# $\mathcal{P}$-APEX GRAPHS 

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#### Abstract

Let $\mathcal{P}$ be an arbitrary class of graphs that is closed under taking induced subgraphs and let $\mathcal{C}(\mathcal{P})$ be the family of forbidden subgraphs for $\mathcal{P}$. We investigate the class $\mathcal{P}(k)$ consisting of all the graphs $G$ for which the removal of no more than $k$ vertices results in graphs that belong to $\mathcal{P}$. This approach provides an analogy to apex graphs and apex-outerplanar graphs studied previously. We give a sharp upper bound on the number of vertices of graphs in $\mathcal{C}(\mathcal{P}(1))$ and we give a construction of graphs in $\mathcal{C}(\mathcal{P}(k))$ of relatively large order for $k \geq 2$. This construction implies a lower bound on the maximum order of graphs in $\mathcal{C}(\mathcal{P}(k))$. Especially, we investigate $\mathcal{C}\left(\mathcal{W}_{r}(1)\right)$, where $\mathcal{W}_{r}$ denotes the class of $P_{r}$-free graphs. We determine some forbidden subgraphs for the class $\mathcal{W}_{r}(1)$ with the minimum and maximum number of vertices. Moreover, we give sufficient conditions for graphs belonging to $\mathcal{C}(\mathcal{P}(k))$, where $\mathcal{P}$ is an additive class, and a characterisation of all forests in $\mathcal{C}(\mathcal{P}(k))$. Particularly we deal with $\mathcal{C}(\mathcal{P}(1))$, where $\mathcal{P}$ is a class closed under substitution and obtain a characterisation of all graphs in the corresponding $\mathcal{C}(\mathcal{P}(1))$. In order to obtain desired results we exploit some hypergraph tools and this technique gives a new result in the hypergraph theory.


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## 1. Introduction

We only consider finite and simple graphs and follow [1] for graph-theoretical terminology and notation not defined here. A graph $G$ is an apex graph if it contains a vertex $w$ such that $G-w$ is planar. Although apex graphs seem to be close to planar graphs, some of their properties are far from corresponding properties of planar graphs (for example, see [18]).

A result of Robertson and Seymour (see [19]) says that every proper minorclosed class of graphs $\mathcal{P}$ can be characterized by a finite family of forbidden minors (minor-minimal graphs not in $\mathcal{P}$ ). Evidently, the class of apex graphs is minor-closed but the long-standing problem of finding the complete family of forbidden minors for this class is still open.

However, Dziobak in [9] introduced an apex-outerplanar graph that is a conceptual analogue to an apex graph. Namely, a graph $G$ is apex-outerplanar if there exists $w \in V(G)$ such that $G-w$ is outerplanar. Moreover, Dziobak provided the complete list of 57 forbidden minors for this class.

Another attempt to extend the concept of an apex graph is presented in [20] where an l-apex graph is defined. A graph $G$ is an $l$-apex graph if it can be made planar by removing at most $l$ vertices.

This paper concerns classes of graphs that generalize the aforementioned. Formally, by a class of graphs we mean an arbitrary family of non-isomorphic graphs. The empty class of graphs and the class of all graphs are called trivial. A class of graphs $\mathcal{P}$ is induced hereditary if it is closed with respect to taking induced subgraphs. Such a class $\mathcal{P}$ can be uniquely characterized by the family of forbidden subgraphs $\mathcal{C}(\mathcal{P})$ that is defined as a set
$\{G: G \notin \mathcal{P}$ and $H \in \mathcal{P}$ for each proper induced subgraph $H$ of $G\}$.
By $\mathbf{L}_{\leq}$we denote the class of all non-trivial induced hereditary classes of graphs. Each class $\mathcal{P} \in \mathbf{L}_{\leq}$has a non-empty family of forbidden subgraphs, consisting of graphs with at least two vertices. Moreover, $\mathcal{C}(\mathcal{P})$ contains only connected graphs when $\mathcal{P}$ is additive, i.e., closed under taking the union of disjoint graphs. By $\mathbf{L}_{<}^{a}$ we denote the family of all non-trivial induced hereditary and additive classes of graphs.

Let $\mathcal{P} \in \mathbf{L}_{\leq}$and let $k$ be a non-negative integer. A graph $G$ is a $\mathcal{P}(k)$-apex graph if there is $W \subseteq V(G),|W| \leq k$ ( $W$ is allowed to be the empty set), such that $G-W$ belongs to $\mathcal{P}$. We denote the set of all $\mathcal{P}(k)$-apex graphs by $\mathcal{P}(k)$ for short.

We can see immediately that if $k$ is a non-negative integer and $\mathcal{P} \in \mathbf{L}_{\leq}$, then $\mathcal{P}(k) \in \mathbf{L}_{\leq}$too. On the other hand, the additivity of $\mathcal{P} \in \mathbf{L}_{\leq}$implies the additivity of $\mathcal{P}(k)$ if and only if $k=0$. Indeed, $\mathcal{P}(0)=\mathcal{P}$. Moreover, if $\mathcal{P} \in \mathbf{L}_{\leq}^{a}$, then $\mathcal{C}(\mathcal{P}) \neq \emptyset$ and assuming that $F \in \mathcal{C}(\mathcal{P})$ we can easily see that the union of
$k+1$ disjoint copies of $F$ is in $\mathcal{C}(\mathcal{P}(k))$. Thus, for $k \geq 1$, it yields the existence of at least one disconnected graph that is forbidden for $\mathcal{P}(k)$. Hence, for $k \geq 1$, the class $\mathcal{P}(k)$ is not additive.

Lewis and Yannakakis in [17] have shown that for any non-trivial induced hereditary class $\mathcal{P}$ containing infinitely many graphs and for a given positive integer $k$, the decision problem: "does $G$ belong to $\mathcal{P}(k)$ ?" is NP-complete.

In this paper, we investigate the classes $\mathcal{P}(k)$, in particular we focus on forbidden subgraphs for the classes $\mathcal{P}(k)$ (i.e., we study graphs in $\mathcal{C}(\mathcal{P}(k)))$. Additionally, we use hypergraphs as an effective tool in the research on $\mathcal{P}(k)$.

Let $\mathcal{H}$ be a hypergraph with vertex set $V(\mathcal{H})$ and edge set $\mathcal{E}(\mathcal{H})$ and let $W \subseteq V(\mathcal{H})$. The hypergraph $\mathcal{H}[W]$ induced in $\mathcal{H}$ by $W$ has vertex set $W$ and edge set $\{E \in \mathcal{E}(\mathcal{H}): E \subseteq W\}$. To simplify the notation we write $\mathcal{H}-W$ instead of $\mathcal{H}[V(\mathcal{H}) \backslash W]$ and, moreover, $\mathcal{H}-v$ instead of $\mathcal{H}-\{v\}$ when $v$ is a vertex of $\mathcal{H}$. Analogously, we write $\mathcal{H}-E$ to denote the hypergraph obtained from $\mathcal{H}$ by the deletion of the edge $E$ from $\mathcal{E}(\mathcal{H})$.

