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HAMILTONIAN AND PANCYCLIC GRAPHS IN THE CLASS OF SELF-CENTERED GRAPHS WITH RADIUS TWO

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Abstract

The paper deals with Hamiltonian and pancyclic graphs in the class of all self-centered graphs of radius 2. For both of the two considered classes of graphs we have done the following. For a given number n of vertices, we have found an upper bound of the minimum size of such graphs. For $n \leq 12$ we have found the exact values of the minimum size. On the other hand, the exact value of the maximum size has been found for every n. Moreover, we have shown that such a graph (of order n and) of size m exists for every m between the minimum and the maximum size. For $n \leq 10$ we have found all nonisomorphic graphs of the minimum size, and for $n = 11$ only for Hamiltonian graphs.

Keywords: self-centered graph with radius 2, Hamiltonian graph, pancyclic graph, size of graph.

2010 Mathematics Subject Classification: 05C12, 05C35, 05C45.

1. Introduction

In this paper we investigate the possible number of edges of Hamiltonian graphs, or pancyclic graphs, in the class of all self-centered graphs with n vertices and radius $r = 2$. Recall that the problem of the size of self-centered graphs of given order n and radius r, without restricting to Hamiltonian or pancyclic graphs, has a long history. Buckley [2] has found all possible sizes of self-centered graphs for given n and r. In the present paper we derive analogous results as those of Buckley, for subclasses of the class of all self-centered graphs with radius $r = 2$.[1] Namely, we will consider the subclass of Hamiltonian graphs and the subclass of pancyclic graphs. Hamiltonian and pancyclic graphs are a topic of intensive study, see e.g. [4]. Due to the complexity of the problem, we restrict ourselves to the radius $r = 2$.

We consider finite, connected, undirected graphs without loops and multiple edges. We follow terminology by [3]. Let us only recall some of them. The *order* of a graph G is the cardinality of its vertex set and the *size* of G is the cardinality of its edge set. For a connected graph G, the *distance* $d_G(u, v)$ between vertices u and v is the length of a shortest path joining them, $\deg_G(u)$ is the *degree* of u in G, $\delta(G)$ is the *minimal degree* of G. The *eccentricity* $e_G(u)$ of a vertex $u \in V(G)$ is $\max\{d_G(u, x) : x \in V(G)\}$. The *radius* $r(G)$ an the *diameter* $d(G)$ of G are the minimum and the maximum eccentricity of its vertices, respectively. A graph is *self-centered* if its diameter is equal to its radius, and is *pancyclic* if it contains cycles of all lengths from 3 up to the order of the graph.

We adopt the following terminology and notations:

- a graph is an S^h*-graph* if it is Hamiltonian and self-centered with $r = 2$,
- a graph is an S^p*-graph* if it is pancyclic and self-centered with $r = 2$,
- $f^h(n)$, for $n \geq 4$, is the minimum size of an S^h-graph of order n,
- $f^p(n)$, for $n \geq 5$, is the minimum size of an S^p-graph of order n,
- $F^h(n)$, for $n \geq 4$, is the number of mutually nonisomorphic S^h-graphs with n vertices and $f^h(n)$ edges,
- $F^p(n)$, for $n \geq 5$, is the number of mutually nonisomorphic S^p-graphs with n vertices and $f^p(n)$ edges.

An overview of the main results of the paper for small values of n is given in Table 1 (see Theorems 3 and 5 below). Table 2 completes the previous one, by listing nonisomorphic S^h-graphs and S^p-graphs of minimum size (scattered in the proof of Theorem 5) for $n \leq 12$.

In Theorem 8 we show that the sets of all possible sizes m of S^h-graphs and S^p-graphs are intervals of positive integers. Namely, for S^h-graphs this interval is

$$f^h(n) \leq m \leq \left\lfloor \frac{n^2 - 2n}{2} \right\rfloor, \quad n \geq 4, \tag{1.1}$$

and for S^p-graphs

[1] Nevertheless, the inspiration for the present paper has not come from [2]. In fact, the authors have received the impetus to explore the described issue in connection with investigation of eccentric sequences (see e.g. [5, 6, 7]).

$$f^p(n) \le m \le \left\lfloor \frac{n^2-2n}{2} \right\rfloor, \quad n \ge 5. \tag{1.2}$$

We know the exact values of the left ends of these intervals only for small values of n, see Table 1. In general, by Theorem 6 and Theorem 3(i) it holds:

$$f^h(n) \le f^p(n) \le \left\lceil \frac{7n}{3} \right\rceil - 6, \quad n \ge 5.$$

Notice that for small values of n considered in Table 1, only for $n = 4$ and 6 the upper bound $\lceil \frac{7n}{3} \rceil - 6$ is best possible for $f^h(n)$ and only for $n = 5$ and 6 it is best possible for $f^p(n)$. However, we conjecture that for all sufficiently large n we have $f^h(n) = f^p(n) = \lceil \frac{7n}{3} \rceil - 6$, see Conjecture 14.

n	$f^h(n)$	$f^p(n)$	$F^h(n)$	$F^p(n)$
4	4	–	1	0
5	5	6	1	1
6	8	8	3	3
7	10	10	4	3
8	12	12	6	5
9	14	14	4	3
10	16	16	1	1
11	18	19	1	≥ 8
12	21	21	≥ 2	?

Table 1. Results for small values of n.

We have already mentioned that Buckley [2] has found all possible sizes of self-centered graphs for given n and r. However, he overlooked that for $r = 2$ his method did not work. In fact, he erroneously obtained that in this case the maximum size of self-centered graphs of order n is $(n^2-3n+4)/2$. For every $n \ge 6$ this number is strictly less than the maximum sizes $\lfloor (n^2-2n)/2 \rfloor$ of S^h-graphs and S^p-graphs of order n from (1.1) and (1.2). It is also worth mentioning that the maximum sizes of S^h-graphs and S^p-graphs are the same as the maximum size of all self-centered graphs with radius $r = 2$ (see Remark 9).

