RECONFIGURING MINIMUM DOMINATING SETS: THE γ-GRAPH OF A TREE

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Abstract

We consider γ-graphs, which are reconfiguration graphs of the minimum dominating sets of a graph G. We answer three open questions about γ-graphs of trees by providing upper bounds on the maximum degree, the diameter, and the number of minimum dominating sets. The latter gives an upper bound on the order of the γ-graph.

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1. Introduction

Recall that a set, D, of vertices of a graph G is called a *dominating set* if every vertex in $V - D$ is adjacent to at least one vertex in D. The minimum cardinality of a dominating set of G is the *domination number of G* and is denoted by $\gamma(G)$. A dominating set of minimum cardinality is sometimes called a γ-set.

We consider two different reconfiguration graphs of the minimum dominating sets of a graph G. In both cases the vertex set is the collection of minimum dominating sets of G. In the *single vertex replacement adjacency model*, different minimum dominating sets D_1 and D_2 are adjacent when there are vertices $x \in D_1$ and $y \in D_2$ such that $D_1 - x = D_2 - y$. In the *slide adjacency model* it is required that, in addition, $xy \in E(G)$. The reconfiguration graphs just defined are the *γ-graph in the single vertex replacement adjacency model*, and the *γ-graph in the slide adjacency model*, respectively. Both are denoted by Γ_G, as the model under consideration is always either clear from the context, or not relevant.

The single vertex replacement adjacency model was introduced by Subramanaian and Sridharan [14] in 2008, and the slide adjacency model was introduced independently by Fricke, Hedetniemi, Hedetniemi, and Hutson [4] in 2011. The single vertex replacement adjacency model was further studied in [9] and [13] and the slide adjacency model was further studied in [2].

Reconfiguration graphs for dominating sets which are not necessarily minimum have also been considered, see [1, 5, 6, 12]. For pointers to the literature on various reconfiguration problems and their complexity, see [7, 8].

In this work, we answer the following questions on γ-graphs of trees for both adjacency models. The questions are posed by Fricke *et al.* in [4].

1. Is $\Delta(\Gamma_T) = O(n)$ for every tree T of order n?
2. Is $\text{diam}(\Gamma_T) = O(n)$ for every tree T of order n?
3. Is $|V(\Gamma_T)| \leq 2^{\gamma(T)}$ for every tree T?

We answer question 1 affirmatively by showing that $n - \gamma(T)$ is a sharp upper bound for $\Delta(\Gamma_T)$ in both adjacency models; see Corollary 3. We also answer question 2 affirmatively by providing an upper bound for $\text{diam}(\Gamma_T)$. For the single vertex replacement adjacency model we show that $\text{diam}(\Gamma_T) \leq 2\gamma(T)$, and for the slide adjacency model we show that $\text{diam}(\Gamma_T) \leq 2(2\gamma(T) - s)$ where s is the number of vertices of T that are adjacent to a leaf of T (often called *support vertices*); see Theorem 11. We show that the inequality in question 3 does not hold for all trees by describing an infinite family of trees with the property that $|V(\Gamma_T)| > 2^{\gamma(T)}$ for each tree T in the collection; see Section 4. Interestingly, these trees also have $\Delta(\Gamma_T) = n - \gamma(T)$. We also provide an upper bound for $|V(\Gamma_T)|$, by showing that $|V(\Gamma_T)| \leq ((1 + \sqrt{13})/2)^{\gamma(T)}$. Almost all of the work presented here also appears in the PhD thesis of the first author; see [3].

We conclude this section with some definitions.

For a vertex $x \in V(G)$, the *open neighbourhood of x* is the set $N_G(x) = \{y \mid xy \in E(G)\}$, and the *closed neighbourhood of x* is the set $N_G[x] = N_G(x) \cup \{x\}$. For a set $S \subseteq V(G)$, the *open neighbourhood of S* is the set $N_G(S) = \{x \mid xy \in E \text{ for some } y \in S\}$, and the *closed neighbourhood of S* is the set $N_G[S] = N_G(S) \cup S$. When the graph G is obvious from context, we simply write

$N(x)$, $N[x]$, $N(S)$, and $N[S]$.

Let D be a dominating set of a graph G. A vertex $x \in D$ is said to *dominate* every vertex in $N[x]$, and also to *dominate* any subset of $N[x]$. Similarly, a subset $X \subseteq V$ is said to *dominate* every vertex in $N[X]$, and also to *dominate* any subset of $N[X]$.

For a set of vertices $D \subseteq V$ and a vertex $x \in D$, the *private neighbourhood of x with respect to D* is the set $\text{pn}(x, D) = N[x] - N[D - \{x\}]$, that is, the set of vertices that are in the closed neighbourhood of x, but are not in the closed neighbourhood of any other vertex in D. Likewise, for sets $S, D \subseteq V$, the *private neighbourhood of S with respect to D*, denoted $\text{pn}(S, D)$, is the set $\text{pn}(S, D) = N[S] - N[D - S]$. A subset $D \subseteq V$ is a *minimal dominating set* if and only if $\text{pn}(x, D) \neq \emptyset$; for all vertices $x \in D$.