By $\mathcal{H}_{1} \cup \mathcal{H}_{2}$ we mean the union of disjoint hypergraphs $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$, i.e., the hypergraph with vertex set $V\left(\mathcal{H}_{1}\right) \cup V\left(\mathcal{H}_{2}\right)$ and edge set $\mathcal{E}\left(\mathcal{H}_{1}\right) \cup \mathcal{E}\left(\mathcal{H}_{2}\right)$. Moreover, notations $2 \mathcal{H}_{1}, \mathcal{H}_{1} \cup \mathcal{H}_{1}$, and their generalization are used interchangeably. The symbol $\mathcal{H}_{1} \leq \mathcal{H}_{2}$ denotes that the hypergraph $\mathcal{H}_{1}$ is isomorphic to a subhypergraph of $\mathcal{H}_{2}$ induced by some of its vertex subset. Let $r$ be a non-negative integer. A hypergraph $\mathcal{H}$ is $r$-uniform if each edge in $\mathcal{E}(\mathcal{H})$ has exactly $r$ vertices. A set $T \subseteq V(\mathcal{H})$ is called a transversal of the hypergraph $\mathcal{H}$ if $T \cap E \neq \emptyset$ for each $E \in \mathcal{E}(\mathcal{H})$. By $\tau(\mathcal{H})$ we denote the cardinality of the minimum transversal of $\mathcal{H}$, i.e.,

$$
\tau(\mathcal{H})=\min \{|T|: T \text { is a transversal of } \mathcal{H}\} .
$$

A hypergraph $\mathcal{H}$ is $\tau$-vertex critical if for any $v \in V(\mathcal{H})$ the inequality $\tau(\mathcal{H}-v) \leq$ $\tau(\mathcal{H})-1$ holds. If a $\tau$-vertex critical hypergraph $\mathcal{H}$ satisfies $\tau(\mathcal{H})=l$ for some positive integer $l$, then we call it $\tau$-vertex $l$-critical.

Recall that each graph is a hypergraph, which allows us to use these notations also for graphs. The symbols $K_{n}, P_{n}, C_{n}$ are used only for graphs and denote the complete graph, the path and the cycle with $n$ vertices, respectively.

This paper is organized as follows. We start with $\tau$-vertex $l$-critical hypergraphs in Section 2. We prove an upper bound on the order of a $\tau$-vertex 2 -critical hypergraph and describe the construction of $\tau$-vertex $l$-critical hypergraphs with large number of vertices. Next, in Section 3, we prove some results on relations between $\tau$-vertex ( $k+1$ )-critical hypergraphs and graphs in $\mathcal{C}(\mathcal{P}(k))$ for $\mathcal{P} \in \mathbf{L}_{\leq}$. In Section 4, for $\mathcal{P} \in \mathbf{L}_{\leq}^{a}$ we show some sufficient conditions that have to be satisfied by a graph to be in $\mathcal{C}(\mathcal{P}(k))$ and we characterize all forests in $\mathcal{C}(\mathcal{P}(k))$. Section 5 deals with the class $\mathcal{P}$ of graphs that does not contain $P_{r}$ as an induced subgraph. We determine some forbidden subgraphs for $\mathcal{P}(1)$ with minimum and maximum order in this case. In Section 6 we characterize all graphs in $\mathcal{C}(\mathcal{P}(1))$,
where $\mathcal{P}$ is a class of graphs that is induced hereditary and closed under substitution (for the definition see Section 6).

## 2. $\tau$-Vertex Critical Hypergraphs

A hypergraph $\mathcal{H}$ is $\tau$-edge $l$-critical if $\tau(\mathcal{H})=l$ and the deletion of an edge decreases the transversal number of the resulting hypergraph. It is clear that the class of $\tau$-edge $l$-critical hypergraphs without isolated vertices forms a subclass of the class of $\tau$-vertex $l$-critical hypergraphs. On the other hand, it is easy to prove that the maximum order of hypergraphs in both classes is the same. In this section we prove that an $r$-uniform $\tau$-vertex 2 -critical hypergraph has at most $\left\lfloor\frac{(r+2)^{2}}{4}\right\rfloor$ vertices. Our proof is different than Tuza's proof in [21] concerning a corresponding theorem for $r$-uniform $\tau$-edge 2-critical hypergraphs.

Next, for $l \geq 3$ we give the construction of an $r$-uniform $\tau$-vertex $l$-critical hypergraph with a large order. Gyárfás et al. [15] proved that each $r$-uniform $\tau$ vertex $l$-critical hypergraph has order bounded from above by $\binom{l+r-2}{r-2} l+l^{r-1}$. This bound is probably far from the exact value of the maximum number of vertices in a hypergraph that is $r$-uniform $\tau$-vertex $l$-critical. Our construction gives a large lower bound on the maximum order of a hypergraph that is $r$-uniform $\tau$-vertex l-critical.

Theorem 1. Let $r$ be an integer, $r \geq 2$, and let $\mathcal{H}$ be a $\tau$-vertex 2-critical hypergraph. If for each $E \in \mathcal{E}(\mathcal{H})$ we have $|E| \leq r$, then

$$
|V(\mathcal{H})| \leq\left\lfloor\frac{(r+2)^{2}}{4}\right\rfloor
$$

Moreover, the bound is sharp.
Proof. Denote by $\mathcal{H}^{\prime}$ a hypergraph obtained from $\mathcal{H}$ by the optional deletion of some edges in such a way that $\tau(\mathcal{H})=\tau\left(\mathcal{H}^{\prime}\right)=2$ and $\tau\left(\mathcal{H}^{\prime}-E^{\prime}\right) \leq 1$ for each edge $E^{\prime}$ of $\mathcal{H}^{\prime}$. Let $\mathcal{E}^{\prime}=\mathcal{E}\left(\mathcal{H}^{\prime}\right)$ and assume $\mathcal{E}^{\prime}=\left\{E_{1}^{\prime}, \ldots, E_{m}^{\prime}\right\}$. Observe that each vertex of $\mathcal{H}^{\prime}$ is contained in at least one of the edges in $\mathcal{E}\left(\mathcal{H}^{\prime}\right)$. Otherwise, if there is $x \in V\left(\mathcal{H}^{\prime}\right)$ such that $x$ belongs to no edge in $\mathcal{E}\left(\mathcal{H}^{\prime}\right)$, then $\tau(\mathcal{H}-x)=2$ giving a contradiction to the $\tau$-vertex criticality of $\mathcal{H}$.

Let a bipartite graph $B$ be the incidence graph of the hypergraph $\mathcal{H}^{\prime}$. Thus $B=\left(V(\mathcal{H}), \mathcal{E}^{\prime} ; E(B)\right)$, where $v E^{\prime} \in E(B)$ if and only if $v \in E^{\prime}$. The previous consideration says that $d_{B}(v) \geq 1$ for all $v \in V(\mathcal{H})$ and $d_{B}\left(E_{i}^{\prime}\right) \leq r$ for all $i \in\{1, \ldots, m\}$. The last condition implies $|E(B)| \leq m r$.
Claim 2. For every $E_{i}^{\prime}$ there is a vertex, say $v_{i} \in V(\mathcal{H}) \subseteq V(B)$, such that $v_{i} \notin E_{i}^{\prime}$ but $v_{i} \in E_{j}^{\prime} \in \mathcal{E}^{\prime}$ for all $j \neq i$.

Proof. Delete a vertex $E_{i}^{\prime}$ from the graph $B$. The graph $B-E_{i}^{\prime}$ is an incidence graph of the hypergraph $\mathcal{H}^{\prime}-E_{i}^{\prime}$, so $\tau\left(\mathcal{H}^{\prime}-E_{i}^{\prime}\right)=1$, i.e., there is a vertex, say $x$, which is adjacent in $B$ to every $E_{j}^{\prime}, j \neq i$. Obviously the vertex $x$ is not adjacent to $E_{i}^{\prime}$, otherwise in the hypergraph $\mathcal{H}^{\prime}$ there would be a 1-element transversal $\{x\}$, which is impossible. Thus $x$ can play the role of $v_{i}$ from the statement.

