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GRAPHIC AND COGRAPHIC Γ-EXTENSIONS OF BINARY MATROIDS

Y.M. Borse

Department of Mathematics
Savitribai Phule Pune University
Pune-411007, India

e-mail: ymborse11@gmail.com

and

Ganesh Mundhe

Army Institute of Technology
Pune-411015, India

e-mail: ganumundhe@gmail.com

Abstract

Slater introduced the point-addition operation on graphs to characterize 4-connected graphs. The Γ-extension operation on binary matroids is a generalization of the point-addition operation. In general, under the Γ-extension operation the properties like graphicness and cographicness of matroids are not preserved. In this paper, we obtain forbidden minor characterizations for binary matroids whose Γ-extension matroids are graphic (respectively, cographic).

Keywords: splitting, Γ-extension, graphic, cographic, minor.

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1. Introduction

We refer to [5] for standard terminology in graphs and matroids. The matroids considered here are loopless and coloopless. Slater [9] introduced the point-addition operation on graphs and used it to classify 4-connected graphs. Azanchiler [1] extended this operation to binary matroids as follows.

Definition 1 [1]**.** Let M be a binary matroid with ground set S and standard matrix representation A over the field $GF(2)$. Let $X = \{x_1, x_2, \ldots, x_m\} \subset S$ be an independent set in M and let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ be a set such that $S \cap \Gamma = \emptyset$. Suppose A' is the matrix obtained from the matrix A by adjoining m columns labeled by $\gamma_1, \gamma_2, \ldots, \gamma_m$ such that the column labeled by γ_i is same as the column labeled by x_i for $i = 1, 2, \ldots, m$. Let A^X be the matrix obtained by adjoining one extra row to A' which has entry 1 in the column labeled by γ_i for $i = 1, 2, \ldots, m$ and zero elsewhere. The vector matroid of the matrix A^X, denoted by M^X, is called as the Γ-extension of M with respect to X and the transition from M to M^X is called as the Γ-extension operation on M.

Note that the ground set of the matroid M^X is $S \cup \Gamma$ and $M^X \setminus \Gamma = M$. Therefore M^X is an extension of M. Some basic properties of M^X are studied in [1] and [2].

The Γ-extension operation is related to the *splitting operation* on binary matroids which is defined by Shikare *et al.* [8] as follows.

Definition 2 [8]**.** Let M be a binary matroid with standard matrix representation A over the field $GF(2)$ and let X be a set of elements of M. Let A_X be the matrix obtained by adjoining one extra row to the matrix A whose entries are 1 in the columns labeled by the elements of the set X and zero otherwise. The vector matroid of the matrix A_X, denoted by M_X, is called as the splitting matroid of M with respect to X, and the transition from M to M_X is called as the splitting operation.

Let M be a binary matroid with ground set S and let $X = \{x_1, x_2, \ldots, x_m\}$ be an independent set in M. Obtain the extension M' of M with ground set $S \cup \Gamma$, where $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ is disjoint from S, such that $\{x_i, \gamma_i\}$ is a 2-circuit in M' for each i. The matroid M'_Γ obtained from M' by splitting with respect to the set Γ is the Γ-extension matroid M^X.

Earlier, the splitting with respect to a pair of elements, which is a special case of Definition 2, was defined by Raghunathan *et al.* [6] for binary matroids as an extension of the corresponding graph operation due to Fleischner [4].

In general, under the splitting operation the properties like graphicness and cographicness of matroids are not preserved. Shikare and Waphare [7] obtained the following characterization for the class of graphic matroids M whose splitting matroids M_X, with $|X| = 2$, are again graphic.

Theorem 3 [7]**.** *Let M be a graphic matroid. For any $X \subset S$ with $|X| = 2$, the splitting matroid M_X is graphic if and only if M has no minor isomorphic to any of the circuit matroids $M(G_1), M(G_2), M(G_3)$ and $M(G_4)$, where G_1, G_2, G_3 and G_4 are the graphs as shown in Figure* 1.

