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$\mathcal{P}$-APEX GRAPHS

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Dedicated to the memory
of Professor Horst Sachs (1927 − 2017)

Abstract

Let $\mathcal{P}$ be an arbitrary class of graphs that is closed under taking induced subgraphs and let $\mathcal{C}(\mathcal{P})$ be the family of forbidden subgraphs for $\mathcal{P}$. We investigate the class $\mathcal{P}(k)$ consisting of all the graphs G for which the removal of no more than k vertices results in graphs that belong to $\mathcal{P}$. This approach provides an analogy to apex graphs and apex-outerplanar graphs studied previously. We give a sharp upper bound on the number of vertices of graphs in $\mathcal{C}(\mathcal{P}(1))$ and we give a construction of graphs in $\mathcal{C}(\mathcal{P}(k))$ of relatively large order for $k \geq 2$. This construction implies a lower bound on the maximum order of graphs in $\mathcal{C}(\mathcal{P}(k))$. Especially, we investigate $\mathcal{C}(\mathcal{W}_r(1))$, where $\mathcal{W}_r$ denotes the class of P_r-free graphs. We determine some forbidden subgraphs for the class $\mathcal{W}_r(1)$ with the minimum and maximum number of vertices. Moreover, we give sufficient conditions for graphs belonging to $\mathcal{C}(\mathcal{P}(k))$, where $\mathcal{P}$ is an additive class, and a characterisation of all forests in $\mathcal{C}(\mathcal{P}(k))$. Particularly we deal with $\mathcal{C}(\mathcal{P}(1))$, where $\mathcal{P}$ is a class closed under substitution and obtain a characterisation of all graphs in the corresponding $\mathcal{C}(\mathcal{P}(1))$. In order to obtain desired results we exploit some hypergraph tools and this technique gives a new result in the hypergraph theory.

Keywords: induced hereditary classes of graphs, forbidden subgraphs, hypergraphs, transversal number.

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1. Introduction

We only consider finite and simple graphs and follow [1] for graph-theoretical terminology and notation not defined here. A graph G is an *apex graph* if it contains a vertex w such that $G - w$ is planar. Although apex graphs seem to be close to planar graphs, some of their properties are far from corresponding properties of planar graphs (for example, see [18]).

A result of Robertson and Seymour (see [19]) says that every proper minor-closed class of graphs $\mathcal{P}$ can be characterized by a finite family of *forbidden minors* (minor-minimal graphs not in $\mathcal{P}$). Evidently, the class of apex graphs is minor-closed but the long-standing problem of finding the complete family of forbidden minors for this class is still open.

However, Dziobak in [9] introduced an apex-outerplanar graph that is a conceptual analogue to an apex graph. Namely, a graph G is *apex-outerplanar* if there exists $w \in V(G)$ such that $G - w$ is outerplanar. Moreover, Dziobak provided the complete list of 57 forbidden minors for this class.

Another attempt to extend the concept of an apex graph is presented in [20] where an *l-apex graph* is defined. A graph G is an *l-apex graph* if it can be made planar by removing at most l vertices.

This paper concerns classes of graphs that generalize the aforementioned. Formally, by a *class of graphs* we mean an arbitrary family of non-isomorphic graphs. The empty class of graphs and the class of all graphs are called *trivial.* A class of graphs $\mathcal{P}$ is *induced hereditary* if it is closed with respect to taking induced subgraphs. Such a class $\mathcal{P}$ can be uniquely characterized by the family of *forbidden subgraphs* $\mathcal{C}(\mathcal{P})$ that is defined as a set

$$\{G : \ G \notin \mathcal{P} \text{ and } H \in \mathcal{P} \text{ for each proper induced subgraph } H \text{ of } G\}.$$

By $\mathbf{L}_{\leq}$ we denote the class of all non-trivial induced hereditary classes of graphs. Each class $\mathcal{P} \in \mathbf{L}_{\leq}$ has a non-empty family of forbidden subgraphs, consisting of graphs with at least two vertices. Moreover, $\mathcal{C}(\mathcal{P})$ contains only connected graphs when $\mathcal{P}$ is *additive*, i.e., closed under taking the union of disjoint graphs. By $\mathbf{L}^a_{\leq}$ we denote the family of all non-trivial induced hereditary and additive classes of graphs.

Let $\mathcal{P} \in \mathbf{L}_{\leq}$ and let k be a non-negative integer. A graph G is a $\mathcal{P}(k)$*-apex graph* if there is $W \subseteq V(G)$, $|W| \leq k$ (W is allowed to be the empty set), such that $G - W$ belongs to $\mathcal{P}$. We denote the set of all $\mathcal{P}(k)$-apex graphs by $\mathcal{P}(k)$ for short.

We can see immediately that if k is a non-negative integer and $\mathcal{P} \in \mathbf{L}_{\leq}$, then $\mathcal{P}(k) \in \mathbf{L}_{\leq}$ too. On the other hand, the additivity of $\mathcal{P} \in \mathbf{L}_{\leq}$ implies the additivity of $\mathcal{P}(k)$ if and only if $k = 0$. Indeed, $\mathcal{P}(0) = \mathcal{P}$. Moreover, if $\mathcal{P} \in \mathbf{L}^a_{\leq}$, then $\mathcal{C}(\mathcal{P}) \neq \emptyset$ and assuming that $F \in \mathcal{C}(\mathcal{P})$ we can easily see that the union of

$k+1$ disjoint copies of F is in $\mathcal{C}(\mathcal{P}(k))$. Thus, for $k \geq 1$, it yields the existence of at least one disconnected graph that is forbidden for $\mathcal{P}(k)$. Hence, for $k \geq 1$, the class $\mathcal{P}(k)$ is not additive.

Lewis and Yannakakis in [17] have shown that for any non-trivial induced hereditary class $\mathcal{P}$ containing infinitely many graphs and for a given positive integer k, the decision problem: "does G belong to $\mathcal{P}(k)$?" is NP-complete.

In this paper, we investigate the classes $\mathcal{P}(k)$, in particular we focus on forbidden subgraphs for the classes $\mathcal{P}(k)$ (i.e., we study graphs in $\mathcal{C}(\mathcal{P}(k))$). Additionally, we use hypergraphs as an effective tool in the research on $\mathcal{P}(k)$.

Let $\mathcal{H}$ be a hypergraph with vertex set $V(\mathcal{H})$ and edge set $\mathcal{E}(\mathcal{H})$ and let $W \subseteq V(\mathcal{H})$. The hypergraph $\mathcal{H}[W]$ induced in $\mathcal{H}$ by W has vertex set W and edge set $\{E \in \mathcal{E}(\mathcal{H}) : E \subseteq W\}$. To simplify the notation we write $\mathcal{H} - W$ instead of $\mathcal{H}[V(\mathcal{H}) \setminus W]$ and, moreover, $\mathcal{H} - v$ instead of $\mathcal{H} - \{v\}$ when v is a vertex of $\mathcal{H}$. Analogously, we write $\mathcal{H} - E$ to denote the hypergraph obtained from $\mathcal{H}$ by the deletion of the edge E from $\mathcal{E}(\mathcal{H})$.

By $\mathcal{H}_1 \cup \mathcal{H}_2$ we mean the union of disjoint hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$, i.e., the hypergraph with vertex set $V(\mathcal{H}_1) \cup V(\mathcal{H}_2)$ and edge set $\mathcal{E}(\mathcal{H}_1) \cup \mathcal{E}(\mathcal{H}_2)$. Moreover, notations $2\mathcal{H}_1$, $\mathcal{H}_1 \cup \mathcal{H}_1$, and their generalization are used interchangeably. The symbol $\mathcal{H}_1 \leq \mathcal{H}_2$ denotes that the hypergraph $\mathcal{H}_1$ is isomorphic to a subhypergraph of $\mathcal{H}_2$ induced by some of its vertex subset. Let r be a non-negative integer. A hypergraph $\mathcal{H}$ is *r-uniform* if each edge in $\mathcal{E}(\mathcal{H})$ has exactly r vertices. A set $T \subseteq V(\mathcal{H})$ is called a *transversal* of the hypergraph $\mathcal{H}$ if $T \cap E \neq \emptyset$ for each $E \in \mathcal{E}(\mathcal{H})$. By $\tau(\mathcal{H})$ we denote the cardinality of the minimum transversal of $\mathcal{H}$, i.e.,

$$\tau(\mathcal{H}) = \min\{|T| : T \text{ is a transversal of } \mathcal{H}\}.$$

A hypergraph $\mathcal{H}$ is *τ-vertex critical* if for any $v \in V(\mathcal{H})$ the inequality $\tau(\mathcal{H} - v) \leq \tau(\mathcal{H}) - 1$ holds. If a τ-vertex critical hypergraph $\mathcal{H}$ satisfies $\tau(\mathcal{H}) = l$ for some positive integer l, then we call it *τ-vertex l-critical.*

Recall that each graph is a hypergraph, which allows us to use these notations also for graphs. The symbols K_n, P_n, C_n are used only for graphs and denote the complete graph, the path and the cycle with n vertices, respectively.

This paper is organized as follows. We start with τ-vertex l-critical hypergraphs in Section 2. We prove an upper bound on the order of a τ-vertex 2-critical hypergraph and describe the construction of τ-vertex l-critical hypergraphs with large number of vertices. Next, in Section 3, we prove some results on relations between τ-vertex $(k+1)$-critical hypergraphs and graphs in $\mathcal{C}(\mathcal{P}(k))$ for $\mathcal{P} \in \mathbf{L}_{\leq}$. In Section 4, for $\mathcal{P} \in \mathbf{L}^{a}_{\leq}$ we show some sufficient conditions that have to be satisfied by a graph to be in $\mathcal{C}(\mathcal{P}(k))$ and we characterize all forests in $\mathcal{C}(\mathcal{P}(k))$. Section 5 deals with the class $\mathcal{P}$ of graphs that does not contain P_r as an induced subgraph. We determine some forbidden subgraphs for $\mathcal{P}(1)$ with minimum and maximum order in this case. In Section 6 we characterize all graphs in $\mathcal{C}(\mathcal{P}(1))$, where $\mathcal{P}$ is a class of graphs that is induced hereditary and closed under substitution (for the definition see Section 6).

2. τ-Vertex Critical Hypergraphs

A hypergraph $\mathcal{H}$ is *τ-edge l-critical* if $\tau(\mathcal{H}) = l$ and the deletion of an edge decreases the transversal number of the resulting hypergraph. It is clear that the class of τ-edge l-critical hypergraphs without isolated vertices forms a subclass of the class of τ-vertex l-critical hypergraphs. On the other hand, it is easy to prove that the maximum order of hypergraphs in both classes is the same. In this section we prove that an r-uniform τ-vertex 2-critical hypergraph has at most $\left\lfloor \frac{(r+2)^2}{4} \right\rfloor$ vertices. Our proof is different than Tuza's proof in [21] concerning a corresponding theorem for r-uniform τ-edge 2-critical hypergraphs.

Next, for $l \geq 3$ we give the construction of an r-uniform τ-vertex l-critical hypergraph with a large order. Gyárfás *et al.* [15] proved that each r-uniform τ-vertex l-critical hypergraph has order bounded from above by $\binom{l+r-2}{r-2}l + l^{r-1}$. This bound is probably far from the exact value of the maximum number of vertices in a hypergraph that is r-uniform τ-vertex l-critical. Our construction gives a large lower bound on the maximum order of a hypergraph that is r-uniform τ-vertex l-critical.

Theorem 1. *Let r be an integer, $r \geq 2$, and let $\mathcal{H}$ be a τ-vertex 2-critical hypergraph. If for each $E \in \mathcal{E}(\mathcal{H})$ we have $|E| \leq r$, then*

$$|V(\mathcal{H})| \leq \left\lfloor \frac{(r+2)^2}{4} \right\rfloor.$$

Moreover, the bound is sharp.

Proof. Denote by $\mathcal{H}'$ a hypergraph obtained from $\mathcal{H}$ by the optional deletion of some edges in such a way that $\tau(\mathcal{H}) = \tau(\mathcal{H}') = 2$ and $\tau(\mathcal{H}' - E') \leq 1$ for each edge E' of $\mathcal{H}'$. Let $\mathcal{E}' = \mathcal{E}(\mathcal{H}')$ and assume $\mathcal{E}' = \{E'_1, \ldots, E'_m\}$. Observe that each vertex of $\mathcal{H}'$ is contained in at least one of the edges in $\mathcal{E}(\mathcal{H}')$. Otherwise, if there is $x \in V(\mathcal{H}')$ such that x belongs to no edge in $\mathcal{E}(\mathcal{H}')$, then $\tau(\mathcal{H} - x) = 2$ giving a contradiction to the τ-vertex criticality of $\mathcal{H}$.