The paper is organized as follows. Section 2 contains just two lemmas. In Section 3 we prove Theorems 3 and 5 covering the results displayed in Tables 1 and 2. In Section 4 we prove Theorem 6 giving an upper bound for $f^h(n)$ and $f^p(n)$, and Theorem 8 showing that the sets of all possible sizes of S^h-graphs and S^p-graphs are intervals, both with the right end-point $\lfloor (n^2-2n)/2 \rfloor$. Finally, in Section 5 we present several open problems, including the already mentioned conjecture.

n	S^h-graphs of order n and of minimum size $f^h(n)$		(graphs with size $f^p(n) > f^h(n)$)
	(non-pancyclic graphs)	S^p-graphs of order n and of minimum size $f^p(n)$	
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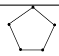
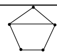
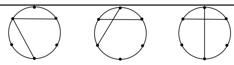
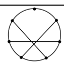
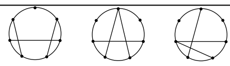
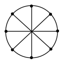
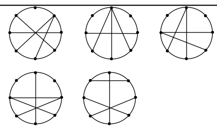
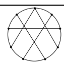
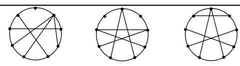
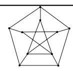
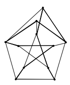
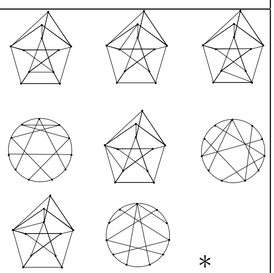
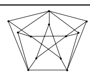
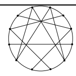
5			
6			
7			
8			
9			
10			
11			 $*$
12	 $*$	 $*$	

$*$ - and possible others

Table 2. S^h-graphs of order n and of minimum size $f^h(n)$, and S^p-graphs of order n and of minimum size $f^p(n)$

2. Preliminaries

When we study how a graph G satisfying some assumptions looks, and we discover that G cannot contain a graph H as a subgraph, we often say that H is a *forbidden subgraph* for G or just that H is *forbidden* for G. If G necessarily contains H then we also say that H is a *forced subgraph* for G or just that H is *forced* for G.

We show that an S^h-graph with at least 7 vertices may contain at most two vertices of degree 2 (and obviously their distance is at most 2). For an S^h-graph G with $\delta(G) = 2$ we give lower bounds for the number of its edges. These bounds are interesting only for small values of n (see Theorems 3, 5).

Lemma 1. *Let G be an S^h-graph. If G has at least 7 vertices, then G has at most two vertices of degree 2.*

Proof. Let C be a Hamiltonian cycle in G and let u, v be vertices of degree 2. Since $d_G(u,v) \leq 2$, we have $d_C(u,v) = d_G(u,v)$. Now it is easy to see that the statement is true. ∎

Lemma 2. *Let G be an S^h-graph with n vertices and $\delta(G) = 2$.*

(i) *If G contains exactly one vertex of degree 2, then*
$|E(G)| \geq 2n - 4$ *for* $n \in \{6, 7, 8, 9\}$,
$|E(G)| \geq 2n - 3$ *for* $n \geq 10$.

(ii) *If G contains two vertices of degree 2 and their distance is 2, then*
$|E(G)| \geq 2n - 4$ *for* $n \in \{7, 8, 9\}$,
$|E(G)| \geq 2n - 3$ *for* $n \geq 10$.

(iii) *If G contains two adjacent vertices of degree 2, then*
$|E(G)| \geq 3n - 10$ *for* $n \geq 6$.

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $v_1 v_2 \cdots v_n v_1$ be a Hamiltonian cycle in G.

(i) Let v_1 be the unique vertex of degree 2 in G and let $n \geq 6$. Then for any vertex v_i, $4 \leq i \leq n-2$, G contains at least one of the edges $v_i v_2$, $v_i v_n$. Since v_3 and v_{n-1} are vertices of degree at least 3, G contains at least one other edge. It follows $|E(G)| \geq n + (n-5) + 1 = 2n - 4$. Moreover, if G contains exactly $2n-4$ edges, then G necessarily contains also the edge $v_3 v_{n-1}$ and $\deg_G(v_i) = 3$, $3 \leq i \leq n-1$. Obviously, if $n \geq 10$, then either $d_G(v_6, v_3) > 2$ (if $v_6 v_n \in E(G)$) or $d_G(v_6, v_{n-1}) > 2$ (if $v_6 v_2 \in E(G)$). It follows $|E(G)| \geq 2n - 3$ for $n \geq 10$.

(ii) Let v_1, v_3 be the vertices of degree 2 in G. Since all other vertices have degree at least 3 (see Lemma 1), for $n = 7$ we trivially get $|E(G)| \geq 10$. Now let $n \geq 8$. Since $e_G(v_1) = e_G(v_3) = 2$, each of the vertices $v_6, v_7, \ldots, v_{n-2}$ is adjacent to at least one of the vertices v_2, v_4, v_n. Moreover, each of the vertices v_4, v_5 is clearly adjacent to at least one of the vertices v_2, v_n and each of the vertices v_n, v_{n-1} is adjacent to at least one of the vertices v_2, v_4. Hence $|E(G)| \geq n+(n-7)+3 = 2n-4$. The equality may occur only if G contains, except the edges of the Hamiltonian cycle, only the following edges: v_2v_i, $6 \leq i \leq n-2$ (if $n \geq 8$), v_4v_n, exactly one of the edges v_5v_2, v_5v_n and exactly one of the edges $v_{n-1}v_2$, $v_{n-1}v_4$. However, if $n \geq 10$, then $d_G(v_4, v_7) > 2$, a contradiction. It follows $|E(G)| \geq 2n-3$ for $n \geq 10$.

(iii) Let the vertices v_1 and v_2 have degree 2 in G. Obviously, $v_{n-1}v_3, v_4v_n \in E(G)$ and if $n > 6$, then $v_iv_3, v_iv_n \in E(G)$ for $5 \leq i \leq n-2$. We get $|E(G)| \geq n+2+2(n-6) = 3n-10$. ∎

3. Exact Values of f^h, f^p, F^h and F^p for Small Numbers of Vertices

This section deals with S^h-graphs and S^p-graphs of order at most 12. Note that the minimum order of an S^h-graph or an S^p-graph is 4 or 5, respectively.

Theorem 3. *The values $f^h(n)$, $4 \leq n \leq 12$, and $f^p(n)$, $5 \leq n \leq 12$, are the following:*

(i) $f^h(4) = 4$, $f^h(5) = 5$, $f^p(5) = 6$,
(ii) $f^h(n) = f^p(n) = 2n-4$ *for* $6 \leq n \leq 10$,
(iii) $f^h(11) = 18$, $f^p(11) = 19$,
(iv) $f^h(12) = f^p(12) = 21$.

Proof. (i) The assertions are obvious (see Figure 3.6).

(ii) When $n = 6$ adding a new edge to C_6 does not give an S^h-graph, but two new edges are enough, see Figure 3.7(a). Hence, we get $f^h(6) = 8$ and $f^p(6) = 8$.

Now let $n \in \{7, 8, 9\}$. A graph of order n and size less than $2n-4$ has to contain a vertex of degree 2. By Lemmas 1 and 2, an S^h-graph with n vertices and with a vertex of degree 2 has to contain at least $2n-4$ edges. Considering the graphs in Figures 3.8(a), 3.10(a), 3.17(a) we get $f^h(n) = f^p(n) = 2n-4$.