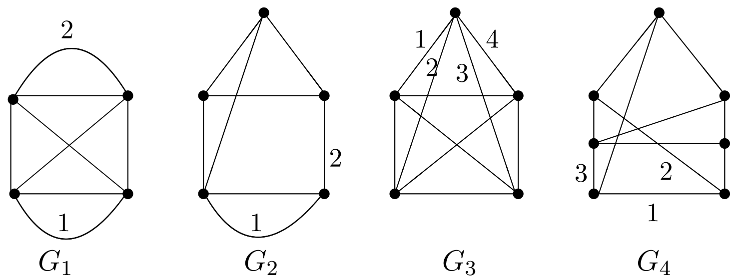

Figure 1

Borse *et al.* [3] obtained a similar characterization for the cographic matroids M whose splitting matroids M_X, with $|X| = 2$, are cographic.

Theorem 4 [3]**.** *Let M be a cographic matroid. For any $X \subset S$ with $|X| = 2$, the splitting matroid M_X is cographic if and only if M has no minor isomorphic to any of the circuit matroids $M(G_1)$ and $M(G_2)$, where G_1 and G_2 are the graphs as shown in Figure* 1*.*

It remains to find the effect of the splitting operation with respect to X where $|X| \geq 3$, on the properties like graphicness and cographicness of a matroid.

Like splitting operation, the Γ-extension operation also does not preserve graphicness and cographicness properties of a given matroid, in general. Azanchiler [2] obtained few results in this direction.

In this paper, we characterize binary matroids M whose Γ-extension matroids M^X with $|X| \geq 2$ are graphic (respectively, cographic).

The following are the main results of the paper.

Theorem 5. *Let M be a binary matroid. Then M^X is graphic* (*respectively, cographic*) *for every independent set X in M with $|X| = 2$ if and only if M does not contain a minor that is isomorphic to $M(K_4)$.*

Corollary 6. *Let M be a graphic* (*respectively, cographic*) *matroid. Then M^X is graphic* (*respectively, cographic*) *for every independent set X in M with $|X| = 2$ if and only if M does not contain a minor that is isomorphic to $M(K_4)$.*

Theorem 7. *Let M be a binary matroid. Then M^X is graphic* (*respectively, cographic*) *for every independent set X in M with $|X| \geq 3$ if and only if M does not contain a minor that is isomorphic to a* 4*-circuit.*

Corollary 8. *Let M be a graphic* (*respectively, cographic*) *matroid. Then M^X is graphic* (*respectively, cographic*) *for every independent set X in M with $|X| \geq 3$ if and only if M does not contain a minor that is isomorphic to a* 4*-circuit.*

2. Case $|X| = 2$

In this section, we prove Theorem 5. First, observe that there should be only three forbidden minors in Theorem 3. For the graphs G_2 and G_4 in Figure 1, $M(G_2) \cong M(G_4)\backslash\{3\}/\{1,2\}$. Therefore $M(G_2)$ is a minor of $M(G_4)$ and hence Theorem 3 can be restated as follows.

Theorem 9. *Let M be a graphic matroid. For any $X \subset S$ with $|X| = 2$ the splitting matroid M_X is graphic if and only if M has no minor isomorphic to any of the circuit matroids $M(G_1), M(G_2)$ and $M(G_3)$, where G_1, G_2 and G_3 are the graphs as shown in Figure 1.*

We need the following well-known characterizations.

Theorem 10 (Oxley [5])**.** *A binary matroid M is graphic if and only if no minor of M is isomorphic to any of the matroids $F_7, F_7^*, M^*(K_{3,3})$ and $M^*(K_5)$.*

Theorem 11 (Oxley [5])**.** *A binary matroid M is cographic if and only if no minor of M is isomorphic to any of the matroids $F_7, F_7^*, M(K_{3,3})$ and $M(K_5)$.*

Theorem 12 (Oxley [5])**.** *A binary matroid M is regular if and only if no minor of M is isomorphic to any of the matroids F_7, F_7^*.*

The proof of the following lemma is trivial.

Lemma 13. *If $\{x, y\}$ is a circuit in a matroid M, then $M\backslash\{x\} \cong M\backslash\{y\}$ and $M/\{x\} \cong M/\{y\}$.*

Lemma 14. *Let M be a binary matroid containing a minor isomorphic to $M(K_4)$. Then there is an independent set X in M with $|X| = 2$ such that the matroid M^X is not regular.*

Proof. Suppose M contains a minor N which is isomorphic to $M(K_4)$. Then there are subsets T_1 and T_2 of the ground set of M such that $N = M\backslash T_1/T_2$. Label the edges of the graph K_4 by the set $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ so that x_1, x_2, x_3, x_4, in order, form a 4-cycle and the edges x_5, x_6 are the chords of this cycle.

