained from $K_{1,3}$ by subdividing exactly one edge of $K_{1,3}$ twice.
$T_{4}$ : the tree obtained from $K_{1, p}(p \geq 4)$ by subdividing exactly one edge of $K_{1, p}$ twice.
$T_{5}$ : the tree obtained from $K_{1,3}$ by subdividing each edge of $K_{1,3}$ twice.


Figure 1. The trees $T_{i}, i=4,5,6,7$.
$T_{6}$ : the tree obtained from $K_{1,3}$ by subdividing two edges of $K_{1,3}$ once and one edge of $K_{1,3}$ twice.
$T_{7}$ : the tree obtained from $K_{1,3}$ by subdividing two edges of $K_{1,3}$ twice and one edge of $K_{1,3}$ three times.
$T_{8}$ : the tree obtained from a path $P$ with $\operatorname{Rad}(P) \geq 3$ by attaching two pendant edges to a support of $P$.
$T_{8}^{\prime}$ : the tree obtained from a path $P_{2 k}$ of even order with $\operatorname{Rad}\left(P_{2 k}\right) \geq 3$ by attaching two pendant edges to a support of $P_{2 k}$.
$T_{8}^{\prime \prime}$ : the tree obtained from a path $P_{2 k+1}$ of odd order with $\operatorname{Rad}\left(P_{2 k+1}\right) \geq 3$ by attaching two pendant edges to a support of $P_{2 k+1}$.
$T_{9}$ : the tree obtained from a path $P_{2 k}$ of even order with $\operatorname{Rad}\left(P_{2 k}\right) \geq 3$ by attaching one pendant edge to any vertex of degree two of $P_{2 k}$.
For convenience, we identify the notation $C(T)$ with the subtree induced by the center $C(T)$ in a tree $T$. We obtain our main result by showing Lemmas 8 , 9 and 10.

Lemma 8. If $T$ is a tree of order $n$ with $\operatorname{Rad}(T) \leq 2$, then the least central subtree of $T$ is $P_{2}$ if and only if $T$ is one of trees $P_{2}, T_{1}, T_{4}$ and a double star.

Proof. Let $T$ be a tree with $\operatorname{Rad}(T) \leq 2$. First, suppose that $T$ is one of the trees $P_{2}$, a double star, $T_{1}$ and $T_{4}$. We show that the least central subtree of $T$ is $P_{2}$. Clearly, the least central subtree of $P_{2}$ is itself. If $T$ is a double star, then $V(T)-C(T)$ are leaves of $T$ since $\operatorname{Rad}(T)=2$. Theorems 6 and 7 imply that the least central subtree of $T$ is $P_{2}$. If $T=T_{1}$, then clearly $\left|C\left(T_{1}\right)\right|=1$. By Theorem 1, the least central subtree of $T_{1}$ is not a single vertex. Hence the least central subtree of $T_{1}$ has at least two vertices and it contains the center $C\left(T_{1}\right)$ by Theorem 6. A direct calculation shows that the least central subtree of $T_{1}$ is
$P_{2}$. If $T=T_{4}$, then $\left|C\left(T_{4}\right)\right|=1$. Let $C\left(T_{4}\right)=\{u\}$ and $v, v^{\prime}$ be two neighbors of $u$ (see Figure 1). Let $S$ be the component of $T_{4}-u$ containing $v$. By Theorem 7 , the possible least central subtrees of $T_{4}$ are $T_{4}\left[\left\{u, v, v^{\prime}\right\}\right]$ or $T_{4}[\{u, v\}]=P_{2}$. It is easy to verify that $e_{L}\left(T_{4}[\{u, v\}]\right)=d_{L}\left(T_{4}[\{u, v\}], T_{4}\right)=n-2$. Note that $e_{L}\left(T_{4}\left[\left\{u, v, v^{\prime}\right\}\right]\right) \geq d_{L}\left(S, T_{4}\left[\left\{u, v, v^{\prime}\right\}\right]\right)=n-2$, so the least subtree of $T_{4}$ is a path $P_{2}$.

Conversely, suppose that the least central subtree of $T$ is $P_{2}$. Let $C_{L}=u u_{1}$ be a least central subtree of $T$. If $\operatorname{Rad}(T)=1$, we see that $T$ is $P_{2}$ or a star $K_{1, p}$ $(p \geq 2)$. By Theorem 1, we see that the central subtree of a star $K_{1, p}(p \geq 2)$ consists of a single vertex, so $T=P_{2}$. Next we may assume that $\operatorname{Rad}(T)=2$.

Suppose $|C(T)|=2$. Then Theorem 6 and $\operatorname{Rad}(T)=2$ imply that $V\left(C_{L}\right)=$ $C(T)=\left\{u, u_{1}\right\}$, and $V(T)-\left\{u, u_{1}\right\}$ are leaves of $T$. Thus $T$ is a double star.