Finally, let $n = 10$. Let G be an S^h-graph with $|V(G)| = 10$ and $|E(G)| \leq 15$. According to Lemmas 1 and 2, G contains no vertex of degree 2 and we get $2|E(G)| \geq 10 \cdot 3 = 30$. It follows that the degree of each vertex of G is 3 and $|E(G)| = 15$. It is easy to check that the cycles C_3 and C_4 are forbidden subgraphs for G, since otherwise the eccentricity, in G, of each vertex of C_3 or C_4 would be greater than 2. Since the degree of any vertex of G is 3, G contains $K_{1,3}$ and it follows that the graph in Figure 3.1 is forced for G ($e_G(u) = 2$). Hence, since C_3 and C_4 are forbidden subgraphs for G, one can easily check that G is isomorphic to the Petersen graph. However, the Petersen graph is not Hamiltonian, thus $f^h(10) \geq 16$. Considering the pancyclic graph in Figure 3.25(b) we get $f^h(10) = f^p(10) = 16$.

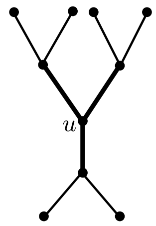


Figure 3.1.

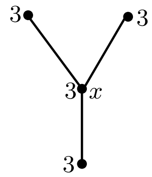


Figure 3.2.

(iii) Let G be an S^h-graph with 11 vertices and at most 18 edges. The graph in Figure 3.2, where the number at each vertex represents its degree in G, is obviously forbidden for G, since otherwise $e_G(x) > 2$.

By Lemmas 1 and 2 ($2 \cdot 11 - 3 = 19$), G contains no vertex of degree 2. Hence $|E(G)| \geq 17$. Let $|E(G)| = 17$. Then G has exactly one vertex of degree 4 and ten vertices of degree 3. So G contains the graph in Figure 3.2, but this graph is forbidden for G, a contradiction. Considering the graph in Figure 3.28 (this graph is not pancyclic), we get $f^h(11) = 18$ and $f^p(11) \geq 18$.

We claim that in fact $f^p(11) > 18$.

Suppose, on the contrary, that there is an S^p-graph G with $|V(G)| = 11$ and $|E(G)| = 18$.

Since the graph in Figure 3.2 (the number at each vertex represents its degree in G) is a forbidden subgraph for G and $d(G) = 2$, it is easy to verify that G has exactly three vertices of degree 4 and eight vertices of degree 3.

The graph in Figure 3.3(a) is forbidden for G, otherwise G has to contain at least two vertices of degree 3 which are not adjacent to a vertex of degree 4. Since the graph in Figure 3.2 is forbidden for G, we have a contradiction.

Obviously, the graphs in Figures 3.3(b), (c) are also forbidden for G, otherwise it would be $e_G(x) > 2$. Since G contains C_3 and the graphs in Figures 3.3(a), (b), (c) are forbidden for G, the graph in Figure 3.4(a) is forced for G. It follows (since $e_G(u) = 2$), the graph in Figure 3.4(b) is forced for G, too.

Since the graph in Figure 3.3(b) is forbidden for G and the graph in Figure 3.4(b) is forced for G, G has to contain the graph in Figure 3.3(d). It is easy to see that the graph in Figure 3.3(d) is forbidden for G, a contradiction.

Once we know that $f^p(11) > 18$, any of the graphs in Figure 3.29 gives $f^p(11) = 19$.

(iv) Let G be an S^h-graph of order 12 and $|E(G)| \leq 20$. By Lemmas 1, 2, $\delta(G) \geq 3$ and it is easy to check that G contains at least eight vertices of degree 3 ($2|E(G)| \leq 2 \cdot 20 = 8 \times 3 + 4 \times 4$).

If G contains at least ten vertices of degree 3, then G has to contain the graph in Figure 3.2 (note that $d(G) = 2$). This graph is forbidden for G, a contradiction.

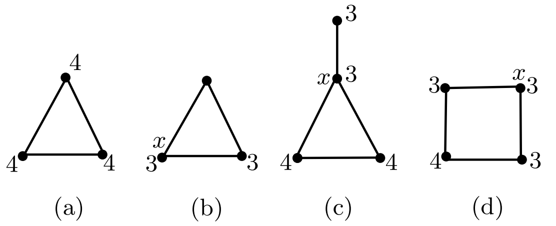


Figure 3.3.

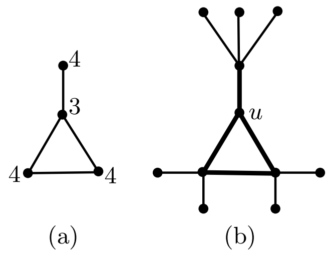


Figure 3.4.

If G contains exactly nine vertices of degree 3, then $2|E(G)| \geq 9 \cdot 3 + 2 \cdot 4 + 1 \cdot 5 = 40$ and we get $|E(G)| = 20$. If G contains neither the graph in Figure 3.2 nor the graph in Figure 3.5(a), then each of nine vertices of degree 3 has to be adjacent to at least two vertices of degree greather than 3 and this is impossible ($2|E(G)| = 40 = 9 \cdot 3 + 2 \cdot 4 + 1 \cdot 5$ and $9 \cdot 2 > 2 \cdot 4 + 5$). It follows that G contains at least one of the graphs in Figures 3.2, 3.5(a). These graphs are forbidden for G, a contradiction.

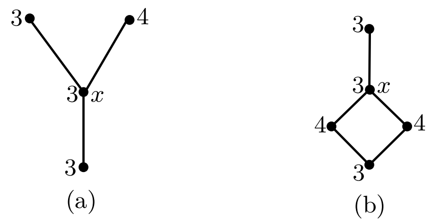


Figure 3.5.

Finally it remains to consider the case when G contains eight vertices of degree 3 and four vertices of degree 4. Since the graphs in Figures 3.2, 3.5(a) are forbidden subgraphs for G, every vertex of degree 3 has to be adjacent to at least two vertices of degree 4. It is only possible when every vertex of degree 3 is adjacent exactly to two vertices of degree 4. Since G contains eight vertices of degree 3 and $\binom{4}{2} = 6$, G has to contain the graph in Figure 3.5(b). This graph is forbidden for G, a contradiction.

Finally, considering the second graph in Figure 3.30, we get $f^h(12) = f^p(12) = 21$. ■

Remark 4. The fact that the Petersen graph is the only self-centered graph with 10 vertices and radius 2 of minimum size and not containing a vertex of degree 2 has been known for a long time (see [1, 8]).