Let $X = \{x_1, x_3\}$. Then X is disjoint from $T_1 \cup T_2$ and is independent in N as well as in M. Further, $N^X = M^X \setminus T_1/T_2$. Moreover, the edges x_1 and x_3 are not adjacent in K_4. Let A be the standard matrix representation of $M(K_4)$ over the field $GF(2)$. Then

$$\begin{array}{cc} & \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ A = & \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

and

$$A^X = \begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \gamma_1 & \gamma_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

Therefore

$$A^X/\{\gamma_1\} = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \gamma_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

Since $A^X/\{\gamma_1\}$ is a matrix representation of the matroid $M(K_4)^X/\{\gamma_1\} \cong N^X/\{\gamma_1\}$, it follows from the standard matrix representation of the matroid F_7 that $N^X/\{\gamma_1\} \cong F_7$. Therefore $M^X \setminus T_1/T_2/\{\gamma_1\} \cong F_7$. This shows that F_7 is a minor of M^X. Hence, by Theorem 12, M^X is not regular. ■

Proposition 15. *Let M be a binary matroid such that no minor of M is isomorphic to $M(K_4)$. Then M^X is graphic as well as cographic for any independent set X in M with $|X| = 2$.*

Proof. Clearly, $M(K_4)$ is a minor of each of the six matroids $F_7, F_7^*, M(K_5)$, $M^*(K_5), M(K_{3,3})$ and $M^*(K_{3,3})$. Since no minor of M is isomorphic to $M(K_4)$, none of these six matroids can be a minor of M. Hence, by Theorems 10 and 11, M is graphic as well as cographic. Thus $M = M(G)$ for some planar graph G. Assume that M^X is not graphic or not cographic for some independent set $X = \{x_1, x_2\}$ in M. We obtain a contradiction by proving that M contains a minor isomorphic to $M(K_4)$.

Let M' be the extension of M obtained by adding two elements $\{\gamma_1, \gamma_2\}$ to the ground set S of M such that $\{x_1, \gamma_1\}$ and $\{x_2, \gamma_2\}$ are circuits in M'. Then $M' \setminus \{\gamma_1, \gamma_2\} = M$. The ground set of M' is $S \cup \{\gamma_1, \gamma_2\}$. Since M is graphic and cographic, so is M'. Therefore M' does not contain a minor isomorphic to $M(K_5) = M(G_3)$. By definition of M^X, we have $M^X = M'_{\{\gamma_1,\gamma_2\}}$, where $M'_{\{\gamma_1,\gamma_2\}}$ is the matroid obtained from M' by splitting with respect to the pair $\{\gamma_1, \gamma_2\}$. Therefore $M'_{\{\gamma_1,\gamma_2\}}$ is not graphic or not cographic.

By Theorems 4 and 9, there is a minor N' of M' such that $N' \cong M(G_1)$ or $N' \cong M(G_2)$, where G_1 and G_2 are the graphs as shown in Figure 1. Clearly, $M(K_4) \cong M(G_1)\setminus\{1,2\} \cong M(G_2)\setminus\{1\}/\{2\}$. Hence $M(K_4)$ is isomorphic to a minor of N'. If N' is a minor of M, then M has a minor isomorphic to $M(K_4)$, a contradiction. Consequently, N' is not a minor of M. It implies that N' contains γ_1 or γ_2 or both. By Lemma 13, we may assume that N' contains x_i whenever it contains γ_i. Thus N' contains at least one of the two 2-circuit $\{x_1, \gamma_1\}$ and $\{x_2, \gamma_2\}$ of M'. Suppose N' contains both γ_1 and γ_2. Then N' contains both the 2-circuits

$\{x_1, \gamma_1\}$ and $\{x_2, \gamma_2\}$. Therefore N' is isomorphic to $M(G_1)$ and the two 2-cycles present in G_1 corresponds to $\{x_1, \gamma_1\}$ and $\{x_2, \gamma_2\}$. Thus $M(K_4) \cong N' \backslash \{\gamma_1, \gamma_2\}$ is minor of $M' \backslash \{\gamma_1, \gamma_2\} = M$, a contradiction. Hence N' contains exactly one of γ_1 and γ_2.