Let a bipartite graph B be the incidence graph of the hypergraph $\mathcal{H}'$. Thus $B = (V(\mathcal{H}), \mathcal{E}'; E(B))$, where $vE' \in E(B)$ if and only if $v \in E'$. The previous consideration says that $d_B(v) \geq 1$ for all $v \in V(\mathcal{H})$ and $d_B(E'_i) \leq r$ for all $i \in \{1, \ldots, m\}$. The last condition implies $|E(B)| \leq mr$.

Claim 2. *For every E'_i there is a vertex, say $v_i \in V(\mathcal{H}) \subseteq V(B)$, such that $v_i \notin E'_i$ but $v_i \in E'_j \in \mathcal{E}'$ for all $j \neq i$.*

Proof. Delete a vertex E'_i from the graph B. The graph $B - E'_i$ is an incidence graph of the hypergraph $\mathcal{H}' - E'_i$, so $\tau(\mathcal{H}' - E'_i) = 1$, i.e., there is a vertex, say x, which is adjacent in B to every E'_j, $j \neq i$. Obviously the vertex x is not adjacent to E'_i, otherwise in the hypergraph $\mathcal{H}'$ there would be a 1-element transversal $\{x\}$, which is impossible. Thus x can play the role of v_i from the statement. $\square$

By Claim 2, in the graph B there is a set of m vertices $\{v_1, \ldots, v_m\}$ with $d_B(v_i) = m - 1$, for $i \in \{1, \ldots, m\}$. Since $d_B(v) \geq 1$ for each $v \in V(\mathcal{H})$ we have $m(m-1) + (n-m) \leq |E(B)| \leq mr$, where $n = |V(\mathcal{H})|$. It leads to the inequality $n \leq -m^2 + (r+2)m$. Thus for fixed r, the maximum n is $\left\lfloor \frac{(r+2)^2}{4} \right\rfloor$ and it is achieved at $m = \left\lfloor \frac{r}{2} \right\rfloor + 1$ or at $m = \left\lceil \frac{r}{2} \right\rceil + 1$.

Finally, we prove that the bound is sharp. All the previous arguments imply that the structure of the τ-vertex 2-critical hypergraph with maximum number of vertices must be defined in the following way. For $m = \left\lfloor \frac{r}{2} \right\rfloor + 1$ or $\left\lceil \frac{r}{2} \right\rceil + 1$ let $U = \{1, \ldots, m\}$ and let $A_i = \{a^i_1, \ldots, a^i_{r+1-m}\}$ with $i \in U$. The r-uniform hypergraph $\mathcal{H}$ such that $V(\mathcal{H}) = U \cup \bigcup_{i=1}^m A_i$ and $E(\mathcal{H}) = \{E_1, \ldots, E_m\}$ where $E_i = (U \setminus \{i\}) \cup A_i$ for $i \in \{1, \ldots, m\}$, confirms the sharpness of the inequality given in the assertion. ■

The construction from the proof of Theorem 1 can be generalized in an easy way resulting in the following r-uniform τ-vertex l-critical hypergraph with a large number of vertices.

Construction 1. Let k, r, x be integers, $k \geq 1$, $r \geq 3$ and $r \geq x \geq 1$ and let $U = \{1, \ldots, k, k+1, \ldots, k+x\}$. Next let $m = \binom{k+x}{x}$ and let $\{U_1, \ldots, U_m\}$ be the family of all x-element subsets of U. Additionally, let $A_i = \{a^i_1, \ldots, a^i_{r-x}\}$ with $i \in \{1, \ldots, m\}$ be m pairwise disjoint sets each of which is also disjoint with U.

We define an r-uniform hypergraph $\mathcal{H}^* = \mathcal{H}^*(k, r, x)$ in the following way:
$E(\mathcal{H}^*) = \{E_1, \ldots, E_m\}$, where $E_i = U_i \cup A_i$, $i \in \{1, \ldots, m\}$;
$V(\mathcal{H}^*) = \bigcup_{i=1}^m E_i = U \cup A$, where $A = \bigcup_{i=1}^m A_i$.

Theorem 3. *If k, r, x are integers such that $k \geq 1$, $r \geq 3$ and $r \geq x \geq 1$, then $\mathcal{H}^*(k, r, x)$ is τ-vertex $(k+1)$-critical.*

Proof. Let $\mathcal{H}^*(k, r, x) = \mathcal{H}^*$. We use the notations connected with $\mathcal{H}^*$ given in Construction 1. Observe that an arbitrary $(k+1)$-element subset of U is a transversal of $\mathcal{H}^*$. Thus $\tau(\mathcal{H}^*) \leq k + 1$. Suppose, for a contradiction, that T is a transversal of $\mathcal{H}^*$ and $|T| \leq k$. If $T \subseteq U$, then $U \setminus T$ contains at least one x-element subset U_i and consequently E_i is an edge of $\mathcal{H}^* - T$. Hence T is not a transversal of $\mathcal{H}^*$, a contradiction. Thus $T \setminus U = S \neq \emptyset$. Denote $t = |T \cap U|$ and $s = |S|$. There are at least $\binom{k+x-t}{x}$ edges of $\mathcal{H}^*$ each of which has nonempty intersection with S. It follows $\binom{k+x-t}{x} \leq s$. Recall that $s + t \leq k$. It means $\binom{k+x-t}{x} \leq k - t$, which is impossible for any x satisfying $r \geq x \geq 1$.

To observe the τ-vertex criticality of $\mathcal{H}^*$ it is enough to show that for each $v \in V(\mathcal{H}^*)$ the condition $\tau(\mathcal{H}^* - v) \leq k$ holds. If $v \in U$, then the removal of any k vertices of U, all different from v, results in a hypergraph without edges. If $v \in A_i$ for some $i \in \{1, \ldots, m\}$, then the k-element transversal $U \setminus U_i$ realizes the inequality $\tau(\mathcal{H}^* - v) \leq k$. ■

In the next lemma we find the maximum order of $\mathcal{H}^*(k, r, x)$. This result gives a lover bound on the maximum number of vertices in an r-uniform τ-vertex $(k + 1)$-critical hypergraph.

Given k, r we introduce $n(x) = \binom{k+x}{x}(r - x) + k + x = \binom{k+x}{k}(r - x) + k + x$.

Lemma 4. *If k, r are integers such that $k \geq 1$, $r \geq 3$, then*

$$\max_{1 \leq x \leq r} |V(\mathcal{H}^*(k, r, x))| = \max_{1 \leq x \leq r} n(x) = n\left(\left\lceil \frac{k(r-1)}{k+1} \right\rceil\right).$$

Proof. By Construction 1 we have $\max_{1 \leq x \leq r} |V(\mathcal{H}^*(k, r, x))| = \max_{1 \leq x \leq r} n(x)$. Consider the difference function $D(x) = n(x) - n(x+1) = -1 + \binom{k+x}{k}[(r - x) - \frac{k+x+1}{x+1}((r - x) - 1)] = -1 + \binom{k+x}{k}\frac{(r-x)(-k)+k+x+1}{x+1} = -1 + \frac{(k+x)!}{k!(x+1)!}[(x + 1)(k + 1) - kr] = -1 + \frac{1}{x+1}\prod_{i=1}^{k}(1 + \frac{x}{i})[(x + 1)(k + 1) - kr]$.

Since x, k and r are positive integers, $D(x) \geq 0$ if and only if $(x+1)(k+1) - kr \geq 1$ and therefore the maximum $n(x)$ is reached at the smallest x such that $D(x) \geq 0$, i.e., at $x = \left\lceil \frac{k(r-1)}{k+1} \right\rceil$. ■

3. Graph Approach

In this section we formulate some results on relations between τ-vertex $(k + 1)$-critical hypergraphs and forbidden subgraphs for $\mathcal{P}(k)$. They are preceded by the helpful lemmas.

Lemma 5. *Let k be a non-negative integer and $\mathcal{P} \in \mathbf{L}_{\leq}$. If $F \in \mathcal{C}(\mathcal{P}(k))$, then $F \in \mathcal{P}(k + 1) \setminus \mathcal{P}(k)$.*

Proof. By the definition of $\mathcal{C}(\mathcal{P}(k))$ it follows that $F \notin \mathcal{P}(k)$. Moreover, for an arbitrary $v \in V(F)$ we have $F - v \in \mathcal{P}(k)$. It means that there exists a set W, contained in $V(F - v)$, such that $|W| \leq k$ and $(F - v) - W \in \mathcal{P}$. Because $|W \cup \{v\}| \leq k + 1$ it leads to $F \in \mathcal{P}(k + 1)$. ■

Let $\mathcal{P} \in \mathbf{L}_{\leq}$ and G be a graph. By $\mathcal{H}_{\mathcal{P}}(G)$ we denote a hypergraph whose vertex set is $V(G)$ and whose edge set is $\{W \subseteq V(G) : G[W] \in \mathcal{C}(\mathcal{P})\}$. Note the following facts.

Remark 1. Let k be a non-negative integer, $\mathcal{P} \in \mathbf{L}_{\leq}$ and G be a graph.

(i) $G \in \mathcal{P}(k)$ if and only if $\tau(\mathcal{H}_{\mathcal{P}}(G)) \leq k$.

(ii) $G \in \mathcal{P}(k+1) \setminus \mathcal{P}(k)$ if and only if $\tau(\mathcal{H}_{\mathcal{P}}(G)) = k+1$.

Lemma 6. *Let k be a non-negative integer and $\mathcal{P} \in \mathbf{L}_{\leq}$. A graph G is a forbidden subgraph for $\mathcal{P}(k)$ if and only if $\mathcal{H}_{\mathcal{P}}(G)$ is τ-vertex $(k+1)$-critical.*

Proof. Suppose that $G \in \mathcal{C}(\mathcal{P}(k))$. By Lemma 5 and Remark 1, $\tau(\mathcal{H}_{\mathcal{P}}(G)) = k+1$. Moreover, for each $v \in V(G)$ we have $G - v \in \mathcal{P}(k)$, which again by Remark 1 implies $\tau(\mathcal{H}_{\mathcal{P}}(G-v)) \leq k$. Since $\mathcal{H}_{\mathcal{P}}(G-v) = \mathcal{H}_{\mathcal{P}}(G) - v$ we conclude that $\mathcal{H}_{\mathcal{P}}(G)$ is τ-vertex $(k+1)$-critical.

Now assume that $\mathcal{H}_{\mathcal{P}}(G)$ is τ-vertex $(k+1)$-critical. Remark 1 and the equality $\mathcal{H}_{\mathcal{P}}(G-v) = \mathcal{H}_{\mathcal{P}}(G) - v$ yield $G \in \mathcal{P}(k+1) \setminus \mathcal{P}(k)$ and $G - v \in \mathcal{P}(k)$ for each $v \in V(G)$. Hence $G \in \mathcal{C}(\mathcal{P}(k))$. ■

Lemma 6 and Theorem 3 make it easy to formulate one more observation.

Corollary 1. *Let k, r, x be integers such that $k \geq 1$, $r \geq 3$, $r \geq x \geq 1$ and let $\mathcal{P} \in \mathbf{L}_{\leq}$. If G is a graph such that $\mathcal{H}_{\mathcal{P}}(G)$ is isomorphic to $\mathcal{H}^*(k, r, x)$ defined in Construction 1, then G is a forbidden subgraph for $\mathcal{P}(k)$.*

A graph G is a *host-graph of a hypergraph* $\mathcal{H}$ if $V(G) = V(\mathcal{H})$ and for each edge e of G there is an edge E of $\mathcal{H}$ satisfying $e \subseteq E$. For an arbitrary family $\mathcal{F}$ of graphs, a graph G is an *$\mathcal{F}$-host-graph of a hypergraph* $\mathcal{H}$ when it is a host-graph of $\mathcal{H}$ such that $G[E] \in \mathcal{F}$ for each edge E of $\mathcal{H}$.