By Claim 2, in the graph $B$ there is a set of $m$ vertices $\left\{v_{1}, \ldots, v_{m}\right\}$ with $d_{B}\left(v_{i}\right)=m-1$, for $i \in\{1, \ldots, m\}$. Since $d_{B}(v) \geq 1$ for each $v \in V(\mathcal{H})$ we have $m(m-1)+(n-m) \leq|E(B)| \leq m r$, where $n=|V(\mathcal{H})|$. It leads to the inequality $n \leq-m^{2}+(r+2) m$. Thus for fixed $r$, the maximum $n$ is $\left\lfloor\frac{(r+2)^{2}}{4}\right\rfloor$ and it is achieved at $m=\left\lfloor\frac{r}{2}\right\rfloor+1$ or at $m=\left\lceil\frac{r}{2}\right\rceil+1$.

Finally, we prove that the bound is sharp. All the previous arguments imply that the structure of the $\tau$-vertex 2 -critical hypergraph with maximum number of vertices must be defined in the following way. For $m=\left\lfloor\frac{r}{2}\right\rfloor+1$ or $\left\lceil\frac{r}{2}\right\rceil+1$ let $U=\{1, \ldots, m\}$ and let $A_{i}=\left\{a_{1}^{i}, \ldots, a_{r+1-m}^{i}\right\}$ with $i \in U$. The $r$-uniform hypergraph $\mathcal{H}$ such that $V(\mathcal{H})=U \cup \bigcup_{i=1}^{m} A_{i}$ and $E(\mathcal{H})=\left\{E_{1}, \ldots, E_{m}\right\}$ where $E_{i}=(U \backslash\{i\}) \cup A_{i}$ for $i \in\{1, \ldots, m\}$, confirms the sharpness of the inequality given in the assertion.

The construction from the proof of Theorem 1 can be generalized in an easy way resulting in the following $r$-uniform $\tau$-vertex $l$-critical hypergraph with a large number of vertices.

Construction 1. Let $k, r, x$ be integers, $k \geq 1, r \geq 3$ and $r \geq x \geq 1$ and let $U=\{1, \ldots, k, k+1, \ldots, k+x\}$. Next let $m=\binom{k+x}{x}$ and let $\left\{U_{1}, \ldots, U_{m}\right\}$ be the family of all $x$-element subsets of $U$. Additionally, let $A_{i}=\left\{a_{1}^{i}, \ldots, a_{r-x}^{i}\right\}$ with $i \in\{1, \ldots, m\}$ be $m$ pairwise disjoint sets each of which is also disjoint with $U$.

We define an $r$-uniform hypergraph $\mathcal{H}^{*}=\mathcal{H}^{*}(k, r, x)$ in the following way: $E\left(\mathcal{H}^{*}\right)=\left\{E_{1}, \ldots, E_{m}\right\}$, where $E_{i}=U_{i} \cup A_{i}, i \in\{1, \ldots, m\} ;$ $V\left(\mathcal{H}^{*}\right)=\bigcup_{i=1}^{m} E_{i}=U \cup A$, where $A=\bigcup_{i=1}^{m} A_{i}$.

Theorem 3. If $k, r, x$ are integers such that $k \geq 1, r \geq 3$ and $r \geq x \geq 1$, then $\mathcal{H}^{*}(k, r, x)$ is $\tau$-vertex $(k+1)$-critical.

Proof. Let $\mathcal{H}^{*}(k, r, x)=\mathcal{H}^{*}$. We use the notations connected with $\mathcal{H}^{*}$ given in Construction 1. Observe that an arbitrary $(k+1)$-element subset of $U$ is a transversal of $\mathcal{H}^{*}$. Thus $\tau\left(\mathcal{H}^{*}\right) \leq k+1$. Suppose, for a contradiction, that $T$ is a transversal of $\mathcal{H}^{*}$ and $|T| \leq k$. If $T \subseteq U$, then $U \backslash T$ contains at least one $x$-element subset $U_{i}$ and consequently $E_{i}$ is an edge of $\mathcal{H}^{*}-T$. Hence $T$ is not a transversal of $\mathcal{H}^{*}$, a contradiction. Thus $T \backslash U=S \neq \emptyset$. Denote $t=|T \cap U|$ and $s=|S|$. There are at least $\binom{k+x-t}{x}$ edges of $\mathcal{H}^{*}$ each of which has nonempty
intersection with $S$. It follows $\binom{k+x-t}{x} \leq s$. Recall that $s+t \leq k$. It means $\binom{k+x-t}{x} \leq k-t$, which is impossible for any $x$ satisfying $r \geq x \geq 1$.

To observe the $\tau$-vertex criticality of $\mathcal{H}^{*}$ it is enough to show that for each $v \in V\left(\mathcal{H}^{*}\right)$ the condition $\tau\left(\mathcal{H}^{*}-v\right) \leq k$ holds. If $v \in U$, then the removal of any $k$ vertices of $U$, all different from $v$, results in a hypergraph without edges. If $v \in A_{i}$ for some $i \in\{1, \ldots, m\}$, then the $k$-element transversal $U \backslash U_{i}$ realizes the inequality $\tau\left(\mathcal{H}^{*}-v\right) \leq k$.

In the next lemma we find the maximum order of $\mathcal{H}^{*}(k, r, x)$. This result gives a lover bound on the maximum number of vertices in an $r$-uniform $\tau$-vertex ( $k+1$ )-critical hypergraph.

Given $k, r$ we introduce $n(x)=\binom{k+x}{x}(r-x)+k+x=\binom{k+x}{k}(r-x)+k+x$. Lemma 4. If $k, r$ are integers such that $k \geq 1, r \geq 3$, then

$$
\max _{1 \leq x \leq r}\left|V\left(\mathcal{H}^{*}(k, r, x)\right)\right|=\max _{1 \leq x \leq r} n(x)=n\left(\left\lceil\frac{k(r-1)}{k+1}\right\rceil\right) .
$$

Proof. By Construction 1 we have $\max _{1 \leq x \leq r}\left|V\left(\mathcal{H}^{*}(k, r, x)\right)\right|=\max _{1 \leq x \leq r} n(x)$. Consider the difference function $D(x)=n(x)-n(x+1)=-1+\binom{k+x}{k}[(r-x)-$ $\left.\frac{k+x+1}{x+1}((r-x)-1)\right]=-1+\binom{k+x}{k} \frac{(r-x)(-k)+k+x+1}{x+1}=-1+\frac{(k+x)!}{k!(x+1)!}[(x+1)(k+1)-$ $k r]=-1+\frac{1}{x+1} \prod_{i=1}^{k}\left(1+\frac{x}{i}\right)[(x+1)(k+1)-k r]$.

Since $x, k$ and $r$ are positive integers, $D(x) \geq 0$ if and only if $(x+1)(k+1)-$ $k r \geq 1$ and therefore the maximum $n(x)$ is reached at the smallest $x$ such that $D(x) \geq 0$, i.e., at $x=\left\lceil\frac{k(r-1)}{k+1}\right\rceil$.