We may assume that N' contains γ_1 but not γ_2. Then $N' \backslash \gamma_1$ is a minor of $M' \backslash \gamma_1$ and hence is a minor of M. Suppose N' is isomorphic to $M(G_2)$. Then the 2-cycle present in G_2 corresponds to the 2-circuit $\{x_1, \gamma_1\}$ in N'. Hence $M(K_4) \cong N' \backslash \{\gamma_1\}/\{2\}$. But $N' \backslash \{\gamma_1\}/\{2\}$ is minor of $N' \backslash \{\gamma_1\}$ and so is a minor of M. Consequently, $M(K_4)$ is isomorphic to a minor of M, a contradiction. Therefore $N' \cong M(G_1)$. We may assume that the 2-circuit $\{x_1, \gamma_1\}$ of N' corresponds to the 2-cycle of G_1 containing the edge labeled by 1. Clearly, $M(K_4) \cong N' \backslash \{\gamma_1, 2\}$. Thus $M(K_4)$ is isomorphic to a minor of $N' \backslash \{\gamma_1\}$ and so is isomorphic to a minor of M, a contradiction. ∎

***Proof of Theorem* 5.** Suppose M contains a minor isomorphic to $M(K_4)$. By Lemma 14, M^X is not regular for some independent set X in M with $|X| = 2$. Therefore, by Theorems 10, 11 and 12, M^X is neither graphic nor cographic. Conversely, if no minor of M is isomorphic to $M(K_4)$, then, by Proposition 15, M^X is graphic as well as cographic for any independent set X in M with $|X| = 2$. ∎

3. Case $|X| \geq 3$

In this section, we prove Theorem 7.

Lemma 16. *Let M be a binary matroid containing a minor isomorphic to a 4-circuit. Then there is an independent set X in M with $|X| \geq 3$ such that M^X is not regular.*

Proof. Suppose M contains a minor N which is isomorphic to a 4-circuit. Let the ground set of N be $\{x_1, x_2, x_3, x_4\}$. Let $X = \{x_1, x_2, x_3\}$. Then X is independent in N and so in M. The following matrix A represents N over the field $GF(2)$.

$$A = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}. \text{ Therefore } A^X = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & \gamma_1 & \gamma_2 & \gamma_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

In A^X, by adding the fourth row to the first row and then interchanging the fourth and fifth columns, we get the following matrix which is the standard matrix representation of the matroid F_7^* over $GF(2)$:

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & \gamma_1 & x_4 & \gamma_2 & \gamma_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

The vector matroid of the matrix A^X is the Γ-extension N^X of N. Hence N^X is isomorphic to F_7^*. Since N is a minor of M, there are disjoint subsets T_1 and T_2 of the ground set of M such that $N = M \setminus T_1/T_2$. Since $X \cap T_1 = \emptyset$ and $X \cap T_2 = \emptyset$, it follows that $N^X = (M \setminus T_1/T_2)^X = M^X \setminus T_1/T_2$. Hence N^X is a minor of M^X. Therefore M^X has a minor isomorphic to F_7^*. By Theorem 12, M^X is not regular. ■

Proposition 17. *Let M be a binary matroid such that no minor of M is isomorphic to a 4-circuit. Then M^X is graphic as well as cographic for any independent set X in M with $|X| \geq 3$.*

Proof. Clearly, each of the six matroids $F_7, F_7^*, M(K_5), M(K_{3,3}), M^*(K_5)$ and $M^*(K_{3,3})$ contains a 4-circuit. Hence none of these six matroids can be a minor of M. Therefore, by Theorems 10 and 11, M is graphic as well as cographic. Hence $M = M(G)$ for some graph G without isolated vertices.

Let X be an independent set in M with $|X| \geq 3$. We prove that M^X is graphic as well as cographic. Let $D_1, D_2, \ldots, D_m$ be components of M. Since M is graphic and cographic, each component D_i is also graphic and cographic. Therefore $D_i = M(H_i)$ for some planar graph H_i for $i = 1, 2, \ldots, m$. If the set X does not intersect a component D_i of M, then D_i is a component of M^X, too. Therefore we may assume that X intersects each D_i. Let $X_i = X \cap D_i$ for $i = 1, 2, \ldots, m$. Then $X = X_1 \cup X_2 \cup \cdots \cup X_m$. Since X is independent in M, each X_i is independent in D_i and so it does not contain parallel edges. Since $M(H_i)$ is component of M for all $i = 1, 2, \ldots, m$, we may assume that graphs $H_i (i = 1, 2, \ldots, m)$ are vertex-disjoint.