Theorem 5. *For the values $F^h(n)$, $4 \leq n \leq 12$, and $F^p(n)$, $5 \leq n \leq 12$, we have the following.*

(i) $F^h(4) = 1,\ F^h(5) = F^p(5) = 1,\ F^h(6) = F^p(6) = 3,$

(ii) $F^h(7) = F^h(9) = 4,\ F^h(8) = 6,$
$F^p(7) = F^p(9) = 3,\ F^p(8) = 5,$

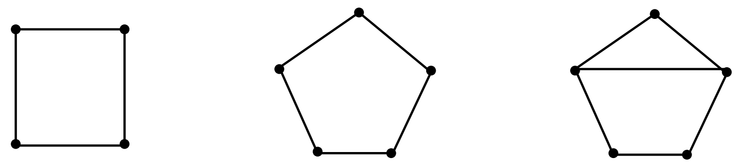
(iii) $F^h(10) = F^p(10) = 1,$

(iv) $F^h(11) = 1,\ F^p(11) \geq 8,$

(v) $F^h(12) \geq 2,\ F^p(12) \geq 1.$

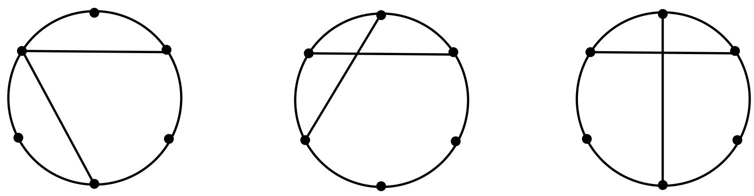
Proof. We will use the values $f^h(n)$ and $f^p(n)$ from Theorem 3.

(i) The assertions for $n \in \{4, 5\}$ are obvious, see Figure 3.6. If $n = 6$, then $f^h(6) = f^p(6) = 8$ and it is easy to verify that $F^h(6) = F^p(6) = 3$, see Figure 3.7.



(a) (b) (c)

Figure 3.6.



(a) (b) (c)

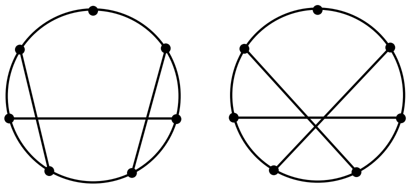
Figure 3.7.

In what follows we suppose that G is an S^h-graph with n vertices and $f^h(n)$ edges. Obviously, the degree of each vertex of G is at least 2.

(ii) ($n = 7$) We have $f^h(7) = 10$ and so $|E(G)| = 10$. Hence G contains at least one vertex of degree 2. By Lemmas 1, 2, it is sufficient to consider two cases.

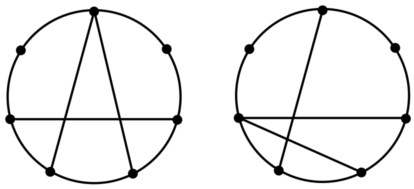
If G contains exactly one vertex of degree 2, then it is easy to see that G is isomorphic either to the graph in Figure 3.8(a) or to the graph in Figure 3.8(b). Only the first of them is pancyclic.

If G contains exactly two vertices of degree 2 and their distance is 2, then it is easy to check that G is isomorphic either to the graph in Figure 3.9(a) or to the graph in Figure 3.9(b). Both are pancyclic.



(a) (b)

Figure 3.8.



(a) (b)

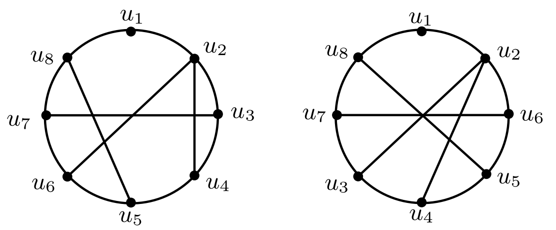
Figure 3.9.

We get $F^h(7) = 4$ and $F^p(7) = 3$.

(ii) ($n = 8$) We have $f^h(8) = 12$ and so $|E(G)| = 12$. According to Lemmas 1, 2, it is sufficient to consider three cases.

If G contains exactly one vertex of degree 2, then it is easy to verify that we get two graphs depicted in Figure 3.10(a), (b). These graphs are pancyclic, but they are isomorphic.

If G contains exactly two vertices of degree 2 and their distance is 2 then, obviously, the graph in Figure 3.11 is a subgraph of G with $\deg_G(v_1) = \deg_G(v_3) = 2$ (see the proof of Lemma 2(ii)). Now it is easy to check that G is isomorphic to one of the graphs in Figure 3.12. They all are pancyclic.



(a) (b)

Figure 3.10.

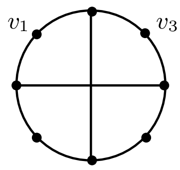
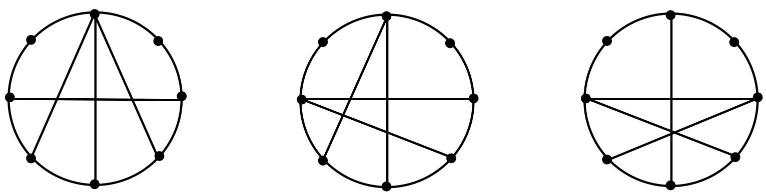


Figure 3.11.

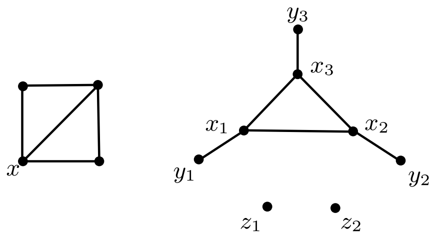


(a) (b) (c)

Figure 3.12.

Finally it remains to consider the case when $\delta(G) > 2$. Since $|E(G)| = 12$, the degree of every vertex of G is 3. We distinguish two subcases according to whether G contains a cycle C_3 or not.

First suppose that G contains C_3. The graph in Figure 3.13(a) is forbidden for G because otherwise it would be $e_G(x) > 2$. Using this, one can see that the graph in Figure 3.13(b) is forced for G. Since $e_G(x_i) = 2$ for $i \in \{1, 2, 3\}$, G contains all edges $z_i y_j$, $i \in \{1, 2\}$, $j \in \{1, 2, 3\}$. Hence G is isomorphic to the graph in Figure 3.14. This graph is pancyclic.



(a) (b)

Figure 3.13.

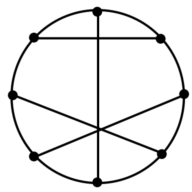


Figure 3.14.