## 3. Graph Approach

In this section we formulate some results on relations between $\tau$-vertex $(k+1)$ critical hypergraphs and forbidden subgraphs for $\mathcal{P}(k)$. They are preceded by the helpful lemmas.
Lemma 5. Let $k$ be a non-negative integer and $\mathcal{P} \in \mathbf{L}_{\leq}$. If $F \in \mathcal{C}(\mathcal{P}(k))$, then $F \in \mathcal{P}(k+1) \backslash \mathcal{P}(k)$.
Proof. By the definition of $\mathcal{C}(\mathcal{P}(k))$ it follows that $F \notin \mathcal{P}(k)$. Moreover, for an arbitrary $v \in V(F)$ we have $F-v \in \mathcal{P}(k)$. It means that there exists a set $W$, contained in $V(F-v)$, such that $|W| \leq k$ and $(F-v)-W \in \mathcal{P}$. Because $|W \cup\{v\}| \leq k+1$ it leads to $F \in \mathcal{P}(k+1)$.

Let $\mathcal{P} \in \mathbf{L}_{\leq}$and $G$ be a graph. By $\mathcal{H}_{\mathcal{P}}(G)$ we denote a hypergraph whose vertex set is $V \overline{( } G)$ and whose edge set is $\{W \subseteq V(G): G[W] \in \mathcal{C}(\mathcal{P})\}$. Note the following facts.

Remark 1. Let $k$ be a non-negative integer, $\mathcal{P} \in \mathbf{L}_{\leq}$and $G$ be a graph.
(i) $G \in \mathcal{P}(k)$ if and only if $\tau\left(\mathcal{H}_{\mathcal{P}}(G)\right) \leq k$.
(ii) $G \in \mathcal{P}(k+1) \backslash \mathcal{P}(k)$ if and only if $\tau\left(\mathcal{H}_{\mathcal{P}}(G)\right)=k+1$.

Lemma 6. Let $k$ be a non-negative integer and $\mathcal{P} \in \mathbf{L}_{\leq}$. A graph $G$ is a forbidden subgraph for $\mathcal{P}(k)$ if and only if $\mathcal{H}_{\mathcal{P}}(G)$ is $\tau$-vertex $(k+1)$-critical.

Proof. Suppose that $G \in \mathcal{C}(\mathcal{P}(k))$. By Lemma 5 and Remark 1, $\tau\left(\mathcal{H}_{\mathcal{P}}(G)\right)=$ $k+1$. Moreover, for each $v \in V(G)$ we have $G-v \in \mathcal{P}(k)$, which again by Remark 1 implies $\tau\left(\mathcal{H}_{\mathcal{P}}(G-v)\right) \leq k$. Since $\mathcal{H}_{\mathcal{P}}(G-v)=\mathcal{H}_{\mathcal{P}}(G)-v$ we conclude that $\mathcal{H}_{\mathcal{P}}(G)$ is $\tau$-vertex $(k+1)$-critical.

Now assume that $\mathcal{H}_{\mathcal{P}}(G)$ is $\tau$-vertex $(k+1)$-critical. Remark 1 and the equality $\mathcal{H}_{\mathcal{P}}(G-v)=\mathcal{H}_{\mathcal{P}}(G)-v$ yield $G \in \mathcal{P}(k+1) \backslash \mathcal{P}(k)$ and $G-v \in \mathcal{P}(k)$ for each $v \in V(G)$. Hence $G \in \mathcal{C}(\mathcal{P}(k))$.

Lemma 6 and Theorem 3 make it easy to formulate one more observation.
Corollary 1. Let $k, r, x$ be integers such that $k \geq 1, r \geq 3, r \geq x \geq 1$ and let $\mathcal{P} \in \mathbf{L}_{\leq}$. If $G$ is a graph such that $\mathcal{H}_{\mathcal{P}}(G)$ is isomorphic to $\mathcal{H}^{*}(k, r, x)$ defined in Construction 1, then $G$ is a forbidden subgraph for $\mathcal{P}(k)$.

A graph $G$ is a host-graph of a hypergraph $\mathcal{H}$ if $V(G)=V(\mathcal{H})$ and for each edge $e$ of $G$ there is an edge $E$ of $\mathcal{H}$ satisfying $e \subseteq E$. For an arbitrary family $\mathcal{F}$ of graphs, a graph $G$ is an $\mathcal{F}$-host-graph of a hypergraph $\mathcal{H}$ when it is a host-graph of $\mathcal{H}$ such that $G[E] \in \mathcal{F}$ for each edge $E$ of $\mathcal{H}$.


Figure 1. The example of a host-graph of a hypergraph.
Observe that for a given family of graphs $\mathcal{F}$ and a hypergraph $\mathcal{H}$ an $\mathcal{F}$-hostgraph of a hypergraph $\mathcal{H}$ does not necessarily exist. However, we can easily find a family $\mathcal{F}$ and a hypergraph $\mathcal{H}$ having an $\mathcal{F}$-host-graph. As an example, for a
fixed positive integer $r$, take $\mathcal{F}=\left\{K_{r}\right\}$ and any $r$-uniform hypergraph $\mathcal{H}$ (see Figure 1).

Furthermore, if $G$ is a $\mathcal{C}(\mathcal{P})$-host-graph of a hypergraph $\mathcal{H}$ then $\mathcal{H}_{\mathcal{P}}(G)$ is not necessarily isomorphic to $\mathcal{H}$ (see Figure 1 again). We use $\mathcal{C}(\mathcal{P})$-host-graphs to describe forbidden subgraphs for $\mathcal{P}(k)$ with large number of vertices. In Section 2 , we have constructed the family of hypergraphs $\mathcal{H}^{*}(k, r, x)$ that are $r$-uniform $\tau$-vertex $(k+1)$-critical and have large number of vertices. So, a $\mathcal{C}(\mathcal{P})$-host-graph of a hypergraph $\mathcal{H}^{*}(k, r, x)$ could be potentially a forbidden subgraph for $\mathcal{P}(k)$. First we give some examples of families $\mathcal{F}$ of graphs for which an $\mathcal{F}$-host-graph of $\mathcal{H}^{*}$ from Construction 1 exists.

Let $G$ be a graph. The symbols $\omega(G)$ and $\alpha(G)$ denote the order of the maximum clique and the cardinality of the maximum independent set of $G$, respectively.
Lemma 7. Let $\mathcal{F}$ be a family of graphs. Next let $k, r, x$ be integers, $k \geq 1, r \geq 3$, $r>x \geq 1$ and $\mathcal{H}^{*}=\mathcal{H}^{*}(k, r, x)$ be a hypergraph from Construction 1 .
(i) If there is $F \in \mathcal{F}$ such that $|V(F)|=r$ and $\omega(F) \geq x$, then there exists an $\mathcal{F}$-host-graph of the hypergraph $\mathcal{H}^{*}$.
(ii) If there is $F \in \mathcal{F}$ such that $|V(F)|=r$ and $\alpha(F) \geq x$, then there exists an $\mathcal{F}$-host-graph of the hypergraph $\mathcal{H}^{*}$.
(iii) If there is $F \in \mathcal{F}$ such that $|V(F)|=r$ and moreover $r \geq x+k$, then there exists an $\mathcal{F}$-host-graph of the hypergraph $\mathcal{H}^{*}$.
Proof. Using the notations from Construction 1 we show how to obtain an $\mathcal{F}$ -host-graph $G$ of the hypergraph $\mathcal{H}^{*}$. First we prove statements (i) and (ii). In the hypergraph $\mathcal{H}^{*}$ we add all the edges between vertices in $U$ to obtain $K_{x+k}$ for (i) and we leave $U$ independent for (ii). Then we choose $F \in \mathcal{F}$ such that $|V(F)|=r$ and $\omega(F) \geq x$ (for (i)) or $\alpha(F) \geq x$ (for (ii)). Now in each set $A_{i}$ from Construction 1 we enter a part of $F$ such that each $E_{i}$ induces $F$ in $G$. Observe that the assumption $\omega(F) \geq x$ or $\alpha(F) \geq x$ guarantees that all steps of this procedure can be done. To construct an $\mathcal{F}$-host-graph $G$ for (iii) we choose an arbitrary vertex subset $W$ of $F$ of the cardinality $k+x$. Such a subset always exists since $r \geq k+x$. Next, we join some of the vertices in $U$ by edges in such a way that the resulting graph is isomorphic to $F[W]$. Then, similarly to above, in each set $A_{i}$ from Construction 1 we enter a part of the graph $F$ such that each $E_{i}$ induces $F$ in the graph $G$.