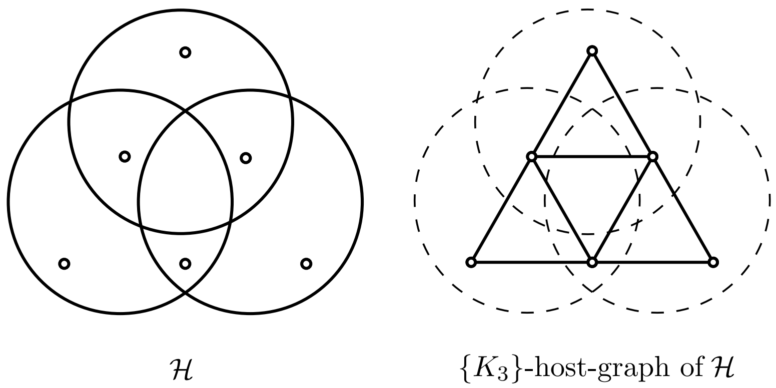


Figure 1. The example of a host-graph of a hypergraph.

Observe that for a given family of graphs $\mathcal{F}$ and a hypergraph $\mathcal{H}$ an $\mathcal{F}$-host-graph of a hypergraph $\mathcal{H}$ does not necessarily exist. However, we can easily find a family $\mathcal{F}$ and a hypergraph $\mathcal{H}$ having an $\mathcal{F}$-host-graph. As an example, for a fixed positive integer r, take $\mathcal{F} = \{K_r\}$ and any r-uniform hypergraph $\mathcal{H}$ (see Figure 1).

Furthermore, if G is a $\mathcal{C}(\mathcal{P})$-host-graph of a hypergraph $\mathcal{H}$ then $\mathcal{H}_{\mathcal{P}}(G)$ is not necessarily isomorphic to $\mathcal{H}$ (see Figure 1 again). We use $\mathcal{C}(\mathcal{P})$-host-graphs to describe forbidden subgraphs for $\mathcal{P}(k)$ with large number of vertices. In Section 2, we have constructed the family of hypergraphs $\mathcal{H}^*(k, r, x)$ that are r-uniform τ-vertex $(k+1)$-critical and have large number of vertices. So, a $\mathcal{C}(\mathcal{P})$-host-graph of a hypergraph $\mathcal{H}^*(k, r, x)$ could be potentially a forbidden subgraph for $\mathcal{P}(k)$. First we give some examples of families $\mathcal{F}$ of graphs for which an $\mathcal{F}$-host-graph of $\mathcal{H}^*$ from Construction 1 exists.

Let G be a graph. The symbols $\omega(G)$ and $\alpha(G)$ denote the order of the maximum clique and the cardinality of the maximum independent set of G, respectively.

Lemma 7. *Let $\mathcal{F}$ be a family of graphs. Next let k, r, x be integers, $k \geq 1$, $r \geq 3$, $r > x \geq 1$ and $\mathcal{H}^* = \mathcal{H}^*(k, r, x)$ be a hypergraph from Construction* 1.

(i) *If there is $F \in \mathcal{F}$ such that $|V(F)| = r$ and $\omega(F) \geq x$, then there exists an $\mathcal{F}$-host-graph of the hypergraph $\mathcal{H}^*$.*

(ii) *If there is $F \in \mathcal{F}$ such that $|V(F)| = r$ and $\alpha(F) \geq x$, then there exists an $\mathcal{F}$-host-graph of the hypergraph $\mathcal{H}^*$.*

(iii) *If there is $F \in \mathcal{F}$ such that $|V(F)| = r$ and moreover $r \geq x + k$, then there exists an $\mathcal{F}$-host-graph of the hypergraph $\mathcal{H}^*$.*

Proof. Using the notations from Construction 1 we show how to obtain an $\mathcal{F}$-host-graph G of the hypergraph $\mathcal{H}^*$. First we prove statements (i) and (ii). In the hypergraph $\mathcal{H}^*$ we add all the edges between vertices in U to obtain K_{x+k} for (i) and we leave U independent for (ii). Then we choose $F \in \mathcal{F}$ such that $|V(F)| = r$ and $\omega(F) \geq x$ (for (i)) or $\alpha(F) \geq x$ (for (ii)). Now in each set A_i from Construction 1 we enter a part of F such that each E_i induces F in G. Observe that the assumption $\omega(F) \geq x$ or $\alpha(F) \geq x$ guarantees that all steps of this procedure can be done. To construct an $\mathcal{F}$-host-graph G for (iii) we choose an arbitrary vertex subset W of F of the cardinality $k + x$. Such a subset always exists since $r \geq k + x$. Next, we join some of the vertices in U by edges in such a way that the resulting graph is isomorphic to $F[W]$. Then, similarly to above, in each set A_i from Construction 1 we enter a part of the graph F such that each E_i induces F in the graph G. ■

Consider $\mathcal{P} \in \mathbf{L}_{\leq}$ and a hypergraph $\mathcal{H}^* = \mathcal{H}^*(k, r, x)$. As we mentioned before if G is a $\mathcal{C}(\mathcal{P})$-host-graph of a hypergraph $\mathcal{H}$, then $\mathcal{H}_{\mathcal{P}}(G)$ may be non-isomorphic to $\mathcal{H}$. Hence we do not know whether a $\mathcal{C}(\mathcal{P})$-host-graph of $\mathcal{H}^*$ is a forbidden subgraph for $\mathcal{P}(k)$ or not. In the next theorem, we solve this problem positively for some cases, regardless of whether the hypergraphs $\mathcal{H}_{\mathcal{P}}(G)$ and $\mathcal{H}^*$ are isomorphic.

A set S is a *vertex-cut-set* in a connected graph G if $G - S$ has at least two connected components. For a positive integer x, a connected graph G is *x-vertex connected* if it does not contain any vertex-cut-set of the cardinality less than x. As usual, for a given graph G and $v \in V(G)$, we denote by $N_G(v)$ the set of neighbours of v in G.

Theorem 8. *Let k, r, x be integers, $k \geq 1$, $r \geq 3$, $r > x \geq 1$, and let $\mathcal{H}^* = \mathcal{H}^*(k, r, x)$ be the hypergraph from Construction 1. If $\mathcal{P} \in \mathbf{L}_{\leq}$ is a class of graphs such that $\mathcal{C}(\mathcal{P})$ consists only of x-vertex connected graphs of order at least r, then each $\mathcal{C}(\mathcal{P})$-host-graph of the hypergraph $\mathcal{H}^*$ is a forbidden subgraph for $\mathcal{P}(k)$.*

Proof. In the proof we refer to the notations from Construction 1. Let G be an arbitrary $\mathcal{C}(\mathcal{P})$-host-graph of the hypergraph $\mathcal{H}^*$. Applying Lemma 6, the aim is to show that $\mathcal{H}_{\mathcal{P}}(G)$ is τ-vertex $(k+1)$-critical.

First we prove that any $(k+1)$-element subset W of U is a transversal of $\mathcal{H}_{\mathcal{P}}(G)$, i.e., for any $(k+1)$-element subset W of U, the graph $G - W$ does not contain any induced subgraph F satisfying $F \in \mathcal{C}(\mathcal{P})$. Suppose that this is not the case and let F be a subgraph of $G - W$ such that $F \in \mathcal{C}(\mathcal{P})$. Denote by $U'_1, \ldots, U'_m$ the subsets of $V(G - W)$ that correspond to $U_1, \ldots, U_m$ in G. Thus, $|U'_i| \leq x - 1$ for each $i \in \{1, \ldots, m\}$. Furthermore, since $r > x$, it follows that $V(F)$ is not contained in $U - W$ and consequently F must contain at least one vertex of some A_i with $i \in \{1, \ldots, m\}$. Because of the symmetry, we may assume that $A' = A_1 \cap V(F) \neq \emptyset$. Since $|A' \cup U'_1| < r$, there is a vertex of F that does not belong to $A' \cup U'_1$. Hence, we can divide vertices of F into three parts $V_1 = V(F) \cap A', V_2 = V(F) \cap U'_1$ and $V_3 = V(F) \setminus (V_1 \cup V_2)$. By our earlier observation $V_3 \neq \emptyset$. Since $N_G(A_1) \subseteq U_1$, it follows that $N_F(V_1) \subseteq V_2$. Thus, V_2 is a vertex-cut-set of F. Furthermore, $|V_2| \leq |U'_1| \leq x - 1$, which contradicts that F is x-vertex connected and proves $\tau(\mathcal{H}_{\mathcal{P}}(G)) \leq k + 1$. Recall that, by the construction of G, each edge of $\mathcal{H}^*$ is an edge of $\mathcal{H}_{\mathcal{P}}(G)$. It means, by Theorem 3, that $\tau(\mathcal{H}_{\mathcal{P}}(G)) \geq k + 1$ and consequently $\tau(\mathcal{H}_{\mathcal{P}}(G)) = k + 1$.

Now, we prove the τ-vertex criticality of $\mathcal{H}_{\mathcal{P}}(G)$. By Remark 1 and the fact that $\mathcal{H}_{\mathcal{P}}(G - v) = \mathcal{H}_{\mathcal{P}}(G) - v$, we have to argue that for any $i \in \{1, \ldots, m\}$ and for any $v \in A_i$ we obtain $G - v \in \mathcal{P}(k)$. Let $W' = U - U_i$. Observe that $|W'| = k$ and $U_j \cap W' \neq \emptyset$ for $j \neq i$. We show that $(G - v) - W' \in \mathcal{P}$ or equivalently that $(G - v) - W'$ does not contain an induced subgraph isomorphic to any $F \in \mathcal{C}(\mathcal{P})$. Let $U''_1, \ldots, U''_m$ be subsets of $V(G - W')$ that correspond to $U_1, \ldots, U_m$ in G. Thus, $|U''_j| \leq x - 1$ for each $j \neq i$ and $|U''_i| = x$. Suppose that there is $F \in \mathcal{C}(\mathcal{P})$ such that $F \leq (G - v) - W'$. It is clear that there is $j \neq i$ such that F contains at least one vertex of A_j. Therefore, similarly as above, we can divide $V(F)$ into three parts $V_1 = V(F) \cap A_j, V_2 = V(F) \cap U'_j$ and $V_3 = V(F) \setminus (V_1 \cup V_2)$ with $V_3 \neq \emptyset$. Since $N_F(V_1) \subseteq V_2$, the set V_2 is a vertex cut-set of F, contrary to the x-vertex connectivity of F. ■

Theorem 8 gives us a very fruitful tool to construct forbidden subgraphs for $\mathcal{P}(k)$.

Corollary 2. *Let k, x be positive integers and let $\mathcal{P} \in \mathbf{L}_{\leq}$ be a class of graphs such that each graph in $\mathcal{C}(\mathcal{P})$ is x-vertex connected of order at least r. If r is the order of some $F \in \mathcal{C}(\mathcal{P})$ and $r \geq 3$, and $r \geq k + x$, then there exists G that is a forbidden subgraph for $\mathcal{P}(k)$ and $|V(G)| = k + x + \binom{k+x}{x}(r - x)$.*

Theorem 9. *Let $\mathcal{P} \in \mathbf{L}_{\leq}$. If $r = \max\{|F| : F \in \mathcal{C}(\mathcal{P})\}$ and $G \in \mathcal{C}(\mathcal{P}(1))$, then $|V(G)| \leq \left\lfloor \frac{(r+2)^2}{4} \right\rfloor$. Moreover, this bound is achieved for infinitely many classes $\mathcal{P} \in \mathbf{L}_{\leq}$.*

Proof. By Lemma 6 and Theorem 1 we only need to show the last sentence of the statement. However, if we put $k = 1$ and $x = \left\lceil \frac{r-1}{2} \right\rceil$ in Corollary 2, then for $r \geq 3$ we obtain a forbidden subgraph for $\mathcal{P}(k)$ with $\left\lfloor \frac{(r+2)^2}{4} \right\rfloor$ vertices and hence the theorem follows. ∎

The next remark is an immediate consequence of Theorem 9 and the fact that $(\mathcal{P}(k))(1) = \mathcal{P}(k + 1)$.

Remark 2. Let k be a non-negative integer and $\mathcal{P} \in \mathbf{L}_{\leq}$. If $\mathcal{C}(\mathcal{P})$ is finite, then the family $\mathcal{C}(\mathcal{P}(k))$ is also finite.

4. The Structure of Forbidden Subgraphs

At the beginning of this section we describe connected forbidden subgraphs for $\mathcal{P}(k)$ in terms of connected forbidden subgraphs for $\mathcal{P}(l)$, where $l < k$. To do it we use the following hypergraph tool.

Remark 3. If $\mathcal{H}_1 \cup \mathcal{H}_2$ is the union of disjoint hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$, then

$$\tau(\mathcal{H}_1 \cup \mathcal{H}_2) = \tau(\mathcal{H}_1) + \tau(\mathcal{H}_2).$$

Note that the definition of the τ-vertex criticality of a hypergraph and Remark 3 imply the following observation.