Now suppose that G does not contain C_3. Then $K_{2,3}$ is a forbidden subgraph for G. Since $K_{1,3}$ is a subgraph of G and $|V(G)| < 10$, C_4 is forced for G. Therefore, the graph in Figure 3.15 is also forced for G. Since $d_G(y_1, x_3) \le 2$, we have $y_1 y_3 \in E(G)$. Analogously, $y_2 y_4 \in E(G)$. We get that G is isomorphic to the graph in Figure 3.16 and this graph is not pancyclic.

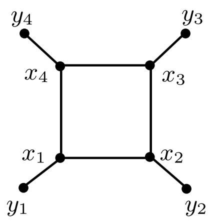


Figure 3.15.

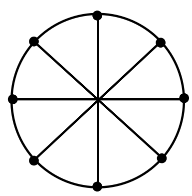


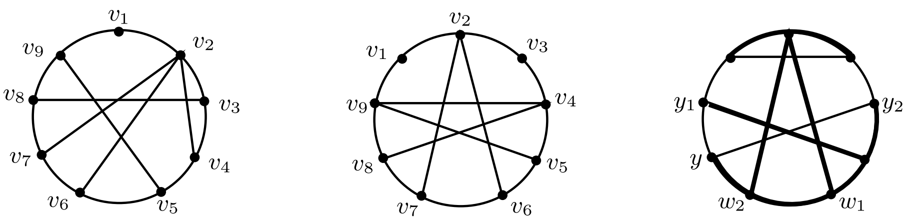
Figure 3.16.

We have thus proved that $F^h(8) = 6$ and $F^p(8) = 5$.

(ii) ($n = 9$) We have $|E(G)| = 14$. Similarly as for $n = 8$, there are three cases to consider.

If G contains exactly one vertex of degree 2, call it v_1, let $v_1 v_2 \cdots v_9 v_1$ be a Hamiltonian cycle in G. By the proof of Lemma 2(i), G contains the edge $v_3 v_8$. Without loss of generality we may assume that $\deg_G(v_9) \le \deg_G(v_2)$ and then $\deg_G(v_9) \in \{3, 4\}$. We show that the assumption $\deg_G(v_9) = 4$ leads to a contradiction. In fact, since $d_G(v_4, v_7) \le 2$, either $v_4 v_9, v_9 v_7 \in E(G)$ (and then

$v_2v_5, v_2v_6 \in E(G)$) or $v_4v_2, v_2v_7 \in E(G)$ (and then $v_9v_5, v_9v_6 \in E(G)$). In the former case $d_G(v_5, v_8) > 2$ and in the latter case $d_G(v_3, v_6) > 2$, a contradiction. Thus we have proved that necessarily $\deg_G(v_9) = 3$. Then, since $d_G(v_9, v_i) \le 2$ for $i \in \{4, 5, 6\}$, we get that $v_9v_5 \in E(G)$. Obviously, G is isomorphic to the graph in Figure 3.17(a) and this graph is pancyclic.



(a) (b) (c)

Figure 3.17.

If G contains exactly two vertices of degree 2 and their distance is 2, let $v_1v_2 \cdots v_9v_1$ be a Hamiltonian cycle of G with $\deg_G(v_1) = \deg_G(v_3) = 2$. By the proof of Lemma 2(ii), G contains the edges v_2v_6, v_2v_7 and v_4v_9. Since $|E(G)| = 14$, G has to contain the edges v_5v_9 ($d_G(v_5, v_1) = d_G(v_5, v_8) = 2$) and v_8v_4 ($d_G(v_8, v_3) = d_G(v_8, v_5) = 2$). We get the pancyclic graph in Figure 3.17(b).

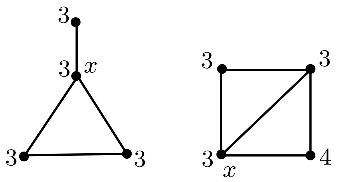
It remains to consider the case when $\delta(G) > 2$. Clearly, the degree of one vertex of G is 4 and the degrees of all other vertices are 3. We distinguish two subcases.

If G contains C_3 then the graphs in Figure 3.18 are forbidden for G, otherwise it would be $e_G(x) > 2$. Hence, the degree of one vertex of C_3 is 4. Therefore, since $e_G(w_1) = 2$, the graph in Figure 3.19(a) is a subgraph of G. Since $e_G(w_2) = 2$, we get $yy_1, yy_2 \in E(G)$. Now it is easy to check that G is isomorphic to the graph in Figure 3.17(c). This graph is pancyclic.

Now suppose G does not contain C_3. The graph in Figure 3.2 is obviously a subgraph of G. Hence, since $e_G(u) = 2$, G contains the graph in Figure 3.19(b). As $e_G(y) = 2$ and G does not contain C_3, we get $yy_1, yy_2 \in E(G)$. Now it is easy to verify that G is isomorphic to the graph in Figure 3.20. This graph is not pancyclic.

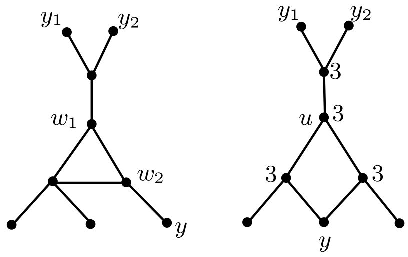
We have finished the proof that $F^h(9) = 4$ and $F^p(9) = 3$.

(iii) ($n = 10$) We have $f^h(10) = 16$ and so $|E(G)| = 16$. According to Lemmas 1, 2, we get that $\delta(G) > 2$. Clearly, G contains at least eight vertices of degree 3. It is easy to see that the graph in Figure 3.2 is forced for G. Then, since $e_G(u) = 2$, also the graph in Figure 3.21 is forced for G. The graph in Figure 3.22(a) is forbidden for G, otherwise $e_G(x) > 2$. It follows that G does not contain a vertex of degree 5. We conclude that G contains eight vertices of



(a) (b)

Figure 3.18.



(a) (b)

Figure 3.19.

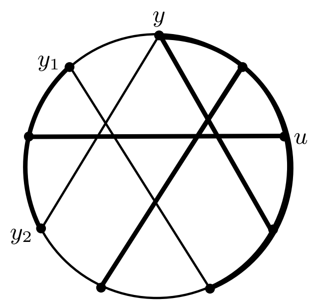


Figure 3.20.

degree 3 and two vertices of degree 4. Since the graph in Figure 3.21 is forced for G and the graphs in Figure 3.22(a), (b) are forbidden for G, G contains C_3 (it is interesting to notice that the graph in Figure 3.22(c) contains the graphs in Figures 3.21, 3.22(b) and it contains neither the graph in Figure 3.22(a) nor C_3). The graphs in Figure 3.23(a), (b) are forbidden for G, otherwise $e_G(x) > 2$. Hence, the graph in Figure 3.23(c) is also forbidden for G. According to the above considerations, the cycle C_3 contains two vertices of degree 4. It follows, since $e_G(w) = 2$, that the graph in Figure 3.24 is forced for G. Taking into account that the graphs in Figures 3.22(a), 3.23(c) are forbidden for G, we get that G is isomorphic to the graph in Figure 3.25(a) which in turn is isomorphic to the graph in Figure 3.25(b). It is easy to check that G is pancyclic.