Suppose $|C(T)|=1$. Then $\operatorname{Rad}(T)=2$ implies that $|T| \geq 5$. Let $L$ be the set of leaves of $T$. If $|L|=2$ then $T=P_{5}$. By Theorem 1, the least central subtree of $P_{5}$ is a single vertex, a contradiction. If $|L|=3, T$ is one of the trees $T_{1}, T_{2}, T_{3}$. It is easy to see that the least central subtrees of $T_{2}$ and $T_{3}$ contain three vertices, respectively. So $T$ is the tree $T_{1}$. If $|L| \geq 4$, we claim that $T$ is the tree $T_{4}$. If not, then either $\omega(T-u) \geq 3$ or $\omega(T-u)=2$ and each component of $T-u$ contains at least two leaves. First note that $e_{L}\left(C_{L}\right) \geq d_{L}\left(C_{L}, T\right)=n-2$. On the other hand, we consider $e_{L}(T-L)$. Let $S$ be any subtree of $T$. If $V(S) \cap(V(T)-L)=\emptyset$, then $d_{L}(S, T-L)=1+n-|L| \leq n-3$ by Lemma 3. If $V(S) \cap(V(T)-L) \neq \emptyset$, then, by Lemma 3, we have

$$
\begin{aligned}
d_{L}(S, T-L) & \leq|S|+n-|L|-2|S \wedge(T-L)| \\
& =|S|-2|S \wedge(T-L)|+n-|L| \\
& \leq(|L|-3)+n-|L| \leq n-3 .
\end{aligned}
$$

Hence $e_{L}(T-L)=\max \left\{d_{L}(S, T-L) \mid S\right.$ is any subtree of $\left.T\right\} \leq n-3$. This implies that $e_{L}(T-L)<e_{L}\left(C_{L}\right)$, contradicting the fact that $C_{L}$ is the least central subtree of $T$. So $T$ is the tree $T_{4}$. The assertion follows.

Lemma 9. Let $T$ be a tree of order $n$ with $|C(T)|=2$ and $\operatorname{Rad}(T) \geq 3$. The least central subtree of $T$ is $P_{2}$ if and only if $T$ is one of trees $T_{6}, T_{8}^{\prime}, T_{9}$ and a path of even order with $\operatorname{Rad}(T) \geq 3$.

Proof. Suppose that $T$ is one of trees $T_{6}, T_{8}^{\prime}, T_{9}$ and a path of even order with $\operatorname{Rad}(T) \geq 3$. Clearly, $|C(T)|=2$ and $\operatorname{Rad}(T) \geq 3$. By Theorem 6 , the least central subtree of $T$ contains the center $C(T)$. By a direct calculation, one can verify that the least central subtree of $T$ is $P_{2}$.

Conversely, let $T$ be a tree with $|C(T)|=2$ and $\operatorname{Rad}(T) \geq 3$. Suppose that the least central subtree of $T$ is $P_{2}$. Let $C_{L}=u u_{1}$ be a least central subtree of $T$. By Theorem $6, C(T)=\left\{u, u_{1}\right\}$. Then there exist two vertices $y, y_{1}$ of $T$ such
that $d_{T}(u, y)=d_{T}\left(u_{1}, y_{1}\right)=\operatorname{Rad}(T)-1$. Let $v$ be the vertex adjacent to $u$ on the $u-y$ path and $v_{1}$ be the vertex adjacent to $u_{1}$ on the $u_{1}-y_{1}$ path, respectively. Define

$$
K=\left\{x \in V(T) \mid d_{T}\left(x, C_{L}\right)=\operatorname{Rad}(T)-1\right\} .
$$

For any $x \in K$, let $w$ be the vertex adjacent to $C_{L}$ on the $x-C_{L}$ path. As defined in Section 2, let $S_{L}$ be the component of $T-w$ containing $C_{L}$, and let $S_{w}=T \backslash V\left(S_{L}\right)$ and $e_{L}\left(C_{w}\right)=d_{L}\left(C_{w}, S_{w}^{*}\right)$, where $C_{w}$ is the subtree of $T$ induced by $V\left(C_{L}\right) \cup\{w\}$.
Claim 1. $S_{w}$ contains at most two vertices besides the vertices on $w-x$ path.
Proof. Since $x \in K$ and the $w-x$ path is contained in $S_{w},\left|S_{w}\right| \geq \operatorname{Rad}(T)-1$. By Lemmas 4 and 5, we have $\operatorname{Rad}(T)+1 \geq d_{T}\left(S_{w}, S_{w}^{*}\right) \geq\left|S_{w}\right|$. Then $\operatorname{Rad}(T)+1 \geq$ $d_{T}\left(S_{w}, S_{w}^{*}\right) \geq\left|S_{w}\right| \geq \operatorname{Rad}(T)-1$. Hence $S_{w}$ contains at most two vertices besides the vertices on the $w-x$ path, as claimed.

Let $r$ be the number of all vertices of $T$ that do not lie in the path $y-y_{1}$. Then

$$
\begin{equation*}
\operatorname{Rad}(T) \leq \frac{n-r}{2} \tag{1}
\end{equation*}
$$

Claim 2. $r \leq 3$.
Proof. We establish the claim by contradiction. Suppose $r \geq 4$. Let $C_{v}$ be the subtree of $T$ induced by $V\left(C_{L}\right) \cup\{v\}$. We shall show that $e_{L}\left(C_{v}\right)<e_{L}\left(C_{L}\right)$. For this purpose, we next show that $d_{L}\left(S, C_{v}\right) \leq n-3$ for any subtree $S$ of tree $T$.

Suppose $V(S) \cap V\left(C_{v}\right) \neq \emptyset$. Then $\left|V(S) \cap V\left(C_{v}\right)\right| \leq 3$. If $\left|V(S) \cap V\left(C_{v}\right)\right|=1$, then $|S| \leq|T|-4=n-4$ by $\operatorname{Rad}(T) \geq 3$, thus $d_{L}\left(S, C_{v}\right)=3+|S|-2 \leq n-3$ by Lemma 3. If $\left|V(S) \cap V\left(C_{v}\right)\right|=2$, then $|S| \leq n-3$ by $\operatorname{Rad}(T) \geq 3$, thus $d_{L}\left(S, C_{v}\right)=3+|S|-4 \leq n-3$ by Lemma 3. If $\left|V(S) \cap V\left(C_{v}\right)\right|=3$, then $d_{L}\left(S, C_{v}\right)=|S|-3 \leq n-3$.