Consider $\mathcal{P} \in \mathbf{L}_{\leq}$and a hypergraph $\mathcal{H}^{*}=\mathcal{H}^{*}(k, r, x)$. As we mentioned before if $G$ is a $\mathcal{C}(\mathcal{P})$-host-graph of a hypergraph $\mathcal{H}$, then $\mathcal{H}_{\mathcal{P}}(G)$ may be nonisomorphic to $\mathcal{H}$. Hence we do not know whether a $\mathcal{C}(\mathcal{P})$-host-graph of $\mathcal{H}^{*}$ is a forbidden subgraph for $\mathcal{P}(k)$ or not. In the next theorem, we solve this problem positively for some cases, regardless of whether the hypergraphs $\mathcal{H}_{\mathcal{P}}(G)$ and $\mathcal{H}^{*}$ are isomorphic.

A set $S$ is a vertex-cut-set in a connected graph $G$ if $G-S$ has at least two connected components. For a positive integer $x$, a connected graph $G$ is $x$-vertex connected if it does not contain any vertex-cut-set of the cardinality less than $x$. As usual, for a given graph $G$ and $v \in V(G)$, we denote by $N_{G}(v)$ the set of neighbours of $v$ in $G$.

Theorem 8. Let $k, r, x$ be integers, $k \geq 1, r \geq 3, r>x \geq 1$, and let $\mathcal{H}^{*}=$ $\mathcal{H}^{*}(k, r, x)$ be the hypergraph from Construction 1. If $\mathcal{P} \in \mathbf{L}_{\leq}$is a class of graphs such that $\mathcal{C}(\mathcal{P})$ consists only of $x$-vertex connected graphs of order at least $r$, then each $\mathcal{C}(\mathcal{P})$-host-graph of the hypergraph $\mathcal{H}^{*}$ is a forbidden subgraph for $\mathcal{P}(k)$.

Proof. In the proof we refer to the notations from Construction 1. Let $G$ be an arbitrary $\mathcal{C}(\mathcal{P})$-host-graph of the hypergraph $\mathcal{H}^{*}$. Applying Lemma 6, the aim is to show that $\mathcal{H}_{\mathcal{P}}(G)$ is $\tau$-vertex $(k+1)$-critical.

First we prove that any $(k+1)$-element subset $W$ of $U$ is a transversal of $\mathcal{H}_{\mathcal{P}}(G)$, i.e., for any $(k+1)$-element subset $W$ of $U$, the graph $G-W$ does not contain any induced subgraph $F$ satisfying $F \in \mathcal{C}(\mathcal{P})$. Suppose that this is not the case and let $F$ be a subgraph of $G-W$ such that $F \in \mathcal{C}(\mathcal{P})$. Denote by $U_{1}^{\prime}, \ldots, U_{m}^{\prime}$ the subsets of $V(G-W)$ that correspond to $U_{1}, \ldots, U_{m}$ in $G$. Thus, $\left|U_{i}^{\prime}\right| \leq x-1$ for each $i \in\{1, \ldots, m\}$. Furthermore, since $r>x$, it follows that $V(F)$ is not contained in $U-W$ and consequently $F$ must contain at least one vertex of some $A_{i}$ with $i \in\{1, \ldots, m\}$. Because of the symmetry, we may assume that $A^{\prime}=A_{1} \cap V(F) \neq \emptyset$. Since $\left|A^{\prime} \cup U_{1}^{\prime}\right|<r$, there is a vertex of $F$ that does not belong to $A^{\prime} \cup U_{1}^{\prime}$. Hence, we can divide vertices of $F$ into three parts $V_{1}=V(F) \cap A^{\prime}, V_{2}=V(F) \cap U_{1}^{\prime}$ and $V_{3}=V(F) \backslash\left(V_{1} \cup V_{2}\right)$. By our earlier observation $V_{3} \neq \emptyset$. Since $N_{G}\left(A_{1}\right) \subseteq U_{1}$, it follows that $N_{F}\left(V_{1}\right) \subseteq V_{2}$. Thus, $V_{2}$ is a vertex-cut-set of $F$. Furthermore, $\left|V_{2}\right| \leq\left|U_{1}^{\prime}\right| \leq x-1$, which contradicts that $F$ is $x$-vertex connected and proves $\tau\left(\mathcal{H}_{\mathcal{P}}(G)\right) \leq k+1$. Recall that, by the construction of $G$, each edge of $\mathcal{H}^{*}$ is an edge of $\mathcal{H}_{\mathcal{P}}(G)$. It means, by Theorem 3 , that $\tau\left(\mathcal{H}_{\mathcal{P}}(G)\right) \geq k+1$ and consequently $\tau\left(\mathcal{H}_{\mathcal{P}}(G)\right)=k+1$.

Now, we prove the $\tau$-vertex criticality of $\mathcal{H}_{\mathcal{P}}(G)$. By Remark 1 and the fact that $\mathcal{H}_{\mathcal{P}}(G-v)=\mathcal{H}_{\mathcal{P}}(G)-v$, we have to argue that for any $i \in\{1, \ldots, m\}$ and for any $v \in A_{i}$ we obtain $G-v \in \mathcal{P}(k)$. Let $W^{\prime}=U-U_{i}$. Observe that $\left|W^{\prime}\right|=k$ and $U_{j} \cap W^{\prime} \neq \emptyset$ for $j \neq i$. We show that $(G-v)-W^{\prime} \in \mathcal{P}$ or equivalently that $(G-v)-W^{\prime}$ does not contain an induced subgraph isomorphic to any $F \in \mathcal{C}(\mathcal{P})$. Let $U_{1}^{\prime \prime}, \ldots, U_{m}^{\prime \prime}$ be subsets of $V\left(G-W^{\prime}\right)$ that correspond to $U_{1}, \ldots, U_{m}$ in $G$. Thus, $\left|U_{j}^{\prime \prime}\right| \leq x-1$ for each $j \neq i$ and $\left|U_{i}^{\prime \prime}\right|=x$. Suppose that there is $F \in \mathcal{C}(\mathcal{P})$ such that $F \leq(G-v)-W^{\prime}$. It is clear that there is $j \neq i$ such that $F$ contains at least one vertex of $A_{j}$. Therefore, similarly as above, we can divide $V(F)$ into three parts $V_{1}=V(F) \cap A_{j}, V_{2}=V(F) \cap U_{j}^{\prime}$ and $V_{3}=V(F) \backslash\left(V_{1} \cup V_{2}\right)$ with $V_{3} \neq \emptyset$. Since $N_{F}\left(V_{1}\right) \subseteq V_{2}$, the set $V_{2}$ is a vertex cut-set of $F$, contrary to the $x$-vertex connectivity of $F$.

Theorem 8 gives us a very fruitful tool to construct forbidden subgraphs for $\mathcal{P}(k)$.