Remark 4. Let s be an integer, $s \geq 2$. The union $\mathcal{H}_1 \cup \cdots \cup \mathcal{H}_s$ of disjoint hypergraphs $\mathcal{H}_1, \ldots, \mathcal{H}_s$ is τ-vertex critical if and only if for each $i \in \{1, \ldots, s\}$ the hypergraph $\mathcal{H}_i$ is τ-vertex critical.

The next result is the consequence of Remark 4.

Theorem 10. *Let k, s be integers, $k \geq 0$, $s \geq 1$ and $\mathcal{P} \in \mathbf{L}^a_{\leq}$. The union $F_1 \cup \cdots \cup F_s$ of disjoint connected graphs $F_1, \ldots, F_s$ is a forbidden subgraph for $\mathcal{P}(k)$ if and only if there exist non-negative integers $k_1, \ldots, k_s$ such that $\sum_{i=1}^{s} k_i = k+1-s$ and for each $i \in \{1, \ldots, s\}$ the graph F_i is a forbidden subgraph for $\mathcal{P}(k_i)$.*

Proof. From Lemma 6 we have $F_1 \cup \cdots \cup F_s \in \mathcal{C}(\mathcal{P}(k))$ if and only if $\mathcal{H}_{\mathcal{P}}(F_1 \cup \cdots \cup F_s)$ is τ-vertex $(k+1)$-critical. Since $\mathcal{H}_{\mathcal{P}}(F_1 \cup \cdots \cup F_s) = \mathcal{H}_{\mathcal{P}}(F_1) \cup \cdots \cup \mathcal{H}_{\mathcal{P}}(F_s)$ and because of Remarks 3, 4 we know that it is equivalent to the conditions $\tau(\mathcal{H}_{\mathcal{P}}(F_1)) + \cdots + \tau(\mathcal{H}_{\mathcal{P}}(F_s)) = k+1$ and for each $i \in \{1, \ldots, s\}$ the hypergraph $\mathcal{H}_{\mathcal{P}}(F_i)$ is τ-vertex critical. It means that there exist non-negative integers $k_1, \ldots, k_s$ such that for each $i \in \{1, \ldots, s\}$ the hypergraph $\mathcal{H}_{\mathcal{P}}(F_i)$ is τ-vertex (k_i+1)-critical and moreover $\sum_{i=1}^{s}(k_i+1) = k+1$. From Lemma 6 these conditions are equivalent to the statement $F_i \in \mathcal{C}(\mathcal{P}(k_i))$ for each $i \in \{1, \ldots, s\}$ and $\sum_{i=1}^{s} k_i = k+1-s$. ■

Corollary 3. *Let k be a non-negative integer and $\mathcal{P} \in \mathbf{L}^a_{\leq}$. If F is the union of disjoint connected graphs $F_1, \ldots, F_s$ and $F \in \mathcal{C}(\mathcal{P}(k))$, then $s \leq k+1$.*

Corollary 4. *Let k be a non-negative integer and $\mathcal{P} \in \mathbf{L}^a_{\leq}$ and let $|\mathcal{C}(\mathcal{P})| = p$. The number of forbidden subgraphs for $\mathcal{P}(k)$ that have exactly $k+1$ connected components is equal to $\binom{k+p}{k+1}$.*

Proof. From Theorem 10 we know that forbidden subgraphs for $\mathcal{P}(k)$ with exactly $k+1$ connected components have the form $F_1 \cup \cdots \cup F_{k+1}$, where for each $i \in \{1, \ldots, k+1\}$ the condition $F_i \in \mathcal{C}(\mathcal{P})$ holds. Let $\mathcal{C}(\mathcal{P}) = \{H_1, \ldots, H_p\}$. Thus, if m_i denotes $|\{l : F_l = H_i\}|$, then we actually are interested in the number of sequences $(m_1, \ldots, m_p)$ whose elements are non-negative integers and for which the equality $m_1 + \cdots + m_p = k+1$ holds, which leads to the assertion. ■

The remaining part of this section is devoted to other constructions of forbidden subgraphs for $\mathcal{P}(k)$ in terms of forbidden subgraphs for $\mathcal{P}$. In this consideration the structure of $\mathcal{H}_{\mathcal{P}}(G)$ is unknown. It means that our results are based only on the analysis of graph structures.

Construction 2. Let s be a positive integer, $G_1, \ldots, G_s$ be graphs and T be a forest with the vertex set $\{x_1, \ldots, x_s\}$. By $T(G_1, \ldots, G_s)$ we denote the family of all graphs obtained from disjoint $G_1, \ldots, G_s$ by the addition of exactly $|E(T)|$ new edges, such that a new edge joins an arbitrary vertex of G_i with an arbitrary vertex of G_j when $x_i x_j$ is an edge of T. Next we use a symbol $(G_1, \ldots, G_s)$ to denote the family of all graphs $T(G_1, \ldots, G_s)$ taken over all s-vertex forests T and all possible orderings of their vertices.

Theorem 11. *If k is a non-negative integer and $\mathcal{P} \in \mathbf{L}^a_{\leq}$ and $G_1, \ldots, G_{k+1} \in \mathcal{C}(\mathcal{P})$, then each graph G in $(G_1, \ldots, G_{k+1})$ is a forbidden subgraph for $\mathcal{P}(k)$.*

Proof. Suppose that $G \in (G_1, \dots, G_{k+1})$. It follows that there exists a forest T with $k+1$ vertices $x_1, \dots, x_{k+1}$ such that $G \in T(G_1, \dots, G_{k+1})$. Observe that $G \notin \mathcal{P}(k)$ since it contains $k+1$ disjoint induced subgraphs that are forbidden subgraphs for $\mathcal{P}$.

Next, let $v \in V(G)$. We show that there exist k vertices $u_2, \dots, u_{k+1}$ in $V(G) \setminus \{v\}$ such that the graph resulting from G by the removal of $v, u_2, \dots, u_{k+1}$ is in $\mathcal{P}$.

The construction of G implies the existence of the unique index i such that $v \in V(G_i)$. Let $x_{j_1}, \dots, x_{j_{k+1}}$ be a new ordering of vertices of T such that $x_{j_1} = x_i$ and for $l \geq 2$ each vertex x_{j_l} has at most one neighbour in $\{x_{j_1}, \dots, x_{j_{l-1}}\}$. Such an ordering can be done by brute-force search algorithm. Suppose, without loss of generality, that $x_{j_l} = x_l$ for each $l \in \{1, \dots, k+1\}$. Consequently, $G_{j_l} = G_l$ for each $l \in \{1, \dots, k+1\}$ and especially $G_i = G_1$.

Now we describe how to choose vertices $u_2, \dots, u_{k+1}$. For each $j \in \{2, \dots, k+1\}$ there is at most one edge $x_l x_j$ with $l < j$. Thus when such an edge exist we take as u_j the vertex of G_j that is the end of the unique edge joining G_j with G_l (see the construction of G), otherwise u_j is an arbitrary vertex of G_j. Observe that $G - \{v, u_2, \dots, u_{k+1}\}$ is the union of $k+1$ disjoint graphs $G_1 - v$ and $G_j - u_j$ for $j \in \{2, \dots, k+1\}$. The assertion follows by the additivity of $\mathcal{P}$ and properties of all G_j. ■

Theorem 12. *Let k be a non-negative integer and $\mathcal{P} \in \mathbf{L}^a_{\leq}$. A forest G is a forbidden subgraph for $\mathcal{P}(k)$ if and only if $G \in (G_1, \dots, G_{k+1})$, where $G_1, \dots, G_{k+1}$ are trees that are forbidden subgraphs for $\mathcal{P}$.*

Proof. By Theorem 11, it is enough to prove that if G is simultaneously a forest and a forbidden subgraph for $\mathcal{P}(k)$, then there are graphs $G_1, \dots, G_{k+1}$ belonging to $\mathcal{C}(\mathcal{P})$ and there exists a $(k+1)$-vertex forest T such that $G \in T(G_1, \dots, G_{k+1})$. To do it we use the induction on k.

By the additivity of $\mathcal{P}$, each forest that is a forbidden subgraph for $\mathcal{P}(0) = \mathcal{P}$ is a tree. The conclusion follows from the fact that there is only one 1-vertex forest $T = K_1$ and each graph G can be represented as $K_1(G)$, which means as $T(G)$.

Assume that the implication is true for parameters less than k and $k \geq 1$. First suppose that G has at least two connected components $H_1, \dots, H_s$. Obviously, each of them is a tree. By Theorem 10, $H_i \in \mathcal{C}(\mathcal{P}(k_i))$, where $\sum_{i=1}^{s} k_i = k + 1 - s$. Because all k_i are non-negative integers and $s \geq 2$ we obtain $0 \leq k_i \leq k-1$ for each $i \in \{1, \dots, s\}$. By the induction hypothesis, $H_i \in T_i\left(G_1^i, \dots, G_{k_i+1}^i\right)$, which implies

$$G \in T\left(G_1^1, \dots, G_{k_1+1}^i, \dots, G_1^s, \dots, G_{k_s+1}^s\right),$$

where T is the union of disjoint $T_1, \dots, T_s$ and $G_j^l \in \mathcal{C}(\mathcal{P})$ for each $l \in \{1, \dots, s\}$ and $j \in \{1, \dots, k_l + 1\}$. Since each T_i has exactly $k_i + 1$ vertices, the forest T has

$\sum_{i=1}^{s}(k_i+1)$ vertices, which means T has $k+1$ vertices. Thus G has a required form.

Now suppose that G is connected, which means G is a tree.

Claim 13. *There is $x \in V(G)$ such that $G-x$ has at least one connected component in $\mathcal{P}$ and if $H_1,\dots,H_p$ are all connected components of $G-x$ belonging to $\mathcal{P}$, then the graph induced in G by $V(H_1)\cup\dots\cup V(H_p)\cup\{x\}$ is not in $\mathcal{P}$.*

Proof. We describe the procedure which finds the required x in a finite number of steps.

Let v_0 be an arbitrary vertex of G that is not a leaf (such a vertex always exists because $k \geq 1$, which implies $|V(G)| \geq 3$). Next let G_1 be an arbitrary connected component of $G - v_0$ such that $G_1 \notin \mathcal{P}$ (since G is in $\mathcal{C}(\mathcal{P}(k))$ and $k \geq 1$ such a connected component exists).

Let v_1 be the unique neighbour of v_0 in G_1. If $G_1 - v_1 \in \mathcal{P}$, then $x = v_1$. Otherwise, let G_2 be an arbitrary connected component of $G_1 - v_1$ such that $G_2 \notin \mathcal{P}$ and let v_2 be the unique neighbour of v_1 in G_2. If $G_2 - v_2 \in \mathcal{P}$, then $x = v_2$. Otherwise, since G is finite, we find the finite sequence of vertices $v_0,\dots,v_q$ and the sequence of graphs $G = G_0, G_1,\dots,G_q$ such that $G_i - v_i \notin \mathcal{P}$ for $i \in \{0,\dots,q-1\}$, $G_q \notin \mathcal{P}$ and $G_q - v_q \in \mathcal{P}$. Moreover for $i \in \{1,\dots,q\}$ the graph G_i is a connected component of $G_{i-1} - v_{i-1}$ and v_i is the unique neighbour of v_{i-1} in G_i.

Observe that v_q can play the role of x. Indeed, the procedure implies that the connected components of $G_q - v_q$ are simultaneously the connected components of $G - v_q$.

Let x be a vertex that satisfies the assumptions of Claim 13. Recall that G is a tree, which means that $G - x$ is a forest. Since G is a forbidden subgraph for $\mathcal{P}(k)$ we obtain $G - x \notin \mathcal{P}(k-1)$. It follows that $G - x$ contains an induced subgraph $G' \in \mathcal{C}(\mathcal{P}(k-1))$ that is a forest. By the induction hypothesis $V(G')$ can be partitioned into k sets $V_1,\dots,V_k$ such that for each $i \in \{1,\dots,k\}$ the graph G'_i induced by V_i in $G - x$ is forbidden for $\mathcal{P}$. Because $\mathcal{P}$ is additive, all of the graphs G'_i are connected and as subgraphs of $G - x$ they are trees. Additionally, $(V(G'_1)\cup\dots\cup V(G'_k)\cup\{x\})\cap V(H_i) = \emptyset$ for $i \in \{1,\dots,p\}$ (keep in mind that $H_1,\dots,H_p \in \mathcal{P}$, see Claim 13).