Suppose that $V(S) \cap V\left(C_{v}\right)=\emptyset$ and the subtree $S$ contains at least one vertex on $y-y_{1}$ path. By Claim 1,

$$
d_{T}\left(S, C_{v}\right) \leq \begin{cases}\operatorname{Rad}(T)-|S| & \text { if }|S| \leq 2 \\ \operatorname{Rad}(T)-|S|+1 & \text { if }|S|=3 \\ \operatorname{Rad}(T)-|S|+2 & \text { if }|S| \geq 4\end{cases}
$$

If $|S| \leq 2$, by Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v}\right) & =|S|+3+2\left(d_{T}\left(S, C_{v}\right)-1\right) \\
& \leq|S|+3+2(\operatorname{Rad}(T)-|S|-1) \\
& \leq 1+2 \operatorname{Rad}(T)-|S| \\
& \leq 1+(n-r)-|S| \quad(\text { by }(1)) \\
& \leq n-3 .
\end{aligned}
$$

If $|S|=3$, by Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v}\right) & =|S|+3+2\left(d_{T}\left(S, C_{v}\right)-1\right) \\
& \leq|S|+3+2(\operatorname{Rad}(T)-|S|) \\
& \leq n-r \quad(\text { by }(1)) \\
& \leq n-3 .
\end{aligned}
$$

If $|S| \geq 4$, by Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v}\right) & =|S|+3+2\left(d_{T}\left(S, C_{v}\right)-1\right) \\
& \leq|S|+3+2(\operatorname{Rad}(T)-|S|+1) \\
& \leq n-r+5-|S| \quad(\text { by }(1)) \\
& \leq n-3 .
\end{aligned}
$$

Suppose that $V(S) \cap V\left(C_{v}\right)=\emptyset$ and the subtree $S$ contains no vertex on $y-y_{1}$ path. Then $|S| \leq r$. If $|S|=1$, then $d_{T}\left(S, C_{v}\right) \leq \operatorname{Rad}(T)-1$. By Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v}\right) & =1+3+2\left(d_{T}\left(S, C_{v}\right)-1\right) \\
& \leq 1+3+2(\operatorname{Rad}(T)-2) \\
& \leq n-r \quad(\text { by }(1)) \\
& \leq n-3 .
\end{aligned}
$$

If $|S| \geq 2$, then $d_{T}\left(S, C_{v}\right) \leq \operatorname{Rad}(T)-2$. By Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v}\right) & =|S|+3+2\left(d_{T}\left(S, C_{v}\right)-1\right) \\
& \leq|S|+3+2(\operatorname{Rad}(T)-3) \\
& \leq|S|+(n-r)-3 \quad(\text { by }(1)) \\
& \leq n-3
\end{aligned}
$$

where the last inequality follows from $|S| \leq r$. Thus

$$
e_{L}\left(C_{v}\right)=\max \left\{d_{L}\left(S, C_{v}\right) \mid S \text { is any subtree of } T\right\} \leq n-3
$$

Note that $e_{L}\left(C_{L}\right) \geq d_{L}\left(T, C_{L}\right)=n-2$. But then $e_{L}\left(C_{L}\right)>e_{L}\left(C_{v}\right)$, a contradiction. Consequently, $r \leq 3$.

Let $K_{0}=\left\{x \in K \mid\right.$ the $u-x$ path or $u_{1}-x$ path pass through neither $v$ nor $\left.v_{1}\right\}$.
Claim 3. If $K_{0} \neq \emptyset$, then $T$ is the tree $T_{6}$.
Proof. If $K_{0} \neq \emptyset$, Claim 2 implies $\operatorname{Rad}(T)=3$ or $\operatorname{Rad}(T)=4$. Then $T$ is isomorphic to the tree $T_{6}$ or $T_{7}$. If $T$ is isomorphic to the tree $T_{7}$, let $C\left(T_{7}\right)=$
$\left\{u, u_{1}\right\}$ and $v, v_{1}, v^{\prime}$ be neighbors of $u$ or $u_{1}$, where $v^{\prime}$ is adjacent to the vertex $u_{1}$ on the $u-y^{\prime}$ path (see Figure 1). Set $M=\left\{u, u_{1}, v, v_{1}, v^{\prime}\right\}$. It is easy to check that $e_{L}\left(T_{7}[M]\right)=8=\left|T_{7}\right|-3 \leq n-3$. Note that $e_{L}\left(C_{L}\right) \geq d_{L}\left(T_{7}, C_{L}\right)=$ $\left|T_{7}\right|-2=n-2$. But then $e_{L}\left(C_{L}\right)>e_{L}\left(T_{7}[M]\right)$, a contradiction. Thus $T$ is isomorphic to the tree $T_{6}$.
Claim 4. If $K_{0}=\emptyset$, then $r \leq 2$.
Proof. Suppose not, then $r=3$ by Claim 2. Let $C_{v v_{1}}=T\left[\left\{u, u_{1}, v, v_{1}\right\}\right]$. We show that $e_{L}\left(C_{v v_{1}}\right) \leq n-3$. For this, it suffices to verify that $d_{L}\left(S, C_{v v_{1}}\right) \leq n-3$ for any subtree $S$ of $T$.

Suppose that $V(S) \cap V\left(C_{v v_{1}}\right) \neq \emptyset$. As indicated in the beginning of the proof for Claim 2, one can easily verify that $d_{L}\left(S, C_{v v_{1}}\right) \leq n-3$.