Corollary 2. Let $k, x$ be positive integers and let $\mathcal{P} \in \mathbf{L} \leq$ be a class of graphs such that each graph in $\mathcal{C}(\mathcal{P})$ is $x$-vertex connected of order at least $r$. If $r$ is the order of some $F \in \mathcal{C}(\mathcal{P})$ and $r \geq 3$, and $r \geq k+x$, then there exists $G$ that is a forbidden subgraph for $\mathcal{P}(k)$ and $|V(G)|=\bar{k}+x+\binom{k+x}{x}(r-x)$.

Theorem 9. Let $\mathcal{P} \in \mathbf{L}_{\leq}$. If $r=\max \{|F|: F \in \mathcal{C}(\mathcal{P})\}$ and $G \in \mathcal{C}(\mathcal{P}(1))$, then $|V(G)| \leq\left\lfloor\frac{(r+2)^{2}}{4}\right\rfloor$. Moreover, this bound is achieved for infinitely many classes $\mathcal{P} \in \mathbf{L}_{\leq}$.

Proof. By Lemma 6 and Theorem 1 we only need to show the last sentence of the statement. However, if we put $k=1$ and $x=\left\lceil\frac{r-1}{2}\right\rceil$ in Corollary 2, then for $r \geq 3$ we obtain a forbidden subgraph for $\mathcal{P}(k)$ with $\left\lfloor\frac{(r+2)^{2}}{4}\right\rfloor$ vertices and hence the theorem follows.

The next remark is an immediate consequence of Theorem 9 and the fact that $(\mathcal{P}(k))(1)=\mathcal{P}(k+1)$.

Remark 2. Let $k$ be a non-negative integer and $\mathcal{P} \in \mathbf{L}_{\leq}$. If $\mathcal{C}(\mathcal{P})$ is finite, then the family $\mathcal{C}(\mathcal{P}(k))$ is also finite.

## 4. The Structure of Forbidden Subgraphs

At the beginning of this section we describe connected forbidden subgraphs for $\mathcal{P}(k)$ in terms of connected forbidden subgraphs for $\mathcal{P}(l)$, where $l<k$. To do it we use the following hypergraph tool.

Remark 3. If $\mathcal{H}_{1} \cup \mathcal{H}_{2}$ is the union of disjoint hypergraphs $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$, then

$$
\tau\left(\mathcal{H}_{1} \cup \mathcal{H}_{2}\right)=\tau\left(\mathcal{H}_{1}\right)+\tau\left(\mathcal{H}_{2}\right)
$$

Note that the definition of the $\tau$-vertex criticality of a hypergraph and Remark 3 imply the following observation.

Remark 4. Let $s$ be an integer, $s \geq 2$. The union $\mathcal{H}_{1} \cup \cdots \cup \mathcal{H}_{s}$ of disjoint hypergraphs $\mathcal{H}_{1}, \ldots, \mathcal{H}_{s}$ is $\tau$-vertex critical if and only if for each $i \in\{1, \ldots, s\}$ the hypergraph $\mathcal{H}_{i}$ is $\tau$-vertex critical.

The next result is the consequence of Remark 4.

Theorem 10. Let $k, s$ be integers, $k \geq 0, s \geq 1$ and $\mathcal{P} \in \mathbf{L}_{\leq}^{a}$. The union $F_{1} \cup \cdots \cup F_{s}$ of disjoint connected graphs $F_{1}, \ldots, F_{s}$ is a forbidden subgraph for $\mathcal{P}(k)$ if and only if there exist non-negative integers $k_{1}, \ldots, k_{s}$ such that $\sum_{i=1}^{s} k_{i}=$ $k+1-s$ and for each $i \in\{1, \ldots, s\}$ the graph $F_{i}$ is a forbidden subgraph for $\mathcal{P}\left(k_{i}\right)$.

Proof. From Lemma 6 we have $F_{1} \cup \cdots \cup F_{s} \in \mathcal{C}(\mathcal{P}(k))$ if and only if $\mathcal{H}_{\mathcal{P}}\left(F_{1} \cup \cdots \cup\right.$ $\left.F_{s}\right)$ is $\tau$-vertex $(k+1)$-critical. Since $\mathcal{H}_{\mathcal{P}}\left(F_{1} \cup \cdots \cup F_{s}\right)=\mathcal{H}_{\mathcal{P}}\left(F_{1}\right) \cup \cdots \cup \mathcal{H}_{\mathcal{P}}\left(F_{s}\right)$ and because of Remarks 3,4 we know that it is equivalent to the conditions $\tau\left(\mathcal{H}_{\mathcal{P}}\left(F_{1}\right)\right)+\cdots+\tau\left(\mathcal{H}_{\mathcal{P}}\left(F_{s}\right)\right)=k+1$ and for each $i \in\{1, \ldots, s\}$ the hypergraph $\mathcal{H}_{\mathcal{P}}\left(F_{i}\right)$ is $\tau$-vertex critical. It means that there exist non-negative integers $k_{1}, \ldots, k_{s}$ such that for each $i \in\{1, \ldots, s\}$ the hypergraph $\mathcal{H}_{\mathcal{P}}\left(F_{i}\right)$ is $\tau$-vertex $\left(k_{i}+1\right)$-critical and moreover $\sum_{i=1}^{s}\left(k_{i}+1\right)=k+1$. From Lemma 6 these conditions are equivalent to the statement $F_{i} \in \mathcal{C}\left(\mathcal{P}\left(k_{i}\right)\right)$ for each $i \in\{1, \ldots, s\}$ and $\sum_{i=1}^{s} k_{i}=k+1-s$.

Corollary 3. Let $k$ be a non-negative integer and $\mathcal{P} \in \mathbf{L}_{\leq}^{a}$. If $F$ is the union of disjoint connected graphs $F_{1}, \ldots, F_{s}$ and $F \in \mathcal{C}(\mathcal{P}(k))$, then $s \leq k+1$.

Corollary 4. Let $k$ be a non-negative integer and $\mathcal{P} \in \mathbf{L}_{\leq}^{a}$ and let $|\mathcal{C}(\mathcal{P})|=p$. The number of forbidden subgraphs for $\mathcal{P}(k)$ that have exactly $k+1$ connected components is equal to $\binom{k+p}{k+1}$.

Proof. From Theorem 10 we know that forbidden subgraphs for $\mathcal{P}(k)$ with exactly $k+1$ connected components have the form $F_{1} \cup \cdots \cup F_{k+1}$, where for each $i \in\{1, \ldots, k+1\}$ the condition $F_{i} \in \mathcal{C}(\mathcal{P})$ holds. Let $\mathcal{C}(\mathcal{P})=\left\{H_{1}, \ldots, H_{p}\right\}$. Thus, if $m_{i}$ denotes $\left|\left\{l: F_{l}=H_{i}\right\}\right|$, then we actually are interested in the number of sequences $\left(m_{1}, \ldots, m_{p}\right)$ whose elements are non-negative integers and for which the equality $m_{1}+\cdots+m_{p}=k+1$ holds, which leads to the assertion.

The remaining part of this section is devoted to other constructions of forbidden subgraphs for $\mathcal{P}(k)$ in terms of forbidden subgraphs for $\mathcal{P}$. In this consideration the structure of $\mathcal{H}_{\mathcal{P}}(G)$ is unknown. It means that our results are based only on the analysis of graph structures.

Construction 2. Let $s$ be a positive integer, $G_{1}, \ldots, G_{s}$ be graphs and $T$ be a forest with the vertex set $\left\{x_{1}, \ldots, x_{s}\right\}$. By $T\left(G_{1}, \ldots, G_{s}\right)$ we denote the family of all graphs obtained from disjoint $G_{1}, \ldots, G_{s}$ by the addition of exactly $|E(T)|$ new edges, such that a new edge joins an arbitrary vertex of $G_{i}$ with an arbitrary vertex of $G_{j}$ when $x_{i} x_{j}$ is an edge of $T$. Next we use a symbol $\left(G_{1}, \ldots, G_{s}\right)$ to denote the family of all graphs $T\left(G_{1}, \ldots, G_{s}\right)$ taken over all $s$-vertex forests $T$ and all possible orderings of their vertices.