A *rooted tree* is a pair (T, c), where T is a tree and $c \in V$ is a special vertex called the *root*. Let (T, c) be a rooted tree. A vertex x is called an *ancestor* of a vertex y if x belongs to the unique path joining y and c. If, in addition, $xy \in E$, then x is a *parent* of y. The terms *descendant* of x and *child* of x, respectively, are used to describe such a vertex y. Note that x is both an ancestor and a descendant of itself. We use T_x to describe the subtree of T induced by the descendants of x, and rooted at x.

2. The Maximum Degree of Γ_T

In this section we show that question 1 has a positive answer in both adjacency models.

Proposition 1. *If D is a γ-set of a tree T and there exist a vertex $x \in D$ and a vertex $y \in V$ such that $D' = (D - \{x\}) \cup \{y\}$ is also a γ-set of T, then $d(x, y) \leq 2$.*

Proof. Let D be a γ-set of the tree T. Let $x \in D$ and $y \in V(T)$ such that $d(x, y) \geq 3$. Suppose, for contradiction, that $D' = (D - \{x\}) \cup \{y\}$ is a γ-set of T. Root T at the vertex y and let z be the parent of x and let w be the parent of z. (Notice that z and w exist because $d(x, y) \geq 3$.) Since D' is a dominating set of T, the set $D' \cap V(T_z)$ is a dominating set of T_x. Notice that $|D' \cap V(T_z)| < |D \cap V(T_z)|$. But then $D'' = (D - V(T_z)) \cup (D' \cap V(T_z))$ is a dominating set of T and $|D''| < |D|$, a contradiction. Thus, $d(x, y) \leq 2$. ∎

Lemma 2. *For any γ-set D of a tree T and vertex $z \notin D$, there is at most one vertex $x \in D$ such that $(D - \{x\}) \cup \{z\}$ is also a γ-set of T.*

Proof. Let D be a γ-set of the tree T, and consider a vertex $z \notin D$. Suppose there exists a set $\{x, y\} \subseteq D$ such that $(D - \{x\}) \cup \{z\}$ and $(D - \{y\}) \cup \{z\}$ are both dominating sets of T. By Proposition 1, $d(x, z) \leq 2$ and $d(y, z) \leq 2$.

By minimality of D, the sets $\mathrm{pn}(x, D)$ and $\mathrm{pn}(y, D)$ are both non-empty. Since T is a tree and z dominates both $\mathrm{pn}(x, D)$ and $\mathrm{pn}(y, D)$, the vertices in $\mathrm{pn}(x, D) \cup \mathrm{pn}(y, D) \cup \{z\}$ must all be on the unique path between x and y in T. Also, x and y have at most one common neighbour, and if such a vertex exists, then it is z. It now follows that z dominates $\mathrm{pn}(x, D) \cup \mathrm{pn}(y, D)$, and any common neighbours of x and y. Therefore, $D' = (D - \{x, y\}) \cup \{z\}$ is a dominating set of T with $|D'| < |D|$, a contradiction. ∎

Lemma 2 implies the following result for both adjacency models under consideration.

Corollary 3. *If $|V(T)| = n$, then $\Delta(\Gamma_T) \leq n - \gamma(T)$.*

The inequality in Corollary 3 is sharp. Figure 1 shows an infinite family of trees with $\Delta(\Gamma_T) = n - \gamma(T)$. Suppose the central vertex, t, has $\deg(t) = a \geq 2$, and that every vertex $v \in N(t)$ has $\deg(v) = b + 1$. Then $|V| = n = a + 2ab + 1$ and $\gamma(T) = ab + 1$. The γ-set D comprised of the central vertex t and all the support vertices of T has degree $a + ab = n - \gamma(T)$ in Γ_T.
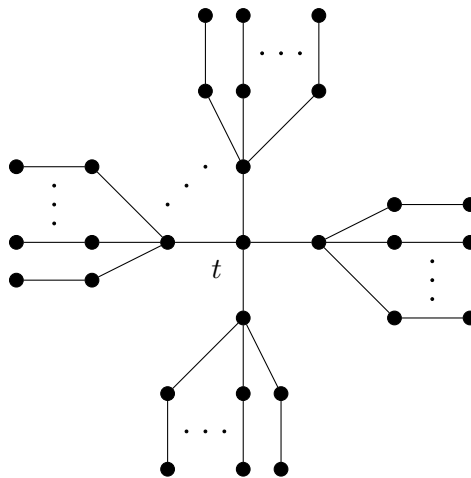

Figure 1. An infinite family of trees with $\Delta(\Gamma_T) = n - \gamma(T)$.

3. The Diameter of Γ_T

In this section we give an upper bound for $\mathrm{diam}(\Gamma_T)$ in each adjacency model, and by doing so show that question 2 has an affirmative answer in each adjacency model.

Let D be a minimum dominating set of the rooted tree (T, c). We define the *height* of D to be the quantity $ht_T(D) = \sum_{x \in D} d(x, c)$. A γ-set D is called a *highest minimum dominating set* if $ht_T(D) \leq ht_T(F)$ for all γ-sets F of T. We shall show later that every tree T has a unique highest minimum dominating set.