Suppose that $V(S) \cap V\left(C_{v v_{1}}\right)=\emptyset$. Hence, by Claim 1,

$$
d_{T}\left(S, C_{v v_{1}}\right) \leq \begin{cases}\operatorname{Rad}(T)-|S|-1 & \text { if }|S| \leq 2 \\ \operatorname{Rad}(T)-|S| & \text { if }|S|=3 \\ \operatorname{Rad}(T)-|S|+1 & \text { if }|S| \geq 4\end{cases}
$$

If $|S| \leq 2$, by Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v v_{1}}\right) & =|S|+4+2\left(d_{T}\left(S, C_{v v_{1}}\right)-1\right) \\
& \leq|S|+4+2(\operatorname{Rad}(T)-|S|-2) \\
& \leq 2 \operatorname{Rad}(T)-|S| \\
& \leq n-r-|S| \quad(\text { by }(1)) \\
& \leq n-3 .
\end{aligned}
$$

If $|S|=3$, by Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v v_{1}}\right) & =|S|+4+2\left(d_{T}\left(S, C_{v v_{1}}\right)-1\right) \\
& \leq|S|+4+2(\operatorname{Rad}(T)-|S|-1) \\
& \leq n-r-1 \quad(\text { by }(1)) \\
& \leq n-3 .
\end{aligned}
$$

If $|S| \geq 4$, by Lemma 3 , we have

$$
\begin{aligned}
d_{L}\left(S, C_{v v_{1}}\right) & =|S|+4+2\left(d_{T}\left(S, C_{v v_{1}}\right)-1\right) \\
& \leq|S|+4+2(\operatorname{Rad}(T)-|S|) \\
& \leq n-r+4-|S| \quad \quad(\text { by }(1)) \\
& \leq n-3 .
\end{aligned}
$$

So $e_{L}\left(C_{v v_{1}}\right) \leq n-3$. But then $e_{L}\left(C_{L}\right) \geq d_{L}\left(T, C_{L}\right)=n-2>e_{L}\left(C_{v v_{1}}\right)$, a contradiction. Thus $r \leq 2$.

We discuss the case $K_{0}=\emptyset$. For $r=0$, clearly $T$ is a path of even order with $\operatorname{Rad}(T) \geq 3$. For $r=1$, it is easy to see that $T$ is the tree $T_{9}$. We next consider the case $r=2$, i.e., $T$ has precisely two vertices that do not lie in the $y-y_{1}$ path. We shall show that the two vertices of $T$ must be adjacent to a support vertex of $y-y_{1}$ path, i.e., $T$ is the tree $T_{8}^{\prime}$.

Suppose not, that is, $T$ is not the tree $T_{8}^{\prime}$. We shall obtain a contradiction by showing $e_{L}\left(C_{L}\right)>e_{L}\left(C_{v v_{1}}\right)$. Note that $e_{L}\left(C_{L}\right) \geq d_{L}\left(T, C_{L}\right)=n-2$. We next show that $e_{L}\left(C_{v v_{1}}\right) \leq n-3$. It suffices to show that $d_{L}\left(S, C_{v v_{1}}\right) \leq n-3$ for any subtree $S$ of $T$. If $V(S) \cap V\left(C_{v v_{1}}\right) \neq \emptyset$, then, as before, one can easily verify that $d_{L}\left(S, C_{v v_{1}}\right) \leq n-3$. Suppose that $V(S) \cap V\left(C_{v v_{1}}\right)=\emptyset$. Then

$$
d_{T}\left(S, C_{v v_{1}}\right) \leq \begin{cases}\operatorname{Rad}(T)-|S|-1 & \text { if }|S| \leq 2 \\ \operatorname{Rad}(T)-|S| & \text { if }|S| \geq 3\end{cases}
$$

So if $|S| \leq 2$, by Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v v_{1}}\right) & =|S|+4+2\left(d_{T}\left(S, C_{v v_{1}}\right)-1\right) \\
& \leq|S|+4+2(\operatorname{Rad}(T)-2-|S|) \\
& \leq(n-r)-|S| \quad(\text { by }(1)) \\
& \leq n-3
\end{aligned}
$$

If $|S| \geq 3$, by Lemma 3 , we have

$$
\begin{aligned}
d_{L}\left(S, C_{v v_{1}}\right) & =|S|+4+2\left(d_{T}\left(S, C_{v v_{1}}\right)-1\right) \\
& \leq|S|+4+2(\operatorname{Rad}(T)-1-|S|) \\
& \leq 2+(n-r)-|S| \quad(\text { by }(1)) \\
& \leq n-3
\end{aligned}
$$

So $e_{L}\left(C_{v v_{1}}\right)=\max \left\{d_{L}\left(S, C_{v v_{1}}\right) \mid S\right.$ is any subtree of $\left.T\right\} \leq n-3$. But then $e_{L}\left(C_{L}\right)>e_{L}\left(C_{v v_{1}}\right)$, a contradiction. Consequently, $T$ is the tree $T_{8}^{\prime}$.

Lemma 10. Let $T$ be a tree of order $n$ with $|C(T)|=1$ and $\operatorname{Rad}(T) \geq 3$. The least central subtree of $T$ is $P_{2}$ if and only if $T$ is the tree $T_{8}^{\prime \prime}$.

Proof. Suppose that $T=T_{8}^{\prime \prime}$. By Theorem 1, the least central subtree of $T_{8}^{\prime \prime}$ is not a single vertex. Note that $\left|C\left(T_{8}^{\prime \prime}\right)\right|=1$, so the least central subtree of $T_{8}^{\prime \prime}$ has at least two vertices and it contains the center $C\left(T_{8}^{\prime \prime}\right)$ by Theorem 6 . By a direct calculation, one may easily verify that the least central subtree of $T_{8}^{\prime \prime}$ is $P_{2}$.