The proof that $F^h(10) = F^p(10) = 1$ is finished.

(iv) ($n = 11$) By Theorem 3, $f^h(11) = 18$ and by Lemmas 1, 2, we have $\delta(G) > 2$. Let G be an S^h-graph of order 11 and size 18. If G contains at least nine vertices of degree 3, then it is easy to see that at least one of them is adjacent only to vertices of degree 3 (note that $d(G) = 2$). Thus G contains the graph in Figure 3.2. Since the graph in Figure 3.2 is forbidden for G, we have a contradiction. Hence G contains eight vertices of degree 3 and three vertices of degree 4 ($2|E(G)| = 36 = 8 \cdot 3 + 3 \cdot 4$). Since G contains exactly three vertices of

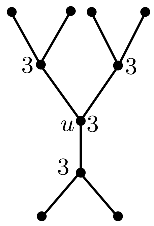
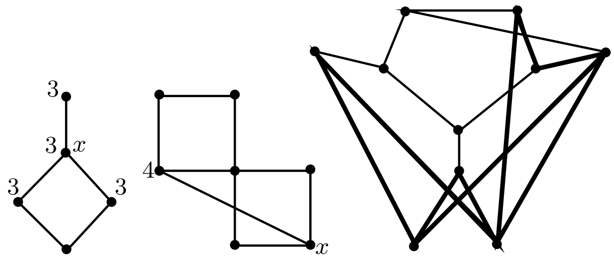
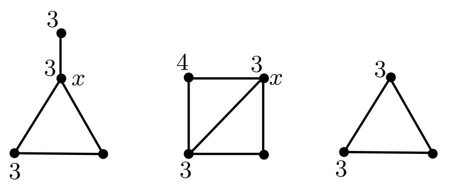


Figure 3.21.



(a) (b) (c)

Figure 3.22.



(a) (b) (c)

Figure 3.23.

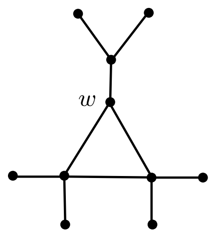
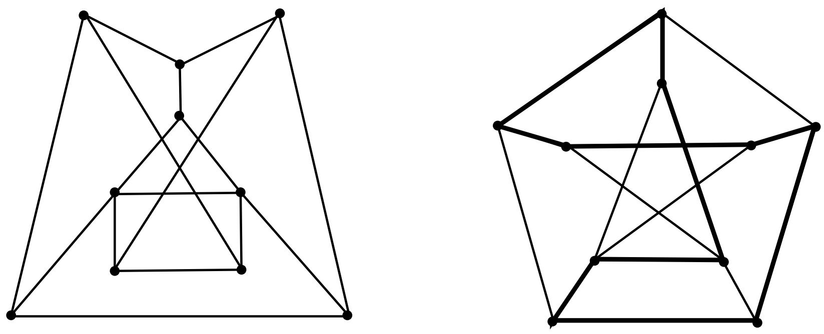


Figure 3.24.



(a) (b)

Figure 3.25.

degree 4, it is easy to verify that the graph in Figure 3.26(a) is forbidden for G ($e_G(x) > 2$ or $e_G(x') > 2$). Obviously, the graph in Figure 3.26(b) is forbidden for G.

Now we show that the graph in Figure 3.26(c) is forbidden for G, too. If G contains this graph, then G would contain the graph in Figure 3.26(d) ($e_G(u) = 2$ and the graph in Figure 3.26(a) is forbidden for G). If G contains the graph in Figure 3.26(d), then G would contain the graph in Figure 3.26(e) ($e_G(u) = 2$

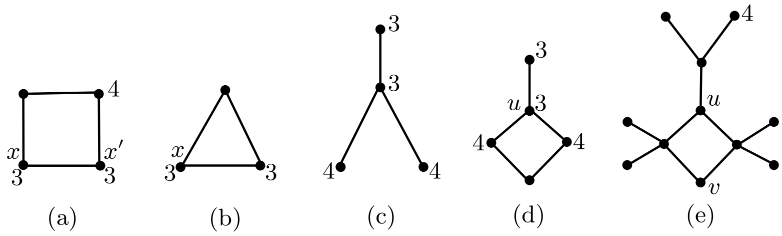


Figure 3.26.

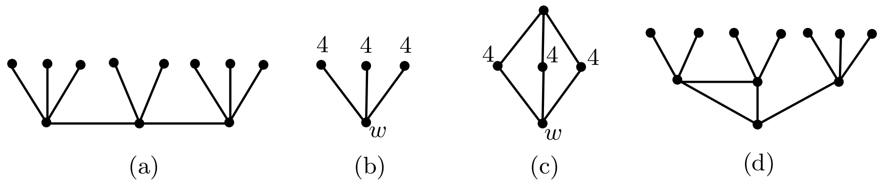


Figure 3.27.

and the graph in Figure 3.2 is forbidden for G). If G contains the graph in Figure 3.26(e), we have $e_G(v) > 2$ (since the graph in Figure 3.26(b) is forbidden for G), a contradiction.

If every vertex of degree 3 in G is adjacent to exactly one vertex of degree 4, G would contain the graph in Figure 3.27(a). Since the graphs in Figure 3.26(b), a) are forbidden for G, we get a contradiction.

According to the above considerations, the graph in Figure 3.27(b) is forced for G. Moreover, the graph in Figure 3.27(c) is forced for G, too. In fact, if G does not contain the graph in Figure 3.27(c), G would contain the graph in Figure 3.27(d). This is impossible, since the graphs in Figure 3.26(b), (a) are forbidden for G. Obviously, $G - w$ is a self-centered graph with radius $r = 2$, so it is isomorphic to the Petersen graph (see proof (ii) of Theorem 3). Hence, G is isomorphic to the graph in Figure 3.28 and this graph is not pancyclic.

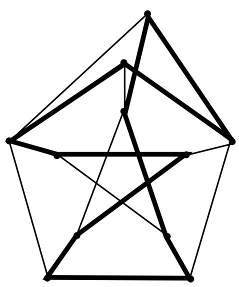


Figure 3.28.