Proposition 4. *Let L be the set of leaves in the tree T and let S be the set of support vertices in T. If D is a highest minimum dominating set, then $S \subseteq D$ and $L \cap D = \emptyset$.*

Proof. Let $x \in L$ and $y \in S$ be such that $xy \in E(T)$. If $x \in D$, then $y \notin D$. Thus, $D' = (D - \{x\}) \cup \{y\}$ is a γ-set of T with $ht_T(D') < ht_T(D)$, a contradiction. Therefore, $L \cap D = \emptyset$. Since $N(L) \subseteq S$, this completes the proof. ∎

Lemma 5. *A γ-set D is a highest minimum dominating set of a rooted tree (T, c) if and only if every vertex in $D - \{c\}$ has a child as a private neighbour.*

Proof. Consider a γ-set D and suppose there is a vertex $x \in D - \{c\}$ such that $y \notin \mathrm{pn}(x, D)$ for every child y of x. Let z be the parent of x. If $z \in D$, then $D - \{x\}$ is a dominating set of T, a contradiction. If $z \notin D$, then $D' = (D - \{x\}) \cup \{z\}$ is a dominating set such that $ht_T(D') < ht_T(D)$, thus D is not a highest minimum dominating set.

To show the converse, we proceed by induction on n, where $n = |V(T)|$. The base cases for $1 \leq n \leq 5$ are easy to verify.

Suppose that $n \geq 6$. Let D be a γ-set of the rooted tree (T, c) such that every vertex $v \in D - \{c\}$ has a child as a private neighbour. Notice that the statement holds for $T = K_{1,n-1}$ and for any tree T with $\mathrm{diam}(T) = 4$ where c is the central vertex of T. Thus, suppose that T is neither $K_{1,n-1}$ nor a tree with $\mathrm{diam}(T) = 4$ where c is the central vertex of T. Let $y \in D$, chosen so that $d(y, c)$ is maximized. Thus, y is a support vertex, and the only children of y are leaves. Let x be the parent of y.

Case 1. Suppose that $x \in \mathrm{pn}(y, D)$. Then $x \notin D$. Let z be the parent of x. (Note that z exists because T is not a graph of diameter 4 with $x = c$.) Then $z \notin D$ since $x \in \mathrm{pn}(y, D)$.

First we show that $N(x) = \{y, z\}$. Suppose x has a child v, $v \neq y$. Then either v is a leaf or v has at least one child and all the children of v are leaves. If v is a leaf, then $x \in D$. If v has a child, then $v \in D$. In both cases we have that $x \notin \mathrm{pn}(y, D)$, a contradiction.

Let T_1 be the tree $T_1 = T_x$, and let T_2 be the tree $T_2 = T - T_1$. Then $D_1 = D \cap V(T_1)$ is a γ-set of T_1 and $D_2 = D \cap V(T_2)$ is a γ-set of T_2.

Consider T_1 to be rooted at x and T_2 to be rooted at c. Then by the induction hypothesis, D_1 is a highest minimum dominating set of T_1 and D_2 is a highest minimum dominating set of T_2. Recall that $N(x) = \{y, z\}$. Let S be a highest

minimum dominating set of T. Then $y \in S$ and $x \notin S$ (otherwise x would not have a child that is a private neighbour) and $S_2 = S \cap V(T_2)$ is a γ-set of T_2. Thus, S_2 is a highest minimum dominating set of T_2. But this implies that $ht_{T_2}(S_2) = ht_{T_2}(D_2)$. Therefore, $ht_T(S) = ht_T(D)$ and so D is a highest minimum dominating set of T.

Case 2. Suppose that $x \notin \mathrm{pn}(y, D)$. Let T_1 be the tree $T_1 = T_y$ and let T_2 be the tree $T_2 = T - T_1$. Then $D_1 = D \cap V(T_1)$ is a γ-set of T_1 and $D_2 = D \cap V(T_2)$ is a γ-set of T_2 (since x is not a private neighbour of y). Consider T_1 to be rooted at y and T_2 to be rooted at c. Obviously, $D_1 = \{y\}$ is a highest minimum dominating set of T_1. By the induction hypothesis, D_2 is a highest minimum dominating set of T_2.

If $x \in D$, then x has a child that is a leaf. Let S be a highest minimum dominating set of T. Then $x \in S$ and $y \in S$ and $S_2 = S \cap V(T_2)$ is a γ-set of T_2. Therefore, $ht_{T_2}(S_2) = ht_{T_2}(D_2)$ and so $ht_T(S) = ht_T(D)$. Thus, D is a highest minimum dominating set of T.

If $x \notin D$, we consider the following two cases.

Case (i). Suppose that x has a child w with $w \in D$. Thus, w is a support vertex of T and the only children of w are leaves. Let S be a highest minimum dominating set of T. Then $y \in S$ and $w \in S$, thus $S_2 = S \cap V(T_2)$ is a γ-set of T_2. Therefore, $ht_{T_2}(S_2) = ht_{T_2}(D_2)$ and so $ht_T(S) = ht_T(D)$. Hence D is a highest minimum dominating set of T.