Recall that, by Claim 13, $V(H_1)\cup\dots\cup V(H_p)\cup\{x\}$ contains at least one subset that induces a graph, say G'_{k+1}, forbidden for $\mathcal{P}$. Hence $G'_1,\dots,G'_{k+1}$ are disjoint induced subgraphs of G, each of which is in $\mathcal{C}(\mathcal{P})$. Suppose, for a contradiction, that there is a vertex $u \in V(G)\setminus\bigcup_{i=1}^{k+1} V(G'_i)$. Since $G \in \mathcal{C}(\mathcal{P}(k))$ we can find at most k different vertices of $G-u$ such that the removal of all of them from $G-u$ results in a graph in $\mathcal{P}$. Because G contains disjoint induced subgraphs $G'_1,\dots,G'_{k+1}$ that are forbidden for $\mathcal{P}$, it is impossible, giving a contradiction.

It means $V(G) = \bigcup_{i=1}^{k+1} V(G'_i)$ and, since G is a tree, there is a tree T with $k+1$ vertices such that $G \in T\left(G'_1, \ldots, G'_{k+1}\right)$. ■

Below we present one more construction of graphs that are forbidden for $\mathcal{P}(k)$.

Construction 3. Let $G_1, \ldots, G_s$ be *rooted graphs*, which means that for each $i \in \{1, \ldots, s\}$ the graph G_i has a marked vertex v_i, called its *root*. Next let H be a graph with $V(H) = \{x_1, \ldots, x_s\}$. We take disjoint $H, G_1, \ldots, G_s$ and identify vertices v_i with x_i for all $i \in \{1, \ldots, s\}$. By $H|G_1, \ldots, G_s|$ we denote the family of all graphs of this type taken over all possible choices of roots $v_1, \ldots, v_s$. More precisely, for each graph G in $H|G_1, \ldots, G_s|$ we have $V(G) = \bigcup_{i=1}^{s} V(G_i)$ and $E(G) = \bigcup_{i=1}^{s} E(G_i) \cup \{v_i v_j : x_i x_j \in E(H)\}$ with a choice of roots $v_1, \ldots, v_s$. Now we use a symbol $|G_1, \ldots, G_s|$ to denote the union of sets $H|G_1, \ldots, G_s|$ taken over all s-vertex graphs H.

Theorem 14. *If k is a non-negative integer and $\mathcal{P} \in \mathbf{L}^a_{\leq}$ and $G_1, \ldots, G_{k+1} \in \mathcal{C}(\mathcal{P})$, then each graph G in $|G_1, \ldots, G_{k+1}|$ is a forbidden subgraph for $\mathcal{P}(k)$.*

Proof. By the assumption $G \in |G_1, \ldots, G_{k+1}|$, we have that $G \in H|G_1, \ldots, G_{k+1}|$ for some $(k+1)$-vertex graph H. Let $x_i = v_i$ be a common vertex of H and G_i, described in Construction 3.

Because G contains disjoint induced subgraphs $G_1, \ldots, G_{k+1}$ it follows that $G \notin \mathcal{P}(k)$. If $v \in V(G)$, then $v \in V(G_j)$ for exactly one index $j \in \{1, \ldots, k+1\}$. The graph obtained from $G - v$ by the removal of the vertex set S, where $S = \{x_l : l \neq j\}$, has at least $k+1$ connected components each of which is in $\mathcal{P}$. The additivity of $\mathcal{P}$ implies $G - v \in \mathcal{P}(k)$. ■

5. P_r-Free Graphs

In this section we focus our attention on the class $\mathcal{W}_r$ of graphs not containing P_r as an induced subgraph. We determine the minimum and maximum number of vertices of a graph in $\mathcal{C}(\mathcal{W}_r(1))$. First we consider $\mathcal{C}(\mathcal{W}_3(1))$. Because of Theorem 9, each graph that is forbidden for $\mathcal{W}_3(1)$ has at most six vertices. Searching all non-isomorphic graphs of this type we can derive that $\mathcal{C}(\mathcal{W}_3(1))$ has 14 elements: $C_4, C_5, C_6, P_6, 2P_3, F_1, \ldots, F_9$, where the graphs F_i for $i \in \{1, \ldots, 9\}$ are depicted in Figure 2. Similar arguments we apply to the classes $\mathcal{O}$ of edgeless graphs and $\mathcal{K}$ of complete graphs. In this case, the facts $\mathcal{C}(\mathcal{O}) = \{K_2\}$ and $\mathcal{C}(\mathcal{K}) = \{\overline{K_2}\}$ yield $\mathcal{C}(\mathcal{O}(1)) = \{K_3, P_4, C_4, 2K_2\}$ and $\mathcal{C}(\mathcal{K}(1)) = \{\overline{K_3}, P_4, C_4, 2K_2\}$.

Of course the brute searching method is not too effective if forbidden subgraphs have big orders. Thus for $r \geq 4$ we start with determining forbidden subgraphs for $\mathcal{W}_r(1)$ with the minimum number of vertices. If $G \in \mathcal{C}(\mathcal{W}_r(1))$,

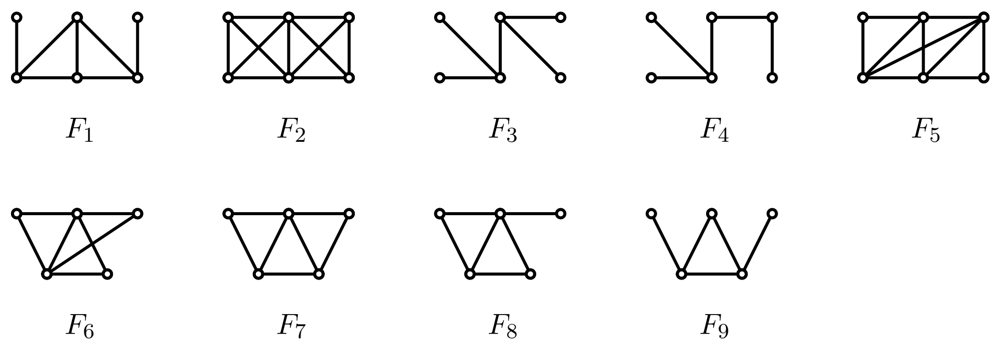


Figure 2. All the graphs in $\mathcal{C}(\mathcal{W}_3(1)) \setminus \{C_4, C_5, C_6, P_6, 2P_3\}$.

then G must contain an induced subgraph P_r after deletion of any vertex. Thus $r+1$ is the lower bound on the number of vertices of a graph in $\mathcal{C}(\mathcal{W}_r(1))$. We conclude the following fact.

Proposition 1. *If r is an integer, $r \geq 3$, then C_{r+1} is a forbidden subgraph for $\mathcal{W}_r(1)$ with the minimum number of vertices.*

By Theorem 9 we have that the upper bound on the number of vertices of a graph in $\mathcal{C}(\mathcal{W}_r(1))$ is $\left\lfloor \frac{(r+2)^2}{4} \right\rfloor$. However, for $r = 4$ we find no graph that realizes this bound. For any $r \geq 5$ there exists a graph in $\mathcal{C}(\mathcal{W}_r(1))$ of order $\left\lfloor \frac{(r+2)^2}{4} \right\rfloor$. To prove this fact we use the class of graphs that contains all the complements of graphs in $\mathcal{W}_r$.

For a given class of graphs $\mathcal{P} \in \mathbf{L}_{\leq}$ let us define $\overline{\mathcal{P}} = \{\overline{G} : G \in \mathcal{P}\}$. It is a known fact that if $\mathcal{P} \in \mathbf{L}_{\leq}$, then $\overline{\mathcal{P}}$ is also in $\mathbf{L}_{\leq}$. Moreover, there is a coincidence between forbidden subgraphs for $\mathcal{P}$ and $\overline{\mathcal{P}}$ given by the equality $\mathcal{C}(\overline{\mathcal{P}}) = \{\overline{F} : F \in \mathcal{C}(\mathcal{P})\}$ [2]. Let $\mathcal{P}_1$, $\mathcal{P}_2$ be classes of graphs. By $\mathcal{P}_1 \circ \mathcal{P}_2$ we denote the class of all graphs G whose vertex set can be partitioned into two parts V_1, V_2 (possible empty) such that, for all $i \in \{1, 2\}$, if V_i is non-empty, then $G[V_i] \in \mathcal{P}_i$. In that case $\mathcal{P}_1 \circ \mathcal{P}_2$ is called a *product* of $\mathcal{P}_1$ and $\mathcal{P}_2$. In [4] it is proved that $F \in \mathcal{C}(\mathcal{P}_1 \circ \mathcal{P}_2)$ if and only if $\overline{F} \in \mathcal{C}(\overline{\mathcal{P}}_1 \circ \overline{\mathcal{P}}_2)$. It is easy to observe that for each class of graphs $\mathcal{P}$ and a positive integer k, the class $\mathcal{P}(k)$ is identical with $\mathcal{P} \circ \mathcal{Q}$, where Q consists of all the graphs of order at most k. Moreover, for such $\mathcal{Q}$ we have $\overline{\mathcal{Q}} = \mathcal{Q}$. Hence, taking into account the previous consideration, we have the following observation.

Proposition 2. *If $\mathcal{P} \in \mathbf{L}_{\leq}$, then*

(i) *$G \in \mathcal{P}(k)$ if and only if $\overline{G} \in \overline{\mathcal{P}}(k)$, and*

(ii) *$F \in \mathcal{C}(\mathcal{P}(k))$ if and only if $\overline{F} \in \mathcal{C}(\overline{\mathcal{P}}(k))$, and*

(iii) $\overline{G} \in \overline{\mathcal{P}}(k)$ *if and only if* $\overline{G} \in \overline{\mathcal{P}(k)}$.

Let us consider $\overline{\mathcal{W}_r}$. Thus, $\mathcal{C}(\overline{\mathcal{W}_r}) = \{\overline{P_r}\}$ and, by Proposition 2, it follows that $G \in \mathcal{C}(\overline{\mathcal{W}_r}(1))$ if and only if $\overline{G} \in \mathcal{C}(\mathcal{W}_r(1))$. As a consequence, the complement of a forbidden subgraph for $\overline{\mathcal{W}_r}(1)$ with the maximum number of vertices is a forbidden subgraph for $\mathcal{W}_r(1)$ with the maximum number of vertices. Since the vertex connectivity of $\overline{P_r}$ is relatively big we will be able to apply Theorem 8. First we give the supporting observation.

Lemma 15. *If r is an integer, $r \geq 5$, then $\overline{P_r}$ is $\left\lceil \frac{r-1}{2} \right\rceil$-connected.*

Proof. Let $G = \overline{P_r}$. Observe that the vertices of G can be divided into two sets W_1, W_2 such that subgraphs induced by W_i for $i \in \{1, 2\}$ are complete graphs and $|W_1| = \left\lceil \frac{r}{2} \right\rceil$, $|W_2| = \left\lfloor \frac{r}{2} \right\rfloor = \left\lceil \frac{r-1}{2} \right\rceil$. Suppose that there is a vertex-cut-set S of G such that $|S| < \left\lfloor \frac{r}{2} \right\rfloor$. Thus $G - S$ has two disjoint subgraphs G_1 and G_2 such that there is no edge joining a vertex of G_1 with a vertex of G_2. Furthermore, observe that $V(G_1) = W_1 \setminus S$ and $V(G_2) = W_2 \setminus S$ and moreover, $V(G_1) \neq \emptyset$ and $V(G_2) \neq \emptyset$. Let us denote $W_1' = W_1 \setminus S$ and $W_2' = W_2 \setminus S$. So, by our assumptions, there is no edge joining a vertex of W_1' with a vertex of W_2' in G. This implies that in $\overline{G}$ each vertex of W_1' is adjacent to each vertex of W_2'. If $|W_1'| \geq 2$ and $|W_2'| \geq 2$, then $\overline{G}$ contains C_4, which contradicts that $\overline{G} = P_r$. If one of the sets W_1', W_2' contains exactly one vertex, then since $|S| < \left\lfloor \frac{r}{2} \right\rfloor$, there are at least three vertices in the second set. Thus $\overline{G}$ has a vertex of degree three, which again gives a contradiction with the assumption that $\overline{G}$ is a path. ■

By Lemma 7 we have the additional fact.