We have finished the proof that $F^h(11) = 1$.

According to the Figure 3.29 we have $F^p(11) \geq 8$.

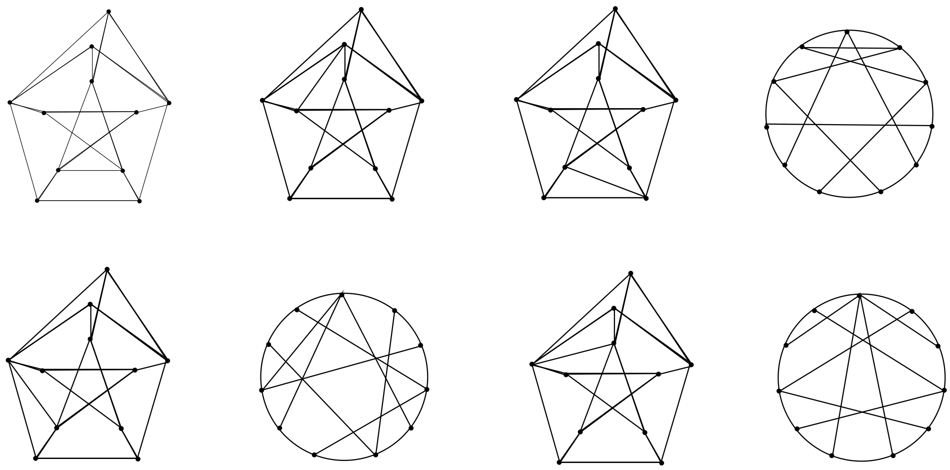


Figure 3.29.

(v) According to the Figure 3.30 we have $F^h(12) \geq 2$ and $F^p(12) \geq 1$.

Figure 3.30. ■

4. Estimates for f^h and f^p. Sizes of S^h and S^p-Graphs

In this section we find the maximum size of S^h-graphs and S^p-graphs of order n. For the minimum size of such graphs we find an upper bound. We conjecture that this upper bound is in fact the exact value of $f^h(n)$ and $f^p(n)$ for almost all values of n.

Theorem 6. *For $n \geq 6$ we have*

$$f^h(n) \leq f^p(n) \leq \left\lceil \frac{7n}{3} \right\rceil - 6.$$

Proof. It is obvious that $f^h(n) \leq f^p(n)$. To prove the other inequality, consider a graph G such that $V(G) = \{v_1, v_2, \dots, v_n\}$, $v_1 v_2 \cdots v_n v_1$ is a Hamiltonian cycle of G and, except for the n edges from this cycle, G contains the following ones (the cases $n = 12$ and $n = 14$ can be seen in Figure 4.1):

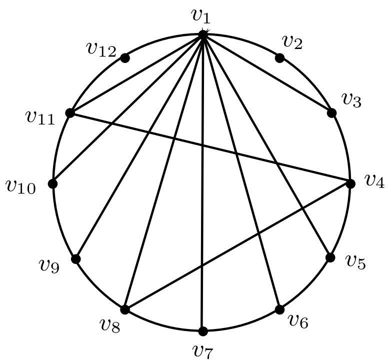
- the edge $v_1 v_3$ and all edges $v_1 v_i$ with $5 \leq i \leq n-1$,
- the edge $v_4 v_n$ if $3 \nmid n$,
- the edges $v_4 v_{3i+5}$ with $1 \leq i \leq \lfloor \frac{n-6}{3} \rfloor$, if $n \geq 9$.

Since clearly no edge of G is listed twice here, we can easily count them. In fact, if $3|n$, then we obtain

$$|E(G)| = n + 1 + (n-5) + \left\lfloor \frac{n-6}{3} \right\rfloor = \frac{7n}{3} - 6 = \left\lceil \frac{7n}{3} \right\rceil - 6,$$

and if $3 \nmid n$, then again

$$|E(G)| = n+1+(n-5)+1+\left\lfloor \frac{n-6}{3} \right\rfloor = 2n-5+\left\lfloor \frac{n}{3} \right\rfloor = \left\lfloor \frac{7n}{3} \right\rfloor - 5 = \left\lceil \frac{7n}{3} \right\rceil - 6.$$



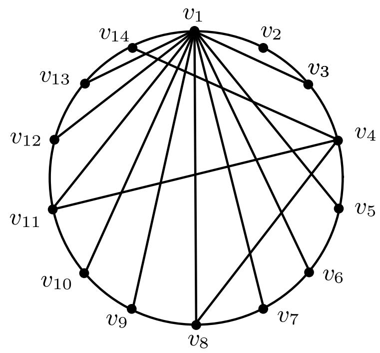


Figure 4.1.

Since G is obviously an S^p-graph, the proof is finished. ■

Remark 7. Comparing the exact values of $f^h(n)$ and $f^p(n)$ for small n from Theorem 3 or Table 1, with the upper bound from the previous theorem, we get

$$\begin{aligned} f^h(6) &= f^p(6) = \left\lceil \frac{7 \cdot 6}{3} \right\rceil - 6, \\ f^h(n) &= f^p(n) < \left\lceil \frac{7n}{3} \right\rceil - 6 \quad \text{for} \quad n \in \{7, 8, 9, 10, 12\}, \\ f^h(11) &= 18 < f^p(11) = 19 < \left\lceil \frac{7 \cdot 11}{3} \right\rceil - 6. \end{aligned}$$

Theorem 8. (a) *Let $n \geq 4$. Then there exists an S^h-graph of order n and size m if and only if*

$$f^h(n) \leq m \leq \left\lfloor \frac{n^2 - 2n}{2} \right\rfloor .$$

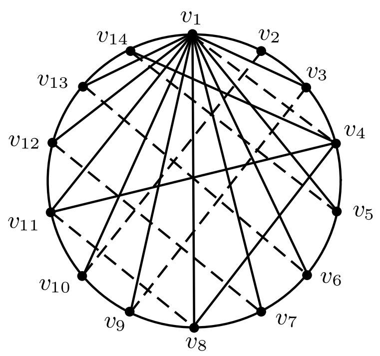
(b) *Let $n \geq 5$. Then there exists an S^p-graph of order n and size m if and only if*

$$f^p(n) \leq m \leq \left\lfloor \frac{n^2 - 2n}{2} \right\rfloor .$$

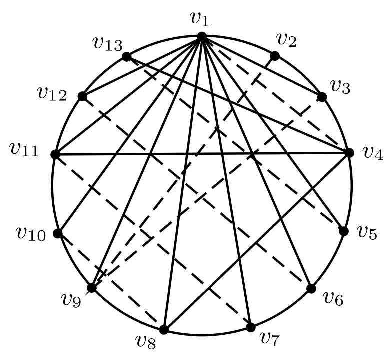
Proof. The assertions are obvious for $n \in \{4, 5\}$. Assume that $n \geq 6$. Let G^* be the graph described in the proof of Theorem 6. Recall that G^* is an S^p-graph with n vertices and $\lceil \frac{7n}{3} \rceil - 6$ edges. Clearly, by adding any new edges to G^* such that the degree of each vertex of G^* is at most $n - 2$, we again get an S^p-graph.