Theorem 11. If $k$ is a non-negative integer and $\mathcal{P} \in \mathbf{L}_{\leq}^{a}$ and $G_{1}, \ldots, G_{k+1} \in$ $\mathcal{C}(\mathcal{P})$, then each graph $G$ in $\left(G_{1}, \ldots, G_{k+1}\right)$ is a forbidden subgraph for $\mathcal{P}(k)$.

Proof. Suppose that $G \in\left(G_{1}, \ldots, G_{k+1}\right)$. It follows that there exists a forest $T$ with $k+1$ vertices $x_{1}, \ldots, x_{k+1}$ such that $G \in T\left(G_{1}, \ldots, G_{k+1}\right)$. Observe that $G \notin \mathcal{P}(k)$ since it contains $k+1$ disjoint induced subgraphs that are forbidden subgraphs for $\mathcal{P}$.

Next, let $v \in V(G)$. We show that there exist $k$ vertices $u_{2}, \ldots, u_{k+1}$ in $V(G) \backslash\{v\}$ such that the graph resulting from $G$ by the removal of $v, u_{2}, \ldots, u_{k+1}$ is in $\mathcal{P}$.

The construction of $G$ implies the existence of the unique index $i$ such that $v \in V\left(G_{i}\right)$. Let $x_{j_{1}}, \ldots, x_{j_{k+1}}$ be a new ordering of vertices of $T$ such that $x_{j_{1}}=x_{i}$ and for $l \geq 2$ each vertex $x_{j_{l}}$ has at most one neighbour in $\left\{x_{j_{1}}, \ldots, x_{j_{l-1}}\right\}$. Such an ordering can be done by brute-force search algorithm. Suppose, without loss of generality, that $x_{j_{l}}=x_{l}$ for each $l \in\{1, \ldots, k+1\}$. Consequently, $G_{j_{l}}=G_{l}$ for each $l \in\{1, \ldots, k+1\}$ and especially $G_{i}=G_{1}$.

Now we describe how to choose vertices $u_{2}, \ldots, u_{k+1}$. For each $j \in\{2, \ldots, k+$ $1\}$ there is at most one edge $x_{l} x_{j}$ with $l<j$. Thus when such an edge exist we take as $u_{j}$ the vertex of $G_{j}$ that is the end of the unique edge joining $G_{j}$ with $G_{l}$ (see the construction of $G$ ), otherwise $u_{j}$ is an arbitrary vertex of $G_{j}$. Observe that $G-\left\{v, u_{2}, \ldots, u_{k+1}\right\}$ is the union of $k+1$ disjoint graphs $G_{1}-v$ and $G_{j}-u_{j}$ for $j \in\{2, \ldots, k+1\}$. The assertion follows by the additivity of $\mathcal{P}$ and properties of all $G_{j}$.

Theorem 12. Let $k$ be a non-negative integer and $\mathcal{P} \in \mathbf{L}_{\leq}^{a}$. A forest $G$ is a forbidden subgraph for $\mathcal{P}(k)$ if and only if $G \in\left(G_{1}, \ldots, G_{k+1}\right)$, where $G_{1}, \ldots, G_{k+1}$ are trees that are forbidden subgraphs for $\mathcal{P}$.

Proof. By Theorem 11, it is enough to prove that if $G$ is simultaneously a forest and a forbidden subgraph for $\mathcal{P}(k)$, then there are graphs $G_{1}, \ldots, G_{k+1}$ belonging to $\mathcal{C}(\mathcal{P})$ and there exists a $(k+1)$-vertex forest $T$ such that $G \in T\left(G_{1}, \ldots, G_{k+1}\right)$. To do it we use the induction on $k$.

By the additivity of $\mathcal{P}$, each forest that is a forbidden subgraph for $\mathcal{P}(0)=\mathcal{P}$ is a tree. The conclusion follows from the fact that there is only one 1 -vertex forest $T=K_{1}$ and each graph $G$ can be represented as $K_{1}(G)$, which means as $T(G)$.

Assume that the implication is true for parameters less than $k$ and $k \geq 1$. First suppose that $G$ has at least two connected components $H_{1}, \ldots, H_{s}$. Obviously, each of them is a tree. By Theorem $10, H_{i} \in \mathcal{C}\left(\mathcal{P}\left(k_{i}\right)\right)$, where $\sum_{i=1}^{s} k_{i}=k+$ $1-s$. Because all $k_{i}$ are non-negative integers and $s \geq 2$ we obtain $0 \leq k_{i} \leq k-1$ for each $i \in\{1, \ldots, s\}$. By the induction hypothesis, $H_{i} \in T_{i}\left(G_{1}^{i}, \ldots, G_{k_{i}+1}^{i}\right)$, which implies

$$
G \in T\left(G_{1}^{1}, \ldots, G_{k_{1}+1}^{i}, \ldots, G_{1}^{s}, \ldots, G_{k_{s}+1}^{s}\right)
$$

where $T$ is the union of disjoint $T_{1}, \ldots, T_{s}$ and $G_{j}^{l} \in \mathcal{C}(\mathcal{P})$ for each $l \in\{1, \ldots, s\}$ and $j \in\left\{1, \ldots, k_{l}+1\right\}$. Since each $T_{i}$ has exactly $k_{i}+1$ vertices, the forest $T$ has
$\sum_{i=1}^{s}\left(k_{i}+1\right)$ vertices, which means $T$ has $k+1$ vertices. Thus $G$ has a required form.

Now suppose that $G$ is connected, which means $G$ is a tree.
Claim 13. There is $x \in V(G)$ such that $G-x$ has at least one connected component in $\mathcal{P}$ and if $H_{1}, \ldots, H_{p}$ are all connected components of $G-x$ belonging to $\mathcal{P}$, then the graph induced in $G$ by $V\left(H_{1}\right) \cup \cdots \cup V\left(H_{p}\right) \cup\{x\}$ is not in $\mathcal{P}$.

Proof. We describe the procedure which finds the required $x$ in a finite number of steps.

Let $v_{0}$ be an arbitrary vertex of $G$ that is not a leaf (such a vertex always exists because $k \geq 1$, which implies $|V(G)| \geq 3$ ). Next let $G_{1}$ be an arbitrary connected component of $G-v_{0}$ such that $G_{1} \notin \mathcal{P}$ (since $G$ is in $\mathcal{C}(\mathcal{P}(k))$ and $k \geq 1$ such a connected component exists).

Let $v_{1}$ be the unique neighbour of $v_{0}$ in $G_{1}$. If $G_{1}-v_{1} \in \mathcal{P}$, then $x=v_{1}$. Otherwise, let $G_{2}$ be an arbitrary connected component of $G_{1}-v_{1}$ such that $G_{2} \notin \mathcal{P}$ and let $v_{2}$ be the unique neighbour of $v_{1}$ in $G_{2}$. If $G_{2}-v_{2} \in \mathcal{P}$, then $x=v_{2}$. Otherwise, since $G$ is finite, we find the finite sequence of vertices $v_{0}, \ldots, v_{q}$ and the sequence of graphs $G=G_{0}, G_{1}, \ldots, G_{q}$ such that $G_{i}-v_{i} \notin \mathcal{P}$ for $i \in\{0, \ldots, q-1\}, G_{q} \notin \mathcal{P}$ and $G_{q}-v_{q} \in \mathcal{P}$. Moreover for $i \in\{1, \ldots, q\}$ the graph $G_{i}$ is a connected component of $G_{i-1}-v_{i-1}$ and $v_{i}$ is the unique neighbour of $v_{i-1}$ in $G_{i}$.