Suppose the rank of D_i is at least 3. Then H_i contains at least four vertices. Since D_i is connected, H_i is 2-connected. It follows that H_i contains an r-circuit and so M contains an r-circuit for some $r \geq 4$. This implies that M has a 4-circuit as a minor, a contradiction. Hence the rank of each D_i is one or two. If the rank of D_i is one, then H_i has exactly two vertices. Therefore H_i is K_2 or a graph in which any two edges are parallel. Thus X_i contains exactly one edge of H_i. Suppose the rank of D_i is two. Then H_i has exactly three vertices and further, H_i is 2-connected and so it contains a triangle, say T. Any edge of H_i which is not in T is parallel to one of the three edges of T. This implies that any two edges of H_i are adjacent. Since X_i is independent, it contains one edge or two non-parallel edges of H_i. Consequently, $|X_i| = 1$ or 2. Let $X_i = \{e_i\}$ if $|X_i| = 1$

and let $X_i = \{e_i, f_i\}$ if $|X_i| = 2$ for $i = 1, 2, \ldots, m$. Let $e_i = u_iv_i$. Then $f_i = u_iw_i$ for some $w_i \neq v_i$.

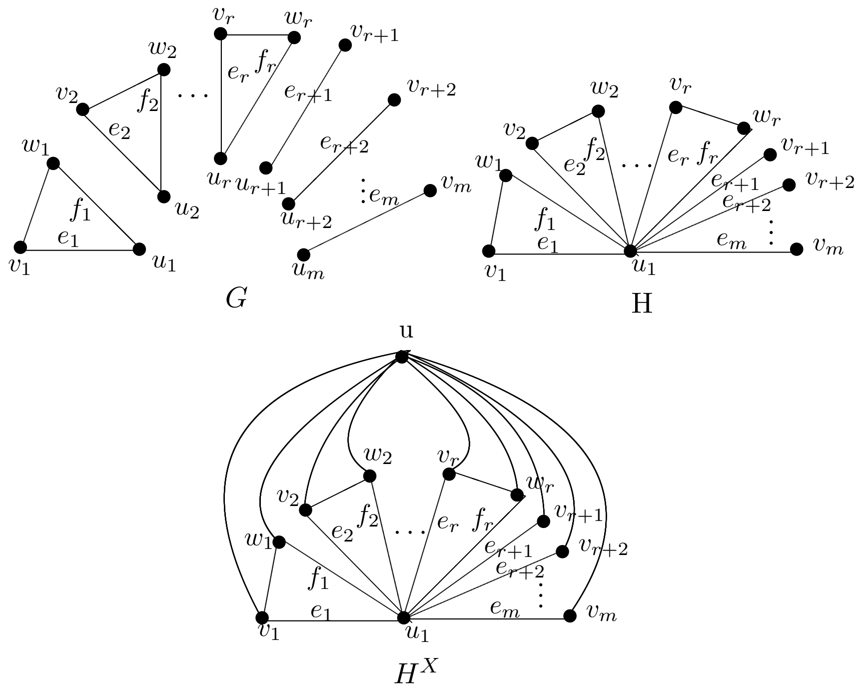

Figure 2

Let H be the graph obtained from $H_1, H_2, \ldots, H_m$ by identifying the vertices $u_2, u_3, \ldots, u_m$ to u_1 (see Figure 2). Then $M(G)$ is isomorphic to $M(H)$. Therefore $M(G)^X$ is isomorphic to $M(H)^X$. Let H^X be the graph obtained from H by adding an additional vertex u and edges uv_i for $i = 1, 2, \ldots, m$, and the edge uw_i if $f_i \in X$ for each i. By Definition 1, $M(H)^X$ is isomorphic to the matroid $M(H^X)$. Thus $M^X = M(G)^X$ is isomorphic to $M(H^X)$. Hence M^X is graphic.