First we are going to construct an S^p-graph of order n and size m with $\lceil \frac{7n}{3} \rceil - 6 \leq m \leq \left\lfloor \frac{n^2-2n}{2} \right\rfloor$. Let us start with G^* and denote by $\overline{G^*}$ the complement of G^*. It is easy to see that if n is even, then there exists a perfect matching E' in $\overline{G^*}$ (see dashed edges in Figure 4.2(a) added to the graph G^* of order 14; obviously, the edge v_1v_4 must be in E'), and if n is odd, then there exists a perfect matching E'' in the graph $\overline{G^*} - \{v_2, v_3, v_k\}$ with $k = \lceil \frac{n}{2} \rceil + 2$. Let $E' = E'' \cup \{v_2v_k, v_3v_k\}$ (see dashed edges in Figure 4.2(b) added to the graph G^* of order 13). Now let us consider a graph G such that

$$V(G) = V(G^*),\ E(G^*) \subsetneqq E(G),\ |E(G)| = m,\ E(G) \cap E' = \emptyset .$$



(a)



(b)

Figure 4.2.

Then G is an S^p-graph (hence also an S^h-graph) of order n and size m and

$$|E(G^*)| = \left\lceil \frac{7n}{3} \right\rceil - 6 \leq m \leq \binom{n}{2} - \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n^2 - 2n}{2} \right\rfloor .$$

Clearly, this upper bound for m is tight. Evidently, if G has more than $\left\lfloor \frac{n^2-2n}{2} \right\rfloor$ edges, then $r(G) = 1$.

Further, for S^h-graphs (respectively, S^p-graphs) it is sufficient to consider the case $f^h(n) \le \left\lceil \frac{7n}{3} \right\rceil - 7$ (respectively, $f^p(n) \le \left\lceil \frac{7n}{3} \right\rceil - 7$). According to Theorem 3 we assume $n \ge 10$.

We are going to construct an S^h-graph G of order n and size m with $f^h(n) \le m \le \left\lceil \frac{7n}{3} \right\rceil - 7$. By definition, there exists an S^h-graph H^* of order n and size $f^h(n)$. H^* has at most two vertices of degree 2 (see Lemma 1). Hence $|E(H^*)| \ge n + \left\lceil \frac{n-2}{2} \right\rceil$. Let u and v be vertices of the minimum and maximum degrees in H^*, respectively. Obviously, $\deg_G(u) \le 4$, since otherwise we would obtain $5n \le 2\left(\left\lceil \frac{7n}{3} \right\rceil - 8\right)$, a contradiction. The vertex v is the only possible vertex of degree $n-2$ in H^*, otherwise we would obtain $|E(H^*)| \ge n + (n-4) + (n-5) = 3n - 9$, a contradiction. Hence, every vertex in H^* different from v has degree at most $n-3$. Adding to H^* at most $n-6$ new edges incident with the vertex u (and different from uv), we obviously obtain an S^h-graph. Since $n + \left\lceil \frac{n-2}{2} \right\rceil + n - 6 > \left\lceil \frac{7n}{3} \right\rceil - 7$ for $n \ge 10$, there exists an S^h-graph with m edges for each m with $f^h(n) \le m \le \left\lceil \frac{7n}{3} \right\rceil - 7$. The proof for S^h-graphs is finished.

For every m, $f^p(n) \le m \le \left\lceil \frac{7n}{3} \right\rceil - 7$, an S^p-graph with n vertices and m edges can be obtained in an analogous way. It is sufficient to assume that H^* is an S^p-graph of order n and size $f^p(n)$. ■

Remark 9. The upper bound for the size of a self-centered graph of order n with radius $r = 2$ is found in [2] (see [3, 4], too). According to Theorem 8, this upper bound $\frac{n^2-3n+4}{2}$ is incorrect. Obviously, for $n > 5$ we have $\frac{n^2-3n+4}{2} < \left\lfloor \frac{n^2-2n}{2} \right\rfloor$. The correct upper bound for self-centered graphs with radius $r = 2$ is $\left\lfloor \frac{n^2-2n}{2} \right\rfloor$. Clearly, if a graph G of order n has more than $\binom{n}{2} - \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n^2-2n}{2} \right\rfloor$ edges, then $r(G) = 1$.

5. Open Problems

We state several open problems and one conjecture.

By Theorem 3, $f^h(11) = 18$ and $f^p(11) = 19$, and by Theorem 5, $F^h(11) = 1$, $F^p(11) \ge 8$.

Problem 10. *Find the value* $F^p(11)$.

By Theorem 3, $f^h(12) = f^p(12) = 21$, and by Theorem 5, $F^h(12) \ge 2$.

Problem 11. *Find the values* $F^h(12)$ *and* $F^p(12)$.

By Theorem 3, $f^h(n) = f^p(n)$ for $n \in \{6, 7, 8, 9, 10, 12\}$, and $f^p(n) = f^h(n) + 1$ for $n \in \{5, 11\}$.

Problem 12. *Does there exist n such that $f^p(n) - f^h(n) > 1$?*

By Theorem 3, we have

$$f^p(n) = \left\lceil \tfrac{7n}{3} \right\rceil - 6 \text{ for } n \in \{5, 6\},$$
$$f^p(n) = \left(\left\lceil \tfrac{7n}{3} \right\rceil - 6\right) - 1 \text{ for } n \in \{7, 8, 9, 11, 12\},$$
$$f^p(n) = \left(\left\lceil \tfrac{7n}{3} \right\rceil - 6\right) - 2 \text{ for } n = 10.$$

Problem 13. *Is the inequality $(\lceil (7n)/3 \rceil - 6) - f^p(n) \geq 2$ true for some $n \neq 10$?*

By Theorem 3, $f^h(n) \neq f^p(n)$ for $n \in \{5, 11\}$, and by Theorem 5, $F^h(n) \neq F^p(n)$ for $n \in \{7, 8, 9, 11, 12\}$. We conjecture that such cases are exceptional.

Conjecture 14. *If $n \geq 30$, then $f^h(n) = f^p(n) = \lceil (7n)/3 \rceil - 6$ and $F^h(n) = F^p(n)$.*

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