Case (ii). Suppose that x has no child w with $w \in D$. Thus, $N(x) = \{y, z\}$. Hence, $z \in D$ and z has a child v, $v \neq x$, such that v is a private neighbour of z. We claim that v has no children. Otherwise v is adjacent to either a leaf or support vertex u where the only children of u are leaves. If v is adjacent to a leaf, then $v \in D$. If v is adjacent to a support vertex u, then $u \in D$. In both cases, $v \notin \mathrm{pn}(z, D)$, a contradiction. Thus, $N(v) = \{z\}$. Let S be a highest minimum dominating set of T. Then, $y \in S$ and $z \in S$ and so $S_2 = S \cap V(T_2)$ is a γ-set of T_2. Therefore, $ht_{T_2}(S_2) = ht_{T_2}(D_2)$ and so $ht_T(S) = ht_T(D)$, which implies D is a highest minimum dominating set of T.   ∎

Proposition 6. *Let T be a rooted tree at c. If D is not a highest minimum dominating set then, independently of the adjacency model, in Γ_T the vertex D is adjacent to a vertex D' with $ht_T(D') < ht_T(D)$.*

Proof. If D is not a highest minimum dominating set, then by Lemma 5, there is an $x \in D$, $x \neq c$, such that x has no child y where $y \in \mathrm{pn}(x, D)$. If the parent w of x is in D, then $D - \{x\}$ is a dominating set of T, a contradiction. If $w \notin D$, then $D' = (D - \{x\}) \cup \{w\}$ is a dominating set of T and $ht_T(D') < ht_T(D)$.   ∎

Applying Proposition 6 repeatedly, as necessary, gives the following results. The second of these was also found by Fricke, Hedetniemi, Hedetniemi, and Hutson [4]. The first result below transpires to be implicit in their proof using the correspondence mentioned following Theorem 10.

Corollary 7. *Let D' be a highest minimum dominating set of a tree T. Then, for any γ-set $D \neq D'$ of T, there is a path in Γ_T from D to D'.*

Corollary 8 [4]. *For any tree T, the γ-graph Γ_T is connected.*

It is possible to define a partial order on the collection of minimum dominating sets of a tree T by $D_1 \preceq D_2$ if and only if $D_1 = D_2$ or $ht_T(D_1) < ht_T(D_2)$. In the slide adjacency model, every edge of Γ_T corresponds to a reconfiguration that changes the height of a dominating set. Thus, Γ_T is the Hasse diagram of $\preceq$. Equivalently, it has a weakly transitive orientation and is the complement of a cylinder graph; for details and definitions, see [11]. It follows that, in the single vertex adjacency model, the Hasse diagram of $\preceq$ is a spanning subgraph of Γ_T.

Lemma 9. *Suppose the rooted tree (T, c) has two highest minimum dominating sets D_1, D_2. Then, either c belongs to both of D_1 and D_2, or c belongs to neither of them.*

Proof. Suppose not, and let D_1 and D_2 be different highest minimum dominating sets such that $c \in D_1$ and $c \notin D_2$. Let $X_i = \{x \in V(T) \mid d(x, c) = i\}$. Let ℓ be the largest distance between c and a leaf of T. Let m be the largest value such that $D_1 \cap X_m \neq D_2 \cap X_m$. Then $D_1 \cap (X_{m+1} \cup X_{m+2} \cup \cdots \cup X_\ell) = D_2 \cap (X_{m+1} \cup X_{m+2} \cup \cdots \cup X_\ell)$. Notice that by Proposition 4, $m \leq \ell - 2$. Consider $x \in X_m$ with $x \in D_1$ and $x \notin D_2$. Since D_1 is a highest minimum dominating set, x has a child y such that y is a private neighbour of x. Thus, $y \in X_{m+1}$. Let the children of y (if they exist) be $y_1, y_2, \ldots, y_r$. Since $y \in \mathrm{pn}(x, D_1)$, we have that $\{y, y_1, y_2, \ldots, y_r\} \cap D_1 = \emptyset$. Since $D_1 \cap (X_{m+1} \cup X_{m+2} \cup \cdots \cup X_l) = D_2 \cap (X_{m+1} \cup X_{m+2} \cup \cdots \cup X_\ell)$, we also have that $\{y, y_1, y_2, \ldots, y_r\} \cap D_2 = \emptyset$. But $x \notin D_2$. Thus, D_2 does not dominate y, a contradiction. Therefore, if $c \in D_1$, then $c \in D_2$. ∎

Theorem 10. *Let T be a tree rooted at vertex c. Then T has a unique highest minimum dominating set.*

Proof. We proceed by induction on $n = |V(T)|$. Again, the base cases of $1 \leq n \leq 5$ are easy to verify. Suppose $n \geq 6$ and let D be a highest minimum dominating set of T on n vertices. Using Lemma 9, either c belongs to all of the highest minimum dominating sets or it does not belong to any of them. This leads to the following two cases.

Suppose $c \notin D$. Let the children of c be $x_1, x_2, \ldots, x_k$. Then $|D \cap \{x_1, x_2, \ldots, x_k\}| \geq 1$. Label $x_1, x_2, \ldots, x_k$ so that $\{x_1, x_2, \ldots, x_i\} \subseteq D$ and $D \cap \{x_{i+1}, x_{i+2}, \ldots, x_k\} = \emptyset$. Let T_j be the tree $T_j = T_{x_j}$ and let T'_j be the tree $T'_j = \langle V(T_j) \cup \{c\}\rangle$, $j \in \{1, 2, \ldots, k\}$. Let $D_j = D \cap V(T_j)$.