Lemma 16. *Let r be an integer, $r \geq 5$. There exists a $\{\overline{P_r}\}$-host-graph of a hypergraph $\mathcal{H}^*\left(1, r, \left\lceil \frac{r-1}{2} \right\rceil\right)$ given in Construction 1.*

Finally, by Theorem 8, Lemma 16 and Proposition 2, we obtain the conclusion.

Theorem 17. *Let r be an integer, $r \geq 5$. The complement of a $\{\overline{P_r}\}$-host-graph of the hypergraph $\mathcal{H}^*\left(1, r, \left\lceil \frac{r-1}{2} \right\rceil\right)$, given in Construction 1, is a forbidden subgraph for $\mathcal{W}_r(1)$ with the maximum number of vertices.*

In Figure 3 we present the complement of a forbidden subgraph for $\mathcal{W}_5(1)$. Theorem 17 says that this graph has the maximum number of vertices among all the graphs in $\mathcal{C}(\mathcal{W}_5(1))$. Moreover, by Proposition 2, the graph in Figure 3 is in $\mathcal{C}\left(\overline{\mathcal{W}_5(1)}\right)$ and also in $\mathcal{C}\left(\overline{\mathcal{W}_5}(1)\right)$ and realizes the maximum order among all the graphs in both these families.

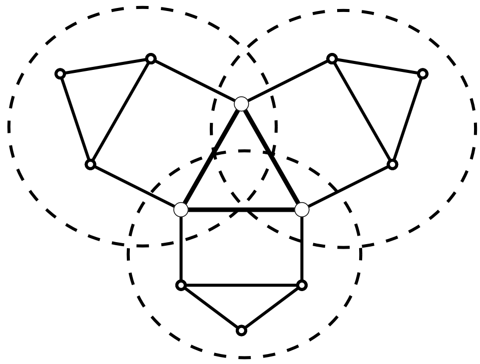


Figure 3. The complement of the graph in $\mathcal{C}(\mathcal{W}_5(1))$ with the maximum order.

6. Classes of Graphs That Are Closed Under Substitution

Let H, $G_1, \ldots, G_n$ be graphs and $v_1, \ldots, v_n$ be an arbitrary ordering of the set $V(H)$. By $H[G_1, \ldots, G_n]$ we denote the graph resulting from H by the simultaneous substitution of each vertex v_i with the graph G_i. Here the *substitution of the vertex v with the graph G in the graph H* means the removal of v and joining all the vertices of G with all the neighbours of v in H. A class $\mathcal{P}$ of graphs is *closed under substitution* if for any graphs $H, G_1, \ldots, G_n \in \mathcal{P}$ and every ordering of $V(H)$, the graph $H[G_1, \ldots, G_n]$, called a *substitution graph*, is also in $\mathcal{P}$. By $\mathbf{L}^*_{\leq}$ we denote the class of all non-trivial induced hereditary classes of graphs that are closed under substitution. The smallest of such ones (in the sense of the number of elements) is $\{K_1\}$, among most notable we should list the classes $\mathcal{O}$ of edgeless graphs, $\mathcal{K}$ of complete graphs, the class of perfect graphs and the classes $\mathcal{W}_r$, where $r = 2$ or $r \geq 4$. Observe that P_4-free graphs are just cographs. In this section we characterize all forbidden subgraphs for $\mathcal{P}(1)$ where $\mathcal{P} \in \mathbf{L}^*_{\leq}$.

A set $W \subseteq V(G)$ is a *module* in a graph G if for each two vertices $x, y \in W$, $N_G(x) \setminus W = N_G(y) \setminus W$. The *trivial modules* in G are $V(G)$, $\emptyset$ and singletons. A graph having only trivial modules is called *prime*. By **PRIME** we denote the class of all prime graphs that have at least two vertices.

In 1997 Giakoumakis [14] proved that for each class of graphs $\mathcal{P} \in \mathbf{L}_{\leq}$ its closure under substitution $\mathcal{P}^*$ consisting of all the graphs in $\mathcal{P}$ and all their substitution graphs can be characterized by $\mathcal{C}(\mathcal{P}^*)$ that consists of all minimal prime extensions of all the graphs in $\mathcal{C}(\mathcal{P})$. It has to be said that G' is a *minimal prime extension of* G if it is a prime induced supergraph of G and it does not contain as a proper induced subgraph any other prime induced supergraph of G.

Since for each class $\mathcal{P} \in \mathbf{L}^*_{\leq}$ we have $\mathcal{P} = \mathcal{P}^*$ (by the definition of $\mathbf{L}^*_{\leq}$), the Giakoumakis consideration leads to the following conclusion.

Remark 5. If $\mathcal{P} \in \mathbf{L}_{\leq}$, then $\mathcal{P} \in \mathbf{L}^*_{\leq}$ if and only if $\mathcal{C}(\mathcal{P}) \subseteq$ **PRIME**.

In [4] the following two theorems concerning $\mathcal{C}(\mathcal{P}_1 \circ \mathcal{P}_2)$ when both $\mathcal{P}_1$, $\mathcal{P}_2$ are in $\mathbf{L}_{\leq}^*$ have been proven.

Theorem 18 [4]**.** *Let $\mathcal{P}_1$, $\mathcal{P}_2 \in \mathbf{L}_{\leq}^*$ and let $H \in \mathbf{PRIME}$ with $V(H) = \{v_1, \ldots, v_n\}$. If $G = H[G_1, \ldots, G_n]$ and $G \in \mathcal{C}(\mathcal{P}_1 \circ \mathcal{P}_2)$, then $H \notin \mathcal{P}_1$ or $H \notin \mathcal{P}_2$ and there exists a partition (A, B, C, D) of $\{1, \ldots, n\}$ (empty parts are allowed), such that*

(i) *$G_i = K_1$ for $i \in A$, and*

(ii) *$G_i \in \mathcal{C}(\mathcal{P}_2) \cap \mathcal{P}_1$ for $i \in B$, and*

(iii) *$G_i \in \mathcal{C}(\mathcal{P}_1) \cap \mathcal{P}_2$ for $i \in C$, and*

(iv) *$G_i \in \mathcal{C}(\mathcal{P}_1 \cup \mathcal{P}_2)$ for $i \in D$.*

A graph G, different from K_1, is *strongly decomposable* if in its description $G = H[G_1, \ldots, G_n]$ with $H \in \mathbf{PRIME}$, all the graphs G_i satisfy $|V(G_i)| \geq 2$. In the next theorem we will restrict our attention to graphs that are strongly decomposable and are forbidden subgraphs for a product of classes of graphs.

Theorem 19 [4]**.** *Let $\mathcal{P} \in \mathbf{L}_{\leq}^* \setminus \{\mathcal{O}, \mathcal{K}, \{K_1\}\}$. A graph G is a forbidden subgraph for $\mathcal{P}_1 \circ \mathcal{P}_2$ and it is strongly decomposable if and only if there exists a representation $H[G_1, \ldots, G_n]$ of G, with $H \in \mathbf{PRIME}$, $V(H) = \{v_1, \ldots, v_n\}$, such that either for $j = 1$ and $l = 2$ or for $j = 2$ and $l = 1$ the following three conditions hold:*

(i) *$H \in \mathcal{C}(\mathcal{P}_j)$, and*

(ii) *for each $i \in \{1, \ldots, n\}$, $G_i \in \mathcal{C}(\mathcal{P}_l)$, and*

(iii) *for $M = \{i \in \{1, \ldots, n\} : G_i \notin \mathcal{P}_j\}$ and for each $s \in \{1, \ldots, n\} \setminus M$ the subgraph of H induced by $\{v_i : i \in M \cup \{s\}\}$ is in $\mathcal{P}_l$; moreover, if $M = \{1, \ldots, n\}$, then $H \in \mathcal{P}_l$.*

Observe that $\mathbf{PRIME}$ includes only two graphs, K_2, $\overline{K_2}$, with two vertices, no graph on three vertices and only one graph, P_4, with four vertices. Next $\mathcal{C}(\mathcal{O}) = \{K_2\}$, $\mathcal{C}(\mathcal{K}) = \{\overline{K_2}\}$, $\mathcal{C}(\{K_1\}) = \{K_2, \overline{K_2}\}$. Thus if $\mathcal{P} \in \mathbf{L}_{\leq}^* \setminus \{\mathcal{O}, \mathcal{K}, \{K_1\}\}$, then the family $\mathcal{C}(\mathcal{P})$ has to contain at least one graph in $\mathbf{PRIME} \setminus \{K_2, \overline{K_2}\}$. Since each graph on at least 4 vertices contains as an induced subgraph K_2 or $\overline{K_2}$ and graphs in $\mathcal{C}(\mathcal{P})$ are not comparable with respect to induced subgraph relation, we conclude that $\mathcal{C}(\mathcal{P}) \cap \{K_2, \overline{K_2}\} = \emptyset$. Hence we have the following fact.

Remark 6. If $\mathcal{P} \in \mathbf{L}_{\leq}^* \setminus \{\mathcal{O}, \mathcal{K}, \{K_1\}\}$, then $\{K_2, \overline{K_2}\} \subseteq \mathcal{P}$.

Recall that $\mathcal{P}(1) = \mathcal{P} \circ \{K_1\}$ and $\{K_1\} \in \mathbf{L}_{\leq}^*$. Hence, from Theorem 19, we obtain the following immediate consequence.

Corollary 5. *If $\mathcal{P} \in \mathbf{L}^*_{\leq} \setminus \{\mathcal{O}, \mathcal{K}, \{K_1\}\}$, then G is a forbidden subgraph for $\mathcal{P}(1)$ that is strongly decomposable if and only if $G = K_2[H_1, H_2]$ or $G = \overline{K_2}[H_1, H_2] = H_1 \cup H_2$ or $G = H_1[G_1, \ldots, G_n]$, where $H_1, H_2 \in \mathcal{C}(\mathcal{P})$ and $G_1, \ldots, G_n \in \{K_2, \overline{K_2}\}$.*

Proof. We apply Theorem 19 together with the notations. If $\mathcal{P} = \mathcal{P}_j$ and $\{K_1\} = \mathcal{P}_l$, then, by Remark 6, $M = \emptyset$ and the graph induced in H by $\{v_i : i \in M \cup \{s\}\}$ is K_1. Consequently we obtain that $H_1[G_1, \ldots, G_n]$ is forbidden for $\mathcal{P} \circ \{K_1\} = \mathcal{P}(1)$. If $\mathcal{P} = \mathcal{P}_l$ and $\{K_1\} = \mathcal{P}_j$, then H is one of the graphs $K_2, \overline{K_2}$. By Remark 6 we have $M = \{1, 2\}$ and we obtain that $K_2[H_1, H_2]$ and $H_1 \cup H_2$ are graphs in $\mathcal{C}(\mathcal{P}(1))$. Theorem 19 guarantees no other strongly decomposable graphs in $\mathcal{C}(\mathcal{P}(1))$. ∎

In [5] the author explained that an arbitrary graph can be obtained from a prime graph by the iterative substitution of some of its vertices by prime graphs. This procedure corresponds to the well-known construction (which has been discovered many times and is based on the Gallai Theorem [13]) called a *tree decomposition of a graph*. For a given graph G, all prime graphs applied in this tree-like iterative procedure and all their prime induced subgraphs create the unique family denoted by $Z^*(G)$. In the next investigation we use the following fact from this field.

Lemma 20 [5]**.** *Let G, G' be graphs. If $G' \in$ **PRIME**, then $G' \leq G$ if and only if $G' \in Z^*(G)$.*

Consequently we have the following observation.

Lemma 21. *If $\mathcal{P} \in \mathbf{L}^*_{\leq}$ and G is a graph, then $G \in \mathcal{P}$ if and only if $Z^*(G) \subseteq \mathcal{P}$.*

Proof. If $G \in \mathcal{P}$, then all induced subgraphs of G are in $\mathcal{P}$, which means $Z^*(G) \subseteq \mathcal{P}$.

Suppose that $Z^*(G) \subseteq \mathcal{P}$ and, for a contradiction, $G \notin \mathcal{P}$. Hence there is an induced subgraph of G, say F, such that $F \in \mathcal{C}(\mathcal{P})$ (obviously $F \notin \mathcal{P}$). Remark 5 implies that F is prime, which by Lemma 20 leads to $F \in Z^*(G)$, and gives a contradiction. ∎

We use Lemma 21 in proofs of forthcoming results.