Conversely, let $T$ be a tree of order $n$ with $|C(T)|=1$ and $\operatorname{Rad}(T) \geq 3$. Let $C_{L}=u u_{1}$ be a least central subtree of $T$. Since $|C(T)|=1$, we may assume that $u \in C(T)$ and $u_{1} \notin C(T)$ by Theorem 6 . Then there exist vertices $y, y_{1}$ such that $d_{T}(u, y)=d_{T}\left(u, y_{1}\right)=\operatorname{Rad}(T)$. Let $v, v_{1}$ be the vertices adjacent to $u$ on the
$u-y$ path and $u-y_{1}$ path, respectively. Let $K=\left\{x \in V(T) \mid d_{T}(u, x)=\operatorname{Rad}(T)\right\}$ and let $r$ be the number of all vertices of $T$ that do not lie in the $y-y_{1}$ path. Then

$$
\begin{equation*}
\operatorname{Rad}(T) \leq \frac{n-r-1}{2} \tag{2}
\end{equation*}
$$

Claim 1. There exists a vertex $y^{\prime} \in K$ such that $u_{1}$ is on the $u-y^{\prime}$ path.
Proof. Suppose not, then $u_{1} \neq v_{1}$. Let the subtrees $S_{v}, S_{v_{1}}$ and $S_{v}^{*}$ of $T$ be defined as in Lemma 4. We have the following fact.

Fact 1. $S_{v}$ and $S_{v_{1}}$ contain at most one vertex besides the vertices on $v-y$ path and $v_{1}-y_{1}$ path, respectively.
Proof. Since $d_{T}(u, y)=\operatorname{Rad}(T),\left|S_{v}\right| \geq \operatorname{Rad}(T)$. By Lemmas 4 and 5, we have

$$
\operatorname{Rad}(T)+1 \geq d_{T}\left(S_{v}, S_{v}^{*}\right) \geq\left|S_{v}\right| \geq \operatorname{Rad}(T)
$$

So $\left|S_{v}\right|=\operatorname{Rad}(T)$ or $\left|S_{v}\right|=\operatorname{Rad}(T)+1$. Thus $S_{v}$ contains at most one vertex besides the vertices of $v-y$ path. Similarly, $S_{v_{1}}$ contains at most one vertex besides the vertices of $v_{1}-y_{1}$ path.

Let $C_{v v_{1}}=T\left[\left\{u, v, v_{1}\right\}\right]$. We shall show that $e_{L}\left(C_{v v_{1}}\right) \leq n-3$. For this, it suffices to show that $d_{L}\left(S, C_{v v_{1}}\right) \leq n-3$ for any subtree $S$ of $T$. Suppose that $V(S) \cap V\left(C_{v v_{1}}\right) \neq \emptyset$. Then, as before, one can easily verify that $d_{L}\left(S, C_{v v_{1}}\right) \leq$ $n-3$. Now we may assume that $V(S) \cap V\left(C_{v v_{1}}\right)=\emptyset$.

Suppose that the subtree $S$ contains at least one vertex on $y-y_{1}$ path. Then $V(S) \subseteq V\left(S_{v}\right)$ or $V(S) \subseteq V\left(S_{v_{1}}\right)$. By Fact 1, we have

$$
d_{T}\left(S, C_{v v_{1}}\right) \leq \begin{cases}\operatorname{Rad}(T)-|S| & \text { if }|S| \leq 2  \tag{3}\\ \operatorname{Rad}(T)-|S|+1 & \text { if }|S| \geq 3\end{cases}
$$

Note that $u_{1}$ is not on $y-y_{1}$ path, so $r \geq 2$ by Theorem 7 . Hence, if $|S| \leq 2$, by Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v v_{1}}\right) & =|S|+3+2\left(d_{T}\left(S, C_{v v_{1}}\right)-1\right) \\
& \leq|S|+3+2(\operatorname{Rad}(T)-|S|-1) \\
& \leq 1+2 \operatorname{Rad}(T)-|S| \\
& \leq n-r-1 \quad(\text { by }(2)) \\
& \leq n-3 .
\end{aligned}
$$

If $|S| \geq 3$, by Lemma 3 , we have

$$
d_{L}\left(S, C_{v v_{1}}\right)=|S|+3+2\left(d_{T}\left(S, C_{v v_{1}}\right)-1\right)
$$

$$
\begin{aligned}
& \leq|S|+3+2(\operatorname{Rad}(T)-|S|) \\
& \leq 3+2 \operatorname{Rad}(T)-|S| \\
& \leq n-r-|S|+2 \quad(\text { by }(2)) \\
& \leq n-3
\end{aligned}
$$

Suppose that the subtree $S$ contains no vertex on $y-y_{1}$ path. Let

$$
K_{0}=\left\{x \in K \mid \text { the } x-u \text { path pass through neither } v \text { nor } v_{1}\right\}
$$

If $K_{0} \neq \emptyset$, note that $u_{1}$ is not on $y-y_{1}$ path, so $r \geq 4$ by Theorem 7. Furthermore, $K_{0} \neq \emptyset$ implies that $|S| \leq r-1$. It is easily seen that

$$
d_{T}\left(S, C_{v v_{1}}\right) \leq \begin{cases}\operatorname{Rad}(T) & \text { if }|S|=1 \\ \operatorname{Rad}(T)-1 & \text { if }|S| \geq 2\end{cases}
$$

So if $|S|=1$, by Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v v_{1}}\right) & =1+3+2\left(d_{T}\left(S, C_{v v_{1}}\right)-1\right) \\
& \leq 1+3+2(\operatorname{Rad}(T)-1) \\
& \leq n-r+1 \quad(\text { by }(2)) \\
& \leq n-3 .
\end{aligned}
$$