Then, $D_1, D_2, \ldots, D_i$ are γ-sets of $T'_1, T'_2, \ldots, T'_i$, respectively, and $D_{i+1}, D_{i+2}, \ldots, D_k$ are γ-sets of $T_{i+1}, T_{i+2}, \ldots, T_k$, respectively. Consider $T'_1, T'_2, \ldots, T'_i$ all to be rooted at c and $T_{i+1}, T_{i+2}, \ldots, T_k$ to be rooted at $x_{i+1}, x_{i+2}, \ldots, x_k$, respectively. Then $D_1, D_2, \ldots, D_k$ are all highest minimum dominating sets in their respective trees. By the induction hypothesis, these highest minimum dominating sets are unique. Therefore, T has only one highest minimum dominating set D with $c \notin D$.

Suppose $c \in D$. Let T_j and T'_j, $j \in \{1, 2, \ldots, k\}$, be defined as before. Let $D_j = D \cap V(T'_j)$, $j \in \{1, 2, \ldots, k\}$. (Notice that $c \in D_j$ for every $j \in \{1, 2, \ldots, k\}$.) Then $D_1, D_2, \ldots, D_k$ are γ-sets of $T'_1, T'_2, \ldots, T'_k$, respectively. Furthermore, $D_1, D_2, \ldots, D_k$ are all highest minimum dominating sets of $T'_1, T'_2, \ldots, T'_k$. By the induction hypothesis, these highest minimum dominating sets are unique. Therefore, T has only one highest minimum dominating set D with $c \in D$. ∎

In the work of Fricke *et al.* [4], a *level vector* $L(D) = [\ell(v_i)] \in \mathbb{R}^{\gamma(T)}$, where $D = \{v_1, v_2, \ldots, v_\gamma\}$ and $\ell(v_i) \leq \ell(v_{i+1})$, is associated with each γ-set D. The height of a dominating set can be computed from its level vector. The minimum level vector in lexicographic order can be seen to correspond to the unique highest minimum dominating set used in this work.

Recall that a *2-packing* of a graph G is a collection of vertices $P \subseteq V(G)$ such that for any two vertices $x, y \in P$, $d(x, y) > 2$. The *2-packing number* of G is the maximum cardinality of a 2-packing of G. Meir and Moon [10] showed that for a tree T the 2-packing number of T is equal to $\gamma(T)$. Furthermore, their proof shows that any 2-packing of T with maximum cardinality can be transformed into a γ-set of T.

Theorem 11. *For any tree T, $\mathrm{diam}(\Gamma_T) \leq 2\gamma(T)$ in the single vertex replacement adjacency model, and $\mathrm{diam}(\Gamma_T) \leq 2(2\gamma(T) - s)$ in the slide adjacency model, where s is the number of support vertices in T.*

Proof. Let D be a γ-set of T, and P be a 2-packing such that $|P| = |D|$ (such a set P exists by [10]). Since D dominates P, for any $x \in P$ there exists a vertex $v \in D$ such that $x \in N[v]$. Since, for $x, y \in P$ we have $N[x] \cap N[y] = \emptyset$, this establishes a useful one-to-one correspondence between the vertices in P and the vertices in D.

Root T at a vertex c and consider two γ-sets, D and D', of T. Let H be the highest minimum dominating set of T. By Corollary 8 we know there is a path from D to H and a path from D' to H in Γ_T. Joining these two paths together gives an upper bound on $d_{\Gamma_T}(D, D')$.

Consider two γ-sets S and S' of T which are adjacent in Γ_T. Then, in either adjacency model, $S' = (S - \{x\}) \cup \{y\}$ for some $x \in S$ and some $y \in S'$. Since every vertex $v \in S$ is in $N[z]$ for some $z \in P$ and every vertex $u \in S'$ is in $N[z']$ for some $z' \in P$ and $|P| = \gamma(T)$, the closed neighbourhood of any vertex $z \in P$ contains exactly one vertex from S and exactly one vertex from S'. Therefore, if $x \in N[z]$ and $y \in N[z']$, $z, z' \in P$, $z \neq z'$, then z is undominated in S', a contradiction. Hence, to move from S to S', where $SS' \in E(\Gamma_T)$, we remove a vertex $x \in S$ and add a vertex $y \in S'$ where $x, y \in N[z]$ for some $z \in P$.

We provide an algorithm that constructs a path from any γ-set D to the highest minimum dominating set H. This provides an upper bound on $d_{\Gamma_T}(D, H)$ and in turn gives an upper bound on $\text{diam}(\Gamma_T)$.

The number of vertices in common to D and H is $|D \cap H|$. We show how to move from D to a γ-set S with $|D \cap H| < |S \cap H|$.

Let $x \in D$ be such that $x \notin H$ and $d(x, c)$ is maximized. If x is a leaf then let y be the parent of x. By Proposition 4, $y \in H$. Then $S = (D - \{x\}) \cup \{y\}$ is a γ-set of T and $|S \cap H| > |D \cap H|$. Hence, suppose that x is not a leaf.