Lemma 22. *Let $\mathcal{P} \in \mathbf{L}^*_{\leq}$ and $H_1, H_2 \in \mathcal{C}(\mathcal{P})$. If $v_1, \ldots, v_n$ is an arbitrary ordering of the set $V(H_1)$, then $H_1[H_2, K_1, \ldots, K_1]$ is a forbidden subgraph for $\mathcal{P}(1)$.*

Proof. Let $G = H_1[H_2, K_1, \ldots, K_1]$ and let $V(G) = \{u_1, \ldots, u_l, v_2, \ldots, v_n\}$, where v_1 is substituted with vertices $u_1, \ldots, u_l$ of H_2. Hence for each $i \in \{1, \ldots, l\}$ the vertices $u_i, v_2, \ldots, v_n$ induce H_1 in G.

First we observe that $G - v \notin \mathcal{P}$ for any vertex $v \in V(G)$. Indeed, if $v = v_i$ for some $i \in \{2, \ldots, n\}$, then H_2 is an induced subgraph of $G - v$. If $v = u_i$ for some $i \in \{1, \ldots, l\}$, then H_1 is an induced subgraph of $G - v$.

Now we argue that for each $v \in V(G)$ there is $x \in V(G) \setminus \{v\}$ such that $G - \{v, x\} \in \mathcal{P}$. If $v \in \{v_2, \ldots, v_n\}$, then we choose as x one of the vertices $u_1, \ldots, u_l$. If $v \in \{u_1, \ldots, u_l\}$, then we choose as x one of the vertices $v_2, \ldots, v_n$. In both cases $Z^*(G - \{v, x\})$ contains only proper prime induced subgraphs of H_1 and H_2, which means $Z^*(G - \{v, x\}) \subseteq \mathcal{P}$ and, by Lemma 21, implies $G - \{v, x\} \in \mathcal{P}$. ∎

Lemma 23. *Let $\mathcal{P} \in \mathbf{L}^*_{\leq}$, $H_1, H_2 \in \mathcal{C}(\mathcal{P})$ and $X \in$ **PRIME**. If $v_1, \ldots, v_n$ is an ordering of the set $V(X)$ such that $X[\{v_2, \ldots, v_n\}] = H_1$ and $X - v_i \in \mathcal{P}$ for each $i \in \{2, \ldots, n\}$, then $X[H_2, K_1, \ldots, K_1]$ is a forbidden subgraph for $\mathcal{P}(1)$.*

Proof. Let $G = X[H_2, K_1, \ldots, K_1]$ and let $V(G) = \{u_1, \ldots, u_l, v_2, \ldots, v_n\}$, where v_1 is substituted with vertices $u_1, \ldots, u_l$ of H_2. Thus G contains two disjoint subgraphs H_1, H_2 induced by vertices $v_2, \ldots, v_n$ and $u_1, \ldots, u_l$, respectively. Hence $G \notin \mathcal{P}(1)$.

Now we argue that each pair of vertices u_i, v_j, with $i \in \{1, \ldots, l\}$ and $j \in \{2, \ldots, n\}$ satisfies the condition $G - \{u_i, v_j\} \in \mathcal{P}$. Indeed, $Z^*(G - \{u_i, v_j\})$ contains only prime graphs that are induced subgraphs of $H_2 - u_i$ and $X - v_j$. Both these graphs are in $\mathcal{P}$, which implies $Z^*(G - \{u_i, v_j\}) \subseteq \mathcal{P}$. Lemma 21 yields $G - \{u_i, v_j\} \in \mathcal{P}$, as we desired.

Now we are ready to prove that $G - v \in \mathcal{P}(1)$ for each $v \in V(G)$, which means that for each vertex $v \in V(G)$ there is $x \in V(G) \setminus \{v\}$ such that $G - \{x, v\} \in \mathcal{P}$. If $v = u_i$ for some $i \in \{1, \ldots, l\}$, then we put $x = v_j$ for an arbitrary $j \in \{2, \ldots, n\}$, and if $v = v_j$ for some $j \in \{2, \ldots, n\}$, then we put $x = u_i$ for an arbitrary $i \in \{1, \ldots, l\}$. The earlier consideration confirms that $G - \{x, v\} \in \mathcal{P}$ in both cases. ∎

Theorem 24. *Let $\mathcal{P} \in \mathbf{L}^*_{\leq} \setminus \{\mathcal{O}, \mathcal{K}, \{K_1\}\}$. A graph G is a forbidden subgraph for $\mathcal{P}(1)$ if and only if G has one of the following forms:*

(i) $G = G_1[H_1, H_2]$, *or*

(ii) $G = H_1\big[G_1, \ldots, G_{|V(H_1)|}\big]$, *or*

(iii) $G = H_1[H_2, K_1, \ldots, K_1]$, *or*

(iv) $G = X[H_2, K_1, \ldots, K_1]$, *or*

(v) $G = Y[G_1, \ldots, G_s, K_1, \ldots, K_1]$,

*where $H_1, H_2 \in \mathcal{C}(\mathcal{P})$ and $G_i \in \{K_2, \overline{K_2}\}$ for all permissible i; further $X, Y \in$ **PRIME** and, assuming that $V(X) = \{v_1, \ldots, v_{n_1}\}$ and $V(Y) = \{u_1, \ldots, u_{n_2}\}$, the following conditions are fulfilled:*

- $X[\{v_2, \ldots, v_{n_1}\}] \in \mathcal{C}(\mathcal{P})$, *and*

- *for each* $i \in \{2, \ldots, n_1\}$*,* $X - v_i \in \mathcal{P}$*, and*
- $n_2 \geq s + 2$*, and*
- *for each* $i \in \{1, \ldots, s\}$*,* $Y - u_i \in \mathcal{P}$*, and*
- *for each* $i \in \{s+1, \ldots, n_2\}$*,* $Y - u_i \notin \mathcal{P}$ *and there exists* $j \in \{s+1, \ldots, n_2\} \setminus \{i\}$ *satisfying* $Y - \{u_i, u_j\} \in \mathcal{P}$*.*

Proof. Lemmas 22, 23 and Corollary 5 show that graphs having forms (i), (ii), (iii) or (iv) are forbidden subgraphs for $\mathcal{P}(1)$. Recall that a graph G belongs to $\mathcal{C}(\mathcal{P}(1))$ if the graph resulting by the removal of any vertex of G does not belong to $\mathcal{P}$ and for each vertex $v \in V(G)$ there exists another vertex $x \in V(G)$ such that $G - \{v, x\} \in \mathcal{P}$. Observe that if a graph has the form (v), then it satisfies these conditions. Namely, if v is one of the vertices of G_i with $i \in \{1, \ldots, s\}$, then we choose another vertex of G_i as x. If v is one of the vertices u_i with $i \geq s + 1$, then the role of x is played by u_j given by the assumptions of the theorem. In both cases the conclusion follows by the construction of G.

Corollary 5 characterizes all strongly decomposable graphs in $\mathcal{C}(\mathcal{P}(1))$. It means that to finish the proof it is enough to show that if G is not strongly decomposable and forbidden for $\mathcal{P}(1)$, then G has either the form (iii) or (iv) or (v). The mentioned earlier observation that graphs in $\mathcal{C}(\mathcal{P}(1))$ are pairwise incomparable with respect to the induced subgraph relation allows us to to simplify analysis. Namely, it is enough to show that such G contains as an induced subgraph a graph of one of the forms (i), (ii), (iii), (iv), (v). As a consequence, we observe that G has to be of the corresponding form.

Assume that G is not strongly decomposable. By Theorem 18, Remark 5 and the iterative construction of graphs via prime graphs, we can assume that G has a form $W[U_1, \ldots, U_l, K_1, \ldots, K_1]$, where $W, U_1, \ldots, U_l \in$ **PRIME** and $V(W) = \{w_1, \ldots, w_l, w_{l+1}, \ldots, w_n\}$ with $n \geq l + 1$ (we adopt the convention that $l = 0$ is equivalent to $G = W[K_1, \ldots, K_1] = W$). Moreover, graphs $U_1, \ldots, U_l$ are forbidden subgraphs for $\mathcal{P}$ or are elements of the set $\{K_2, \overline{K_2}\}$.

Suppose that two of the graphs $U_1, \ldots, U_l$, say U_i, U_j, are forbidden subgraphs for $\mathcal{P}$. Hence $K_2[U_i, U_j]$ or $\overline{K_2}[U_i, U_j]$ is an induced subgraph of G depending on whether or not w_i, w_j are adjacent in W. In both cases it leads to the conclusion that G contains an induced subgraph of the form (i).

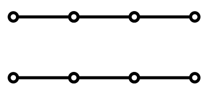
In the next part of the proof we assume that at most one among graphs $U_1, \ldots, U_l$ is in $\mathcal{C}(\mathcal{P})$ and, without loss of generality, only U_1 can be such a graph. Following this assumption $W \notin \mathcal{P}$. If not, then $Z^*(G - v) \subseteq \mathcal{P}$, where v is an arbitrary vertex of U_1 and next, by Remark 6, $G - v \in \mathcal{P}$ giving $G \in \mathcal{P}(1)$, which is impossible. Thus $W \notin \mathcal{P}$.

Now we consider the case $U_1 \in \mathcal{C}(\mathcal{P})$. It means that if $l \geq 2$, then $U_2, \ldots, U_l \in \{K_2, \overline{K_2}\}$. If there is $W' \leq W$ such that $W' \in \mathcal{C}(\mathcal{P})$ with $w_1 \in V(W')$, then G contains an induced subgraph of the form (iii). Otherwise, since $W \notin \mathcal{P}$

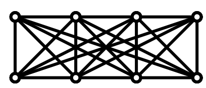
there is $W' \leq W$ such that $W' \in \mathcal{C}(\mathcal{P})$ but $w_1 \notin V(W')$ and moreover, for $W'' = W[\{w_1\} \cup V(W')]$ we have $W'' - x \in \mathcal{P}$ for each $x \in V(W')$. Observe that $W''[U_1, K_1, \ldots, K_1] \leq G$ and $W''[U_1, K_1, \ldots, K_1]$ is of the form (iv), which completes the proof in this case.

Suppose that $U_1 \notin \mathcal{C}(\mathcal{P})$. Hence $G = W[U_1, \ldots, U_l, K_1, \ldots, K_1]$, where $U_1, \ldots, U_l \in \{K_2, \overline{K_2}\}$. Assume that $V(G) = \{w_1^1, w_1^2, \ldots, w_l^1, w_l^2, w_{l+1}, \ldots, w_n\}$, where for $i \in \{1, \ldots, l\}$ w_i is substituted with vertices w_i^1, w_i^2 of either K_2 or $\overline{K_2}$. Next we show that $W - w_i \notin \mathcal{P}$ for $i \in \{l+1, \ldots, n\}$. For a contradiction, let $W - w_i \in \mathcal{P}$ for some i from the range. Hence, because $K_2, \overline{K_2} \in \mathcal{P}$, by Remark 6, we have $Z^*(G - w_i) \subseteq \mathcal{P}$. It implies, by Lemma 21, that $G \in \mathcal{P}(1)$ and gives a contradiction. Therefore $W - w_i \notin \mathcal{P}$ for $i \in \{l+1, \ldots, n\}$. By the definition of $\mathcal{C}(\mathcal{P}(1))$ we know that there exists a vertex $v \in V(G) \setminus \{w_i\}$ such that $G - \{w_i, v\} \in \mathcal{P}$. We ask whether or not v could be w_t^j for some $t \in \{1, \ldots, l\}$ and $j \in \{1, 2\}$. Without loss of generality, let $v = w_t^2$ for some t from the range. Thus $G\big[\{w_1^1, \ldots, w_l^1, w_{l+1}, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n\}\big] = W - w_i$. We observed previously that $W - w_i \notin \mathcal{P}$, which means that $G - \{w_i, w_t^2\} \notin \mathcal{P}$ and excludes this possibility. Thus v must be w_j for some $j \in \{l+1, \ldots, n\} \setminus \{i\}$ and moreover, it implies $n \geq l + 2$. Finally, we show that if $l \geq 1$, then $W - w_i \in \mathcal{P}$ for each $i \in \{1, \ldots, l\}$. If not, then $W - w_i \notin \mathcal{P}$ for some $i \in \{1, \ldots, l\}$. It implies $G - \{w_i^1, w_i^2\} \notin \mathcal{P}$. By the definition of graphs in $\mathcal{C}(\mathcal{P}(1))$ we know that there exists $v \in V(G) \setminus \{w_i^2\}$ such that $G - \{v, w_i^2\} \in \mathcal{P}$. Obviously $v \neq w_i^1$. Moreover, $W - w_t \leq G - \{w_t, w_i^2\}$ for each $t \in \{l+1, \ldots, n\}$ and $W \leq G - \{w_t^j, w_i^2\}$ for each $t \in \{1, \ldots, l\} \setminus \{i\}$ and $j \in \{1, 2\}$. Because $W - w_t \notin \mathcal{P}$ for $t \in \{l+1, \ldots, n\}$ and $W \notin \mathcal{P}$, we obtain a contradiction. Hence we conclude that $W - w_i \in \mathcal{P}$ for each $i \in \{1, \ldots, l\}$. Thus, adopting $l = s$ and $n = n_2$, G satisfies all the conditions that define the form (v) in this case. ∎

In Figures 4, 5(d), 5(e), and 6 we present all possible graphs in $\mathcal{C}(\mathcal{W}_4(1))$ that have forms pointed out in Theorem 24(i), 24(iii) and Theorem 24(iv). Some examples of graphs in $\mathcal{C}(\mathcal{W}_4(1))$ having the construction given by Theorem 24(ii) are shown in Figures 5(a), 5(b), 5(c). Figure 7 illustrates Theorem 24(v). It refers to cases $s = 0$, $s = 1$, $s = 2$, represented by Y being C_5, $\overline{P_5}$, P_6, respectively. It should be mentioned here that the graph in Figure 3 has the form given by Theorem 24(v) with $s = 0$.