If $|S| \geq 2$, by Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v v_{1}}\right) & =|S|+3+2\left(d_{T}\left(S, C_{v v_{1}}\right)-1\right) \\
& \leq|S|+3+2(\operatorname{Rad}(T)-2) \\
& \leq|S|-1+(n-r-1) \quad(\text { by }(2)) \\
& \leq n-3
\end{aligned}
$$

where the last inequality follows from $|S| \leq r-1$. If $K_{0}=\emptyset$, we can see that

$$
d_{T}\left(S, C_{v v_{1}}\right) \leq \begin{cases}\operatorname{Rad}(T)-1 & \text { if }|S|=1 \\ \operatorname{Rad}(T)-2 & \text { if }|S| \geq 2\end{cases}
$$

So if $|S|=1$, by Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v v_{1}}\right) & =1+3+2\left(d_{T}\left(S, C_{v v_{1}}\right)-1\right) \\
& \leq 1+3+2(\operatorname{Rad}(T)-2) \\
& \leq n-r-1 \quad(\text { by }(2)) \\
& \leq n-3 .
\end{aligned}
$$

If $|S| \geq 2$, by Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v v_{1}}\right) & =|S|+3+2\left(d_{T}\left(S, C_{v v_{1}}\right)-1\right) \\
& \leq|S|+3+2(\operatorname{Rad}(T)-3) \\
& \leq|S|-3+(n-r-1) \quad(\text { by }(2)) \\
& \leq n-3,
\end{aligned}
$$

where the last inequality follows from $|S| \leq r$.
Therefore, we obtain $e_{L}\left(C_{v v_{1}}\right) \leq n-3$. But then $e_{L}\left(C_{L}\right) \geq d_{L}\left(T, C_{L}\right)=$ $n-2>e_{L}\left(C_{v v_{1}}\right)$, a contradiction. The claim follows.

By Claim 1, we may assume that $u_{1}=v_{1}$. Let $K_{1}=\{x \in K \mid$ the $u-x$ path does not pass through vertex $\left.u_{1}\right\}, K_{2}=\{x \in K \mid u-x$ path passes through the vertex $\left.u_{1}\right\}$. For any $x \in K_{1}$, let $w$ be the vertex adjacent to $u$ on the $u-x$ path. For any $x \in K_{2}$, let $w_{1}$ be the vertex adjacent to $u_{1}$ on the $u_{1}-x$ path. Let $S_{w}, S_{w}^{*}, S_{w_{1}}$ and $S_{w_{1}}^{*}$ be defined as in Lemma 4. Then we have the following claims.

Claim 2. $S_{w}$ contains at most one vertex besides the vertices on the $w-x$ path.
Proof. By definition, $w$ lies in the $x-u$ path. Since $d_{T}(x, u)=\operatorname{Rad}(T)$ and the vertices on $w-x$ path are contained in $S_{w},\left|S_{w}\right| \geq \operatorname{Rad}(T)$. By Lemmas 4 and 5,

$$
\operatorname{Rad}(T)+1 \geq d\left(S_{w}, S_{w}^{*}\right) \geq\left|S_{w}\right| \geq \operatorname{Rad}(T)
$$

So $\left|S_{w}\right|=\operatorname{Rad}(T)$ or $\left|S_{w}\right|=\operatorname{Rad}(T)+1$. Thus $S_{w}$ contains at most one vertex besides the vertices on the $w-x$ path.

Claim 3. $S_{w_{1}}$ contains at most three vertices besides the vertices on the $w_{1}-y_{1}$ path.
Proof. By definition, $w_{1}$ lies in the $x-u_{1}$ path. Since $d_{T}\left(x, u_{1}\right)=\operatorname{Rad}(T)-1$ and the vertices on the $x-w_{1}$ path are contained in $S_{w_{1}},\left|S_{w_{1}}\right| \geq \operatorname{Rad}(T)-1$. By Lemmas 4 and $5, \operatorname{Rad}(T)+2 \geq d_{T}\left(S_{w_{1}}, S_{w_{1}}^{*}\right) \geq\left|S_{w_{1}}\right| \geq \operatorname{Rad}(T)-1$. Hence $S_{w_{1}}$ contains at most three vertices besides the vertices on the $w_{1}-y_{1}$ path.
Claim 4. $2 \leq r \leq 3$.
Proof. If $r=1$, we observe that $e_{L}\left(C_{L}\right) \geq d_{L}\left(y, C_{L}\right)=1+2+2(\operatorname{Rad}(T)-1)=$ $n-1$ by Lemma 3. On the other hand, it is easy to verify that $e_{L}\left(C_{v v_{1}}\right) \leq n-2$ by Lemma 3. Then $e_{L}\left(C_{L}\right)>e_{L}\left(C_{v v_{1}}\right)$, a contradiction. Thus $r \geq 2$. We claim that $r \leq 3$. Suppose to the contrary that $r \geq 4$. As before, let $C_{v}$ be the subtree of $T$ induced by $V\left(C_{L}\right) \cup\{v\}$. We shall prove that $e_{L}\left(C_{v}\right) \leq n-3$. It suffices to show that $d_{L}\left(S, C_{v}\right) \leq n-3$ for any subtree $S$ of $T$.

Suppose that $V(S) \cap V\left(C_{v}\right) \neq \emptyset$. By Lemma 3, one can easily verify that $d_{L}\left(S, C_{v}\right) \leq n-3$ for any subtree $S$ of $T$.