Case 1. $x \in P$. Let y be the parent of x. Since every vertex $z \in D$ such that $d(z, c) > d(x, c)$ is in H and $x \notin H$, it follows that $y \in H$ and $S = (D - \{x\}) \cup \{y\}$ is a γ-set of T. Notice that $|S \cap H| > |D \cap H|$.

Case 2. $x \notin P$. Let y be the parent of x. We claim that $y \in P$. We know that there is a unique element $p \in P \cap N[x]$, and that p is not dominated by $D - \{x\}$. Thus, there is an element of H in $N[p]$, say x'. By definition of H and definition of x, $d(x', c) < d(x, c)$. Therefore, p cannot be a child of x. Since $x \notin P$, it follows that $y \in P$. This proves the claim.

Suppose $y \in H$. Since every vertex $z \in D$ such that $d(z, c) > d(x, c)$ is in H and $x \notin H$, we have that $S = (D - \{x\}) \cup \{y\}$ is a γ-set of T. Notice that $|S \cap H| > |D \cap H|$.

Suppose that $y \notin H$. Let v be the parent of y. Then $v \in H$. Since every vertex $z \in D$ such that $d(z, c) > d(x, c)$ is in H, the set $S' = (D - \{x\}) \cup \{y\}$ is a γ-set of T and so is $S = (D - \{x\}) \cup \{v\}$. Notice that $DS' \in E(\Gamma_T)$ and that $S'S \in E(\Gamma_T)$. Also notice that $|S \cap H| > |D \cap H|$ and that in the single vertex replacement adjacency model $DS \in E(\Gamma_T)$.

The above argument shows that there is a path from a γ-set D to a γ-set S with $|S \cap H| > |D \cap H|$. In the single vertex replacement adjacency model D and S are adjacent, so that there is a path in Γ_T from D to H where each edge corresponds to a replacement that increases the number of vertices in common with H. Thus, any γ-set D is joined to the highest minimum dominating set H by a path of length at most $\gamma(T)$. Hence $\text{diam}(\Gamma_T) \leq 2\gamma(T)$.

We now consider the slide adjacency model. Let s be the number of support vertices in T. Then at most s leaves of T could belong to D. In a path from

a γ-set D to H, these would be replaced by the s support vertices (that are in H). As outlined in Case 2 above, for each internal vertex of T that belongs to D, there may be two replacements needed to change D into a γ-set S such that $|S \cap H| > |D \cap H|$. Thus, a path from D to H has length at most $2(\gamma(T) - s) + s = 2\gamma(T) - s$. Hence $\operatorname{diam}(\Gamma_T) \leq 2(2\gamma(T) - s)$. ∎

Let T be a tree on n vertices, and let T' be *the corona of T with respect to* K_1, that is, the tree on $2n$ vertices obtained from T by attaching a new leaf to every vertex of T. Then T' has n support vertices and $\gamma(T') = n$. For every leaf ℓ of T', either ℓ or its support vertex must be in any dominating set. It is then easy to see that $\operatorname{diam}(\Gamma_{T'}) = n = \gamma(T')$ in either adjacency model. In either case, the bound in Theorem 11 is $2\gamma(T') = 2n$. We know of no example where the diameter of $\Gamma_{T'}$ is greater than half of the bound given in the theorem.

4. The Order of Γ_T

In this section we show that question 3 has a negative answer, and then provide an upper bound on the number of vertices of the γ-graph of a tree.

Let T be a tree belonging to the infinite family depicted in Figure 1. The number of γ-sets can be derived by counting based on which leaves are in the γ-set, or counting based on which vertex dominates t in the γ-set. A brief calculation shows that T has

$$(a+1)\binom{a}{0}\left(2^b - 1\right)^a + 2\binom{a}{1}\left(2^b - 1\right)^{a-1} + \binom{a}{2}\left(2^b - 1\right)^{a-2} + \cdots + \binom{a}{a}\left(2^b - 1\right)^0$$

γ-sets. This quantity equals $a2^b \left(2^b - 1\right)^{a-1} + 2^{ab}$.

Now $2^{\gamma(T)} = 2^{ab+1} = 2^{ab} + 2^{ab}$, so if $a2^b \left(2^b - 1\right)^{a-1} > 2^{ab}$, a negative answer to question 3 posed by Fricke *et al.* [4] can be given. Thus, consider the inequality

$$\log_2\left[a2^b\left(2^b - 1\right)^{a-1}\right] > \log_2\left[2^{ab}\right].$$

This can equivalently be expressed as

$$\log_2[a] + (a-1)\log_2\left[2^b - 1\right] + b > ab,$$

which in turn can be written as

$$\log_2[a] + (a-1)\log_2\left[2^b - 1\right] > (a-1)b.$$

Dividing by $(a-1)$ and rearranging gives

$$\frac{\log_2[a]}{a-1} > b - \log_2\left[2^b - 1\right]$$

or

$$\frac{\log_2[a]}{a-1} > \log_2\left[1 + \frac{1}{2^b - 1}\right].$$

Since the value of b can be chosen so that $\log_2\left[1 + \frac{1}{2^b-1}\right]$ is arbitrarily close to zero, for any fixed value of a there exists a value of b for which this inequality holds. Thus, there are infinitely many trees T which have more than $2^{\gamma(T)}$ minimum dominating sets. On the other hand, a similar calculation shows that, for any fixed value of a, the number of minimum dominating sets of a tree in this family is asymptotic to 2^γ.