$\overline{K}_2[P_4, P_4] = 2P_4$



$K_2[P_4, P_4]$

Figure 4. All the graphs in $\mathcal{C}(\mathcal{W}_4(1))$ of the form given in Theorem 24(i).

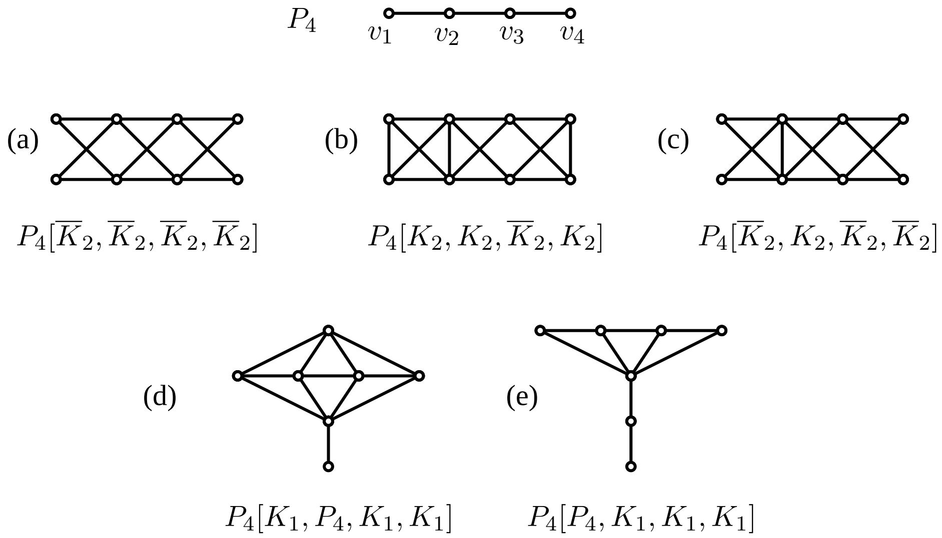


Figure 5. Some examples of graphs in $\mathcal{C}(\mathcal{W}_4(1))$ of the form given in Theorem 24(ii) ((a), (b), (c)) and all the graphs in $\mathcal{C}(\mathcal{W}_4(1))$ of the form given in Theorem 24(iii)) ((d), (e)).

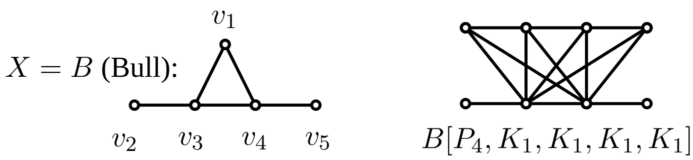


Figure 6. The unique graph in $\mathcal{C}(\mathcal{W}_4(1))$ of the form given in Theorem 24(iv).

7. Concluding Remarks

In this final section we would like to present relations between the concept of a $\mathcal{P}(k)$-apex graph and a concept of an (H, k)-stable graph. According to [12, 16], let H be a fixed graph, a graph G is (H, k)-*stable* whenever the deletion of any set of k edges of G results in a graph that still contains a subgraph isomorphic to H.

An (H, k)-stable graph G is *minimal* if for every $A \subseteq E(G)$, $|A| = k$, there is $e \in E(G) \setminus A$ such that $(G - A) - e$ does not contain a subgraph isomorphic to H. Let us denote by $Stab(H, k)$ the set of all minimal (H, k)-stable graphs.

Proposition 3. *Let k be an integer and H be a graph such that $|V(H)| \geq 4$. Next let $\mathcal{Q}$ be the class of all graphs that do not contain $L(H)$ (the line graph of H) as an induced subgraph. If $G \in Stab(H, k)$, then $L(G) \in \mathcal{C}(\mathcal{Q}(k))$.*

Proof. On the contrary, suppose that $L(G) \notin \mathcal{C}(\mathcal{Q}(k))$. Consider now two cases.

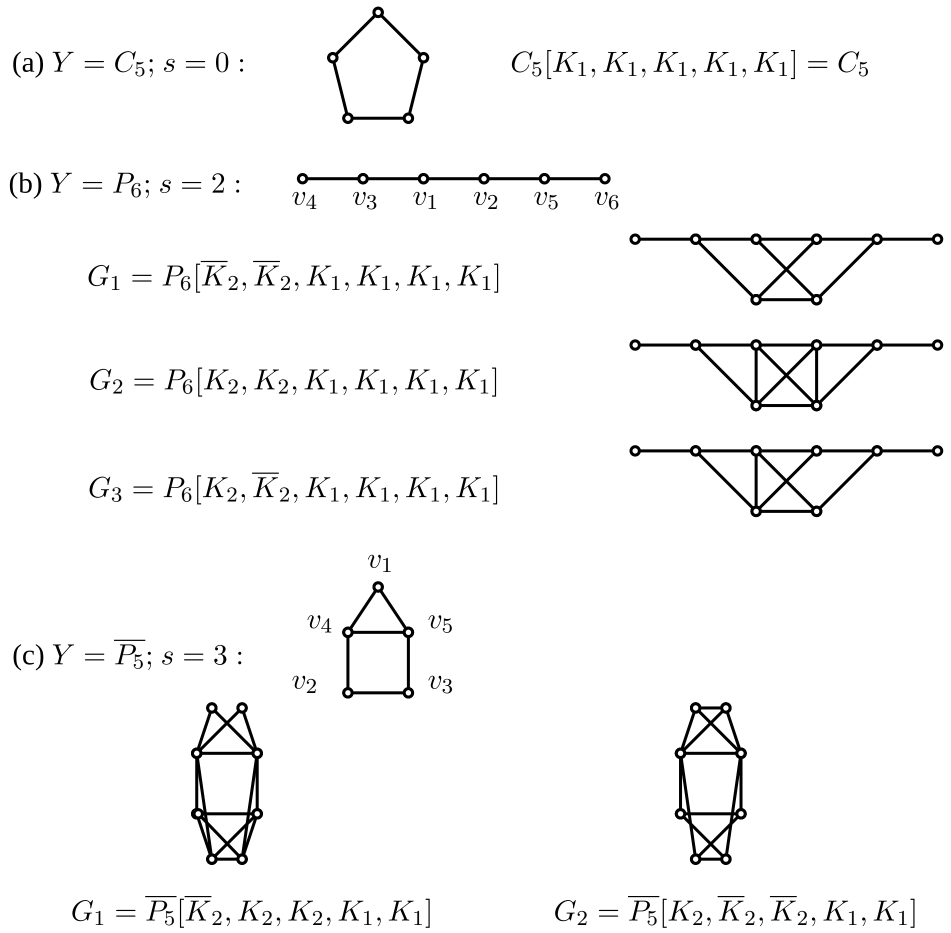


Figure 7. Some examples of graphs in $\mathcal{C}(\mathcal{W}_4(1))$ of the form given in Theorem 24(v).

Case 1. $L(G) \in \mathcal{Q}(k)$. It follows that there is a set $B \subseteq V(L(G)), |B| \leq k$ such that $L(G) - B \in \mathcal{Q}$. The graph $L(G) - B$ is a line graph of some graph G'. Thus $L(G) - B = L(G') \not\geq L(H)$. From Whitney's Theorem [22] and assumptions it follows that $G' \not\geq H$. The graph G' is obtained by removing at most k edges from the graph G which correspond in a unique way to the vertices of the set B. This contradicts our assumption that $G \in Stab(H, k)$.

Case 2. $L(G) \geq F \in \mathcal{C}(\mathcal{Q}(k))$. If $L(G) = F$, then the conclusion is obvious. Suppose that $L(G) \neq F$. Thus F is a line graph of some graph G' which is a proper spanning subgraph of G. Let $e \in E(G) \setminus E(G')$. From the assumption $G \in Stab(H, k)$ it follows that for the edge e there is a set $B' \subseteq E(G) \setminus \{e\}$, $|B'| = k$ such that $(G - e) - B'$ has no subgraph H. Obviously, $|B' \cap E(G')| \leq k$. Since $G' \subseteq G - e$, then $G' - B'$ has no subgraph H. This fact implies that there is a set $A' \subseteq V(F)$, $|A'| = k$ such that $F - A' \in \mathcal{Q}$. This contradicts our assumption that $F \in \mathcal{C}(\mathcal{Q}(k))$ and the proof is complete. ■

In [16] the minimum size of (P_4, k)-stable graphs was determined. In Section 5 of this paper we deal with the minimum and maximum order of graphs in $\mathcal{C}(\mathcal{W}_r(k))$. Since $L(P_{r+1}) = P_r$ we have the following observation.

Corollary 6. *Let k, r be integers, $r \geq 3$. If $G \in Stab(P_{r+1}, k)$, then $L(G) \in \mathcal{C}(\mathcal{W}_r(k))$.*

Let us define a vertex version of the H-stability. Let H be a graph and k be a positive integer. A graph G of order at least k is said to be (H, k)-*vertex stable* if for any set S of k vertices the subgraph $G - S$ contains an induced subgraph isomorphic to H. An (H, k)-vertex stable graph G is *minimal* if for every $W \subseteq V(G)$, $|W| = k$, there is $v \in V(G) \setminus W$ such that $(G - W) - v$ does not contain H. Let us denote by $Stab_V(H, k)$ the set of all minimal (H, k)-vertex stable graphs. Observe the following fact.

Proposition 4. *If k is an integer and H is a connected graph, then $Stab_V(H, k) = \mathcal{C}(\mathcal{P}(k))$, where $\mathcal{P}$ is the class of all graphs that do not contain H as an induced subgraph.*

Proof. If $G \in \mathcal{C}(\mathcal{P}(k))$, then $G - v \in \mathcal{P}(k)$ and $G - v \notin \mathcal{P}(k-1)$ for every $v \in V(G)$. In the case when $G - v \in \mathcal{P}(k-1)$ for an vertex v, then there is a set $A \subseteq V(G)$, $|A| = k - 1$ such that $(G - v) - A \in \mathcal{P}$. This contradicts our assumption that $G \in \mathcal{C}(\mathcal{P}(k))$. It implies that for every set $A \subseteq V(G)$, $|A| = k$ we have $G - A \geq H$, i.e., $G \in Stab_V(H, k)$. Thus, $\mathcal{C}(\mathcal{P}(k)) \subseteq Stab_V(H, k)$.

Now let $G \in Stab_V(H, k)$. Then for every $A \subseteq V(G)$, $|A| = k$, there is $v \in V(G) \setminus A$ such that $(G - A) - v$ does not contain H as an induced subgraph. It follows that for every $v \in V(G)$ there is a set $A \subseteq V(G)$, $|A| = k$ such that $(G - v) - A \in \mathcal{P}$, i.e., $G \in \mathcal{C}(\mathcal{P}(k))$. Hence $Stab_V(H, k) \subseteq \mathcal{C}(\mathcal{P}(k))$. ■

Yet another version of an (H, k)-stable graph was studied in a series of papers [3,6–8,10,11] where the (H, k)-vertex stability was considered taking into account, instead of induced subgraphs, subgraphs of G isomorphic to H. In case of $H = K_q$, both concepts coincide.

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