Suppose that $V(S) \cap V\left(C_{v}\right)=\emptyset$ and the subtree $S$ contains at least one vertex on $y-y_{1}$ path. Then either $V(S) \subseteq V\left(S_{v}\right)$ or there exists a vertex $x \in K_{2}$ such that $V(S) \subseteq V\left(S_{w_{1}}\right) \cup\left\{u_{1}\right\}$. By Claims 2 and 3 ,

$$
d_{T}\left(S, C_{v}\right) \leq \begin{cases}\operatorname{Rad}(T)-|S| & \text { if }|S| \leq 2 \\ \operatorname{Rad}(T)-|S|+2 & \text { if } 3 \leq|S| \leq 4 \\ \operatorname{Rad}(T)-|S|+3 & \text { if }|S| \geq 5\end{cases}
$$

Then if $|S| \leq 2$, by Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v}\right) & =|S|+3+2\left(d_{T}\left(S, C_{v}\right)-1\right) \\
& \leq|S|+3+2(\operatorname{Rad}(T)-|S|-1) \\
& \leq 1+2 \operatorname{Rad}(T)-|S| \\
& \leq n-r-1 \quad(\text { by }(2)) \\
& \leq n-3
\end{aligned}
$$

If $3 \leq|S| \leq 4$, by Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v}\right) & =|S|+3+2\left(d_{T}\left(S, C_{v}\right)-1\right) \\
& \leq|S|+3+2(\operatorname{Rad}(T)-|S|+1) \\
& \leq 5+2 \operatorname{Rad}(T)-|S| \\
& \leq n-r-|S|+4 \quad(\text { by }(2)) \\
& \leq n-3
\end{aligned}
$$

If $|S| \geq 5$, by Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v}\right) & =|S|+3+2\left(d_{T}\left(S, C_{v}\right)-1\right) \\
& \leq|S|+3+2(\operatorname{Rad}(T)-|S|+2) \\
& \leq 7+2 \operatorname{Rad}(T)-|S| \\
& \leq n-r-|S|+6 \quad(\text { by }(2)) \\
& \leq n-3
\end{aligned}
$$

Suppose that $V(S) \cap V\left(C_{v}\right)=\emptyset$ and the subtree $S$ contains no vertex on $y-y_{1}$ path. Then by Claim 2,

$$
d_{T}\left(S, C_{v}\right) \leq \begin{cases}\operatorname{Rad}(T) & \text { if }|S|=1 \\ \operatorname{Rad}(T)-1 & \text { if } 2 \leq|S| \leq 3 \\ \operatorname{Rad}(T)-2 & \text { if }|S| \geq 4\end{cases}
$$

So if $|S|=1$, by Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v}\right) & =1+3+2\left(d_{T}\left(S, C_{v}\right)\right) \\
& \leq 1+3+2(\operatorname{Rad}(T)-1) \\
& \leq n-r+1 \quad(\text { by }(2)) \\
& \leq n-3 .
\end{aligned}
$$

If $2 \leq|S| \leq 3$, by Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v}\right) & =|S|+3+2\left(d_{T}\left(S, C_{v}\right)-1\right) \\
& \leq|S|+3+2(\operatorname{Rad}(T)-2) \\
& \leq|S|-1+(n-r-1) \quad(\text { by }(2)) \\
& \leq n-3,
\end{aligned}
$$

If $|S| \geq 4$, by Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v}\right) & =|S|+3+2\left(d_{T}\left(S, C_{v}\right)-1\right) \\
& \leq|S|+3+2(\operatorname{Rad}(T)-3) \\
& \leq|S|-3+(n-r-1) \quad(\text { by }(2)) \\
& \leq n-3,
\end{aligned}
$$

where the last inequality follows from $|S| \leq r$. Thus $e_{L}\left(C_{v}\right) \leq n-3$. But then $e_{L}\left(C_{L}\right) \geq d_{L}\left(T, C_{L}\right)=n-2>e_{L}\left(C_{v}\right)$, a contradiction. Consequently, $2 \leq r \leq 3$.

Let $K^{\prime}=\left\{x \in K \mid x-u\right.$ path pass through neither $v$ nor $\left.v_{1}\right\}$.
Claim 5. $K^{\prime}=\emptyset$.
Proof. If $K^{\prime} \neq \emptyset$, then there exists a vertex $x \in K^{\prime}$ such that the $x-u$ path pass through neither $v$ nor $v_{1}$. Claim 4 implies that $\operatorname{Rad}(T)=3$ and $T$ is the tree $T_{5}$. Let $C\left(T_{5}\right)=\{u\}$ and $v, v_{1}, v^{\prime}$ be neighbors of $u$, where $v^{\prime}$ is adjacent to $u$ on the $u-x$ path (see Figure 1). Set $M=\left\{u, v, u_{1}, v^{\prime}\right\}$ (where $u_{1}=v_{1}$ ). One may verify that $e_{L}\left(T_{5}[M]\right) \leq n-3$ by Lemma 3 . But then $e_{L}\left(C_{L}\right) \geq d_{L}\left(T_{5}, C_{L}\right)=n-2>e_{L}\left(T_{5}[M]\right)$, a contradiction.

Let $C_{v v^{\prime}}=T\left[\left\{u, u_{1}, v, v^{\prime}\right\}\right]$, where $v^{\prime}$ is the vertex adjacent to $u_{1}$ on the $u_{1}-y_{1}$ path.
Claim 6. $r=2$.
Proof. Suppose not, then Claim 4 implies that $r=3$. Note the fact that $e_{L}\left(C_{L}\right) \geq d_{L}\left(T, C_{L}\right)=n-2$. If we can show $e_{L}\left(C_{v v^{\prime}}\right) \leq n-3$, we will derive a contradiction. It is sufficient to show that $d_{L}\left(C_{v v^{\prime}}\right) \leq n-3$ for any subtree $S$ of $T$.