For any tree T, not necessarily in this infinite family, a straightforward proof by induction on $\gamma(T)$ shows that T has at most $3^{\gamma(T)}$ γ-sets. This bound can be improved, as we now show. The mysterious quantity $\left(\left(1 + \sqrt{13}\right)/2\right)$ which appears in the improved bound is the solution to a recursively defined inequality that appears near the end of the proof.

Theorem 12. *Any forest F has at most $\left(\left(1 + \sqrt{13}\right)/2\right)^{\gamma(F)}$ γ-sets.*

Proof. For ease of notation, say that $b = \left(\left(1 + \sqrt{13}\right)/2\right)$. Let $T_1, T_2, \ldots, T_k$ be the components of F. Notice that $\gamma(F) = \gamma(T_1) + \gamma(T_2) + \cdots + \gamma(T_k)$. Let T be a component of F with maximum order. Consider T to be rooted at the vertex c and let ℓ be a leaf at maximum distance from c and let x be the parent of ℓ. Let y be the parent of x.

Let P be a maximum 2-packing of T. By Meir and Moon [10], we know that $|P| = \gamma(T)$ and that this 2-packing can be transformed into a γ-set of T.

Notice that, for each vertex $v \in P$, $x \in N[v]$ for some $v \in P$, otherwise this implies that $\{x, \ell\} \cap P = \emptyset$. But then $P \cup \{\ell\}$ would be a 2-packing with greater cardinality, a contradiction. We proceed by strong induction on $\gamma(F)$. It is easy to check that the result holds for $\gamma(F) = 1$ and $\gamma(F) = 2$.

Suppose x is adjacent to at least two leaves. Then x is in every γ-set of F. Consider a γ-set D of F. If $y \in \mathrm{pn}(x, D)$ then $D - \{x\}$ is a minimum dominating set of $F' = F - (T_x \cup \{y\})$. In this case $\gamma(F') = \gamma(F) - 1$, so by the induction hypothesis F' has at most $b^{\gamma(F)-1}$ minimum dominating sets. Thus, F has at most $b^{\gamma(F)-1}$ γ-sets D where $y \in \mathrm{pn}(x, D)$. If $y \notin \mathrm{pn}(x, D)$ then $D - \{x\}$ is a minimum dominating set of $F' = F - T_x$. In this case $\gamma(F') = \gamma(F) - 1$, so by the induction hypothesis F' has at most $b^{\gamma(F)-1}$ minimum dominating sets. Thus, F has at most $b^{\gamma(F)-1}$ γ-sets D where $y \notin \mathrm{pn}(x, D)$. In total then, F has at most $b^{\gamma(F)-1} + b^{\gamma(F)-1} = 2b^{\gamma(F)-1} < b^{\gamma(F)}$ γ-sets.

Thus, suppose that x is adjacent to only one leaf, ℓ. That is, suppose that $\deg(x) = 2$.

Consider any γ-set D of F and any maximum 2-packing P of T. Notice that for each $v \in P$, the set $N[v]$ contains exactly one vertex of D (otherwise v is not

dominated or P is not a 2-packing). We consider the following three cases for P: $y \in P$, $x \in P$, or $\ell \in P$. Recall that $\deg(x) = 2$.

Case 1. $y \in P$. Then for any γ-set D of F, $x \in N[y] \cap D$ because ℓ must be dominated. This implies that $\deg(y) = 2$, for otherwise the other children of y are not dominated. Consider the forest $F' = F - (N[y] \cup \{l\})$. Notice that $\gamma(F') = \gamma(F) - 1$. By the induction hypothesis, F' has at most $b^{\gamma(F)-1}$ minimum dominating sets. Hence F has at most $b^{\gamma(F)-1}$ γ-sets.

Case 2. $x \in P$. Notice that for any γ-set D of F, either x or ℓ is in D. As above, y has no children other than x; in particular, it is not adjacent to any leaves.

First suppose that $\ell \in D$. Consider the forest $F' = F - T_x$. Notice that $\gamma(F') = \gamma(F) - 1$ and that $D - \{\ell\}$ is a minimum dominating set of F'. By the induction hypothesis, F' has at most $b^{\gamma(F)-1}$ minimum dominating sets. Therefore there are at most $b^{\gamma(F)-1}$ γ-sets D of F with $\ell \in D$.