Now, we prove that $M(H^X)$ is cographic, that is, H^X is planar. Assume that $M(H^X)$ is not cographic. Then, by Theorem 11, it has $M(K_5)$ or $M(K_{3,3})$ as a minor. Each of K_5 and $K_{3,3}$ are simple graphs. Also, addition or deletion of parallel edges to a graph does not change its planarity. Further, X does not contain parallel edges. Therefore we may assume that each graph H_i is simple. Hence each H_i is a K_2 or a triangle. Clearly, the graph H is planar. All vertices of H^X other than u_1 and u have degree two or three. However, K_5 has five vertices with degree four. Contractions and deletions in H^X does not increase degree of any vertex in H^X other than u and u_1. Hence $M(K_5)$ cannot be a minor of $M(H^X)$.

Thus $M(H^X)$ contains $M(K_{3,3})$ as a minor. If H does not contain a triangle, then it is the star $K_{1,m}$ and hence H^X is $K_{2,m}$. Therefore $M(H^X)$ does not have

$M(K_{3,3})$ as a minor, a contradiction. Suppose H contains a triangle. The vertices of a triangle in H are u_1 and v_i, w_i for some $i \geq 1$. Hence u is adjacent to v_i or w_i or both in H^X. The graph $M(K_{3,3})$ does not contain a triangle and also has all vertices of degree three. Suppose u is not adjacent to w_i in H^X. Then the degree of w_i in H^X is two. In order to get $K_{3,3}$ as a minor of H^X, we need to delete or contract one edge incident to w_i and then delete the other edge incident to w_i. This also can be done by just deleting both edges incident to w_i. But then the degree of v_i becomes two. Suppose u is adjacent to both v_i and w_i. Then u, v_i, w_i induces a triangle in H^X. Since $M(K_{3,3})$ does not contain a triangle, we need to delete or contract one of the edges in this triangle. The contraction creates a parallel edge which is to be deleted later on. Thus, at least one edge of the triangle with vertices u, v_i, w_i is deleted. Hence the degree of v_i or w_i or both becomes two. It follows that in order to remove triangles from H^X we are left with a subgraph isomorphic to $K_{2,r}$ for some $r \geq 1$. However $M(K_{2,r})$ does not contain $M(K_{3,3})$ as a minor and hence $M(H^X)$ does not contain $M(K_{3,3})$ as a minor, a contradiction. Thus $M(H^X)$ is cographic. ∎

***Proof of Theorem* 7.** If M contains a minor isomorphic to a 4-circuit, then, by Lemma 16, M^X is not regular and hence, by Theorems 10, 11 and 12, M^X is neither graphic nor cographic for every independent set X in M with $|X| \geq 3$. Conversely, if no minor of M is isomorphic to a 4-circuit, then, by Proposition 17, M^X is graphic as well as cographic for any independent set X with $|X| \geq 3$. ∎

Remark 18. As pointed out by one of the referees, Theorem 5 can be proved using graph-theoretic approach, as a binary matroid without $M(K_4)$-minor is the cycle matroid of some series-parallel graph. There is no change in the proof of the "only if" part of Theorem 5. The referee outlined the proof of the "if" part as follows.

Suppose $M = M(G)$ for some series-parallel graph G. To show that $M(G)^X$ is graphic and cographic, it suffices to show that $M(G)^X$ is graphic and planar. To show that $M(G)^X$ is graphic, it suffices to show that, for any pair of edges e and f of G, there exists a graph G' that is 2-isomorphic to G in which e and f are adjacent. (Showing e and f are adjacent in G' implies that every matroid splitting operation in $M(G)$ can be realized as a graphic splitting operation in G'.) Showing that such a G' exists is easily done by induction: first reduce to the 2-connected case, which is trivial, and then take a 2-sum $\{G_1, G_2\}$ of G. (Such a 2-sum always exists in series-parallel graphs having at least four edges.) If e and f are in G_1 (say), then just apply induction. If e is in G_1 and f in G_2, then apply induction to e and q in G_1, and f and q in G_2, where q is the edge common to G_1 and G_2. Now, given that e and f are adjacent in G', and G' is series-parallel, it is easy to verify that the graph splitting operation of e and f in G' produces a planar graph, which proves Theorem 5.

Theorem 7 can be handled in a similar fashion. In particular, binary matroids with no 4-circuit minor are graphic, and can be constructed from 1-sums of "fat" triangles (a triangle plus parallel edges) and "fat" edges (an edge plus parallel edges).

Acknowledgments

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