Suppose that $V(S) \cap V\left(C_{v v^{\prime}}\right) \neq \emptyset$. Then one can easily check that $d_{L}\left(S, C_{v v^{\prime}}\right)$ $\leq n-3$ by Lemma 3 .

Suppose that $V(S) \cap V\left(C_{v v^{\prime}}\right)=\emptyset$. By Claims 2, 3, we have

$$
d_{T}\left(S, C_{v v^{\prime}}\right) \leq \begin{cases}\operatorname{Rad}(T)-|S| & \text { if }|S| \leq 2 \\ \operatorname{Rad}(T)-|S|+1 & \text { if } 3 \leq|S| \leq 4 \\ \operatorname{Rad}(T)-|S|+2 & \text { if }|S| \geq 5\end{cases}
$$

So if $|S| \leq 2$, by Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v v^{\prime}}\right) & =|S|+4+2\left(d_{T}\left(S, C_{v v^{\prime}}\right)-1\right) \\
& \leq|S|+4+2(\operatorname{Rad}(T)-|S|-1) \\
& \leq n-r \quad(\text { by }(2)) \\
& \leq n-3 .
\end{aligned}
$$

If $3 \leq|S| \leq 4$, by Lemma 3 , we have

$$
\begin{aligned}
d_{L}\left(S, C_{v v^{\prime}}\right) & =|S|+4+2\left(d_{T}\left(S, C_{v v^{\prime}}\right)-1\right) \\
& \leq|S|+4+2(\operatorname{Rad}(T)-|S|) \\
& \leq(n-r)-|S|+3 \quad(\text { by }(2)) \\
& \leq n-3 .
\end{aligned}
$$

If $|S| \geq 5$, by Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v v^{\prime}}\right) & =|S|+4+2\left(d_{T}\left(S, C_{v v^{\prime}}\right)-1\right) \\
& \leq|S|+4+2(\operatorname{Rad}(T)-|S|+1) \\
& \leq(n-r)-|S|+5 \quad(\text { by }(2)) \\
& \leq n-3 .
\end{aligned}
$$

So $e_{L}\left(C_{v v^{\prime}}\right)<e_{L}\left(C_{L}\right)$, a contradiction. Thus $r=2$.
Claim 6 implies that $T$ has precisely two vertices that do not lie in the $y-y_{1}$ path.
Claim 7. The two vertices of $T$ that do not lie in the $y-y_{1}$ path are adjacent to a support vertex on the $y-y_{1}$ path, i.e., $T$ is the tree $T_{8}^{\prime \prime}$.
Proof. Otherwise, we shall derive a contradiction by showing that $e_{L}\left(C_{L}\right)>$ $e_{L}\left(C_{v u_{1}}\right)$ where $C_{v u_{1}}=T\left[\left\{v, u, u_{1}\right\}\right]$. It is sufficient to show that $d_{L}\left(S, C_{v u_{1}}\right) \leq$ $n-3$ for any subtree $S$ of $T$ since $e_{L}\left(C_{L}\right) \geq d_{L}\left(T, C_{L}\right)=n-2$. Suppose that $V(S) \cap V\left(C_{v u_{1}}\right) \neq \emptyset$. One may directly verify that $d_{L}\left(S, C_{v u_{1}}\right) \leq n-3$ by Lemma 3. Suppose that $V(S) \cap V\left(C_{v u_{1}}\right)=\emptyset$. Then by Claims 2,3 and 5 , we have

$$
d_{T}\left(S, C_{v u_{1}}\right) \leq \begin{cases}\operatorname{Rad}(T)-|S| & \text { if }|S| \leq 2 \\ \operatorname{Rad}(T)-|S|+1 & \text { if }|S| \geq 3\end{cases}
$$

So if $|S| \leq 2$, by Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v u_{1}}\right) & =|S|+3+2\left(d_{T}\left(S, C_{v u_{1}}\right)-1\right) \\
& \leq|S|+3+2(\operatorname{Rad}(T)-|S|-1) \\
& \leq(n-r)-1 \quad(\text { by }(2)) \\
& \leq n-3 .
\end{aligned}
$$

If $|S| \geq 3$, by Lemma 3, we have

$$
\begin{aligned}
d_{L}\left(S, C_{v u_{1}}\right) & =|S|+3+2\left(d_{T}\left(S, C_{v u_{1}}\right)-1\right) \\
& \leq|S|+3+2(\operatorname{Rad}(T)-|S|) \\
& \leq(n-r)-|S|+2 \quad \text { (by (2)) } \\
& \leq n-3 .
\end{aligned}
$$

So $e_{L}\left(C_{v u_{1}}\right)<e_{L}\left(C_{L}\right)$. This contradiction implies that $T$ is the tree $T_{8}^{\prime \prime}$, as desired.

Proof of Theorem 2. Note that $T_{8}=T_{8}^{\prime}$ or $T_{8}^{\prime \prime}$. By Lemmas 8, 9 and 10, we conclude that the least central subtree of a tree $T$ is $P_{2}$ if and only if $T$ is one of the trees $T_{1}, T_{4}, T_{6}, T_{8}, T_{9}$, a double star and a path of even order. Furthermore, we observe that the tree $T_{1}$ has three least central subtrees, while each one of the trees $T_{4}, T_{6}, T_{8}, T_{9}$, a double star and a path of even order has a unique least central subtree $P_{2}$. This completes the proof of Theorem 2.

## 4. Conclusion

In this paper we give a complete characterization of trees with the unique least central subtree consisting of two adjacent vertices. In light of the idea in the proof of Theorem 2, it is possible to characterize the trees having the unique least central subtree with small order.

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