Now suppose that $x \in D$. Consider the forest $F' = F - N[x]$. Suppose $\deg(y) = t + 2$. Since y is not adjacent to any leaves, F' is comprised of a forest F'' and t copies of K_2. Notice that $\gamma(F') = \gamma(F) - 1$ and that $\gamma(F'') = \gamma(F) - (t + 1)$ and that $D - \{x\}$ is a minimum dominating set of F'. By the induction hypothesis, F'' has at most $b^{\gamma(F)-t-1}$ minimum dominating sets and the t copies of K_2 together have 2^t minimum dominating sets. Thus, F' has at most $2^t b^{\gamma(F)-t-1} < b^{\gamma(F)-1}$ minimum dominating sets. Therefore there are at most $b^{\gamma(F)-1} + b^{\gamma(F)-1} = 2b^{\gamma(F)-1} < b^{\gamma(F)}$ γ-sets of F.

Case 3. $\ell \in P$. Consider D, a γ-set of F. If y is adjacent to a leaf, v, then either $y \in D$ or $v \in D$. In either case, $D - \{x, \ell\}$ is a minimum dominating set of $F' = F - \{x, \ell\}$. By the induction hypothesis, F' has at most $b^{\gamma(F)-1}$ minimum dominating sets and so F has at most $2b^{\gamma(F)-1} < b^{\gamma(F)}$ γ-sets. Suppose, then, that y is not adjacent to any leaves. Let $\deg(y) = t + 1$ ($t \geq 1$).

There are three possibilities: (i) $y \in D$; (ii) $y \notin D$ and at least one child of y is in D; and (iii) $y \notin D$ and no children of y are in D. We consider these in turn.

Case (i). $y \in D$. Consider the forest $F' = F - N[y]$. Notice that $\gamma(F') = \gamma(F) - 1$ and that $D - \{y\}$ is a minimum dominating set of F'. By the induction hypothesis, F' has at most $2^t b^{\gamma(F)-t-1}$ minimum dominating sets. Thus, F has at most $2^t b^{\gamma(F)-t-1}$ γ-sets D with $y \in D$.

Case (ii). $y \notin D$ and at least one child of y is in D. Consider the forest $F' = F - T_y$. Notice that $\gamma(F') = \gamma(F) - t$, that $D - (D \cap V(T_y))$ is a minimum dominating set of F', and that $D \cap V(T_y)$ is a minimum dominating set of T_y. By the induction hypothesis, F' has at most $b^{\gamma(F)-t}$ minimum dominating sets. Now T_y has $2^t - 1$ minimum dominating sets that do not contain y. Thus, F has at most $(2^t - 1)b^{\gamma(F)-t}$ minimum dominating sets D with $y \notin D$ and at least one child of y in D.

Case (iii). $y \notin D$ and no children of y are in D. Then there is a vertex $w \in D$ which dominates y. Consider the forest $F' = F - (N[w] \cup T_y)$. Notice that $\gamma(F') = \gamma(F) - t - 1$ and that $D - V(N[w] \cup T_y)$ is a minimum dominating set of F'. Also notice that $D \cap V(N[w] \cup T_y)$ is a minimum dominating set of the induced subgraph $\langle N[w] \cup T_y \rangle$. By the induction hypothesis F' has at most $b^{\gamma(F)-t-1}$ minimum dominating sets. Now $\langle N[w] \cup T_y \rangle$ has one minimum dominating set that contains w and no children of y. Thus, F has at most $b^{\gamma(F)-t-1}$ γ-sets D with $y \notin D$, $w \in D$, and no children of y in D.

Considering these three cases together, we see that F has at most $2^t b^{\gamma(F)-t-1} + (2^t - 1)b^{\gamma(F)-t} + b^{\gamma(F)-t-1}$ γ-sets. Now $2^t b^{\gamma(F)-t-1} + (2^t - 1)b^{\gamma(F)-t} + b^{\gamma(F)-t-1} = b^{\gamma(F)-t-1}[2^t + 1 + b(2^t - 1)]$. Thus, if this value is at most $b^{\gamma(F)}$, the proof is complete.

From the desired inequality $2^t + 1 + b(2^t - 1) \le b^{t+1}$ we obtain the inequality

$$\frac{1}{b}\left(\frac{2}{b}\right)^t + \frac{1}{b^{t+1}} + \frac{2^t - 1}{b^t} \le 1.$$

Notice that since $b = \left((1 + \sqrt{13})/2\right) > 2$, the function

$$f(t) = \frac{1}{b}\left(\frac{2}{b}\right)^t + \frac{1}{b^{t+1}} + \frac{2^t - 1}{b^t}$$

is a decreasing function. That is, $f(t) > f(t+1)$ for $t \ge 1$. Therefore $f(t)$ is maximized for $t = 1$. By evaluating $2^t + 1 + b(2^t - 1) \le b^{t+1}$ at $t = 1$, we see that $0 \le b^2 - b - 3$ and that any $b \ge \left((1 + \sqrt{13})/2\right)$ satisfies this inequality. Hence for $b = \left((1 + \sqrt{13})/2\right)$, we have that $2^t b^{\gamma(F)-t-1} + (2^t - 1)b^{\gamma(F)-t} + b^{\gamma(F)-t-1} \le b^{\gamma(F)}$. This completes the proof. ∎

Corollary 13. *Any tree T has at most $\left((1 + \sqrt{13})/2\right)^{\gamma(T)}$ γ-sets.*

Corollary 14. *For any tree T, $|V(\Gamma_T)| \le \left((1 + \sqrt{13})/2\right)^{\gamma(T)}$.*

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