Observe that $v_{q}$ can play the role of $x$. Indeed, the procedure implies that the connected components of $G_{q}-v_{q}$ are simultaneously the connected components of $G-v_{q}$.

Let $x$ be a vertex that satisfies the assumptions of Claim 13. Recall that $G$ is a tree, which means that $G-x$ is a forest. Since $G$ is a forbidden subgraph for $\mathcal{P}(k)$ we obtain $G-x \notin \mathcal{P}(k-1)$. It follows that $G-x$ contains an induced subgraph $G^{\prime} \in \mathcal{C}(\mathcal{P}(k-1))$ that is a forest. By the induction hypothesis $V\left(G^{\prime}\right)$ can be partitioned into $k$ sets $V_{1}, \ldots, V_{k}$ such that for each $i \in\{1, \ldots, k\}$ the graph $G_{i}^{\prime}$ induced by $V_{i}$ in $G-x$ is forbidden for $\mathcal{P}$. Because $\mathcal{P}$ is additive, all of the graphs $G_{i}^{\prime}$ are connected and as subgraphs of $G-x$ they are trees. Additionally, $\left(V\left(G_{1}^{\prime}\right) \cup \cdots \cup V\left(G_{k}^{\prime}\right) \cup\{x\}\right) \cap V\left(H_{i}\right)=\emptyset$ for $i \in\{1, \ldots, p\}$ (keep in mind that $H_{1}, \ldots, H_{p} \in \mathcal{P}$, see Claim 13).

Recall that, by Claim $13, V\left(H_{1}\right) \cup \cdots \cup V\left(H_{p}\right) \cup\{x\}$ contains at least one subset that induces a graph, say $G_{k+1}^{\prime}$, forbidden for $\mathcal{P}$. Hence $G_{1}^{\prime}, \ldots, G_{k+1}^{\prime}$ are disjoint induced subgraphs of $G$, each of which is in $\mathcal{C}(\mathcal{P})$. Suppose, for a contradiction, that there is a vertex $u \in V(G) \backslash \bigcup_{i=1}^{k+1} V\left(G_{i}^{\prime}\right)$. Since $G \in \mathcal{C}(\mathcal{P}(k))$ we can find at most $k$ different vertices of $G-u$ such that the removal of all of them from $G-u$ results in a graph in $\mathcal{P}$. Because $G$ contains disjoint induced subgraphs $G_{1}^{\prime}, \ldots, G_{k+1}^{\prime}$ that are forbidden for $\mathcal{P}$, it is impossible, giving a contradiction.

It means $V(G)=\bigcup_{i=1}^{k+1} V\left(G_{i}^{\prime}\right)$ and, since $G$ is a tree, there is a tree $T$ with $k+1$ vertices such that $G \in T\left(G_{1}^{\prime}, \ldots, G_{k+1}^{\prime}\right)$.

Below we present one more construction of graphs that are forbidden for $\mathcal{P}(k)$.

Construction 3. Let $G_{1}, \ldots, G_{s}$ be rooted graphs, which means that for each $i \in\{1, \ldots, s\}$ the graph $G_{i}$ has a marked vertex $v_{i}$, called its root. Next let $H$ be a graph with $V(H)=\left\{x_{1}, \ldots, x_{s}\right\}$. We take disjoint $H, G_{1}, \ldots, G_{s}$ and identify vertices $v_{i}$ with $x_{i}$ for all $i \in\{1, \ldots, s\}$. By $H\left|G_{1}, \ldots, G_{s}\right|$ we denote the family of all graphs of this type taken over all possible choices of roots $v_{1}, \ldots, v_{s}$. More precisely, for each graph $G$ in $H\left|G_{1}, \ldots, G_{s}\right|$ we have $V(G)=\bigcup_{i=1}^{s} V\left(G_{i}\right)$ and $E(G)=\bigcup_{i=1}^{s} E\left(G_{i}\right) \cup\left\{v_{i} v_{j}: x_{i} x_{j} \in E(H)\right\}$ with a choice of roots $v_{1}, \ldots, v_{s}$. Now we use a symbol $\left|G_{1}, \ldots, G_{s}\right|$ to denote the union of sets $H\left|G_{1}, \ldots, G_{s}\right|$ taken over all $s$-vertex graphs $H$.

Theorem 14. If $k$ is a non-negative integer and $\mathcal{P} \in \mathbf{L}_{\leq}^{a}$ and $G_{1}, \ldots, G_{k+1} \in$ $\mathcal{C}(\mathcal{P})$, then each graph $G$ in $\left|G_{1}, \ldots, G_{k+1}\right|$ is a forbidden subgraph for $\mathcal{P}(k)$.

Proof. By the assumption $G \in\left|G_{1}, \ldots, G_{k+1}\right|$, we have that $G \in H \mid G_{1}, \ldots$, $G_{k+1} \mid$ for some $(k+1)$-vertex graph $H$. Let $x_{i}=v_{i}$ be a common vertex of $H$ and $G_{i}$, described in Construction 3.

Because $G$ contains disjoint induced subgraphs $G_{1}, \ldots, G_{k+1}$ it follows that $G \notin \mathcal{P}(k)$. If $v \in V(G)$, then $v \in V\left(G_{j}\right)$ for exactly one index $j \in\{1, \ldots, k+1\}$. The graph obtained from $G-v$ by the removal of the vertex set $S$, where $S=$ $\left\{x_{l}: l \neq j\right\}$, has at least $k+1$ connected components each of which is in $\mathcal{P}$. The additivity of $\mathcal{P}$ implies $G-v \in \mathcal{P}(k)$.

## 5. $\quad P_{r}$-Free Graphs

In this section we focus our attention on the class $\mathcal{W}_{r}$ of graphs not containing $P_{r}$ as an induced subgraph. We determine the minimum and maximum number of vertices of a graph in $\mathcal{C}\left(\mathcal{W}_{r}(1)\right)$. First we consider $\mathcal{C}\left(\mathcal{W}_{3}(1)\right)$. Because of Theorem 9 , each graph that is forbidden for $\mathcal{W}_{3}(1)$ has at most six vertices. Searching all non-isomorphic graphs of this type we can derive that $\mathcal{C}\left(\mathcal{W}_{3}(1)\right)$ has 14 elements: $C_{4}, C_{5}, C_{6}, P_{6}, 2 P_{3}, F_{1}, \ldots, F_{9}$, where the graphs $F_{i}$ for $i \in\{1, \ldots, 9\}$ are depicted in Figure 2. Similar arguments we apply to the classes $\mathcal{O}$ of edgeless graphs and $\mathcal{K}$ of complete graphs. In this case, the facts $\mathcal{C}(\mathcal{O})=\left\{K_{2}\right\}$ and $\mathcal{C}(\mathcal{K})=\left\{\overline{K_{2}}\right\}$ yield $\mathcal{C}(\mathcal{O}(1))=\left\{K_{3}, P_{4}, C_{4}, 2 K_{2}\right\}$ and $\mathcal{C}(\mathcal{K}(1))=\left\{\overline{K_{3}}, P_{4}, C_{4}, 2 K_{2}\right\}$.

Of course the brute searching method is not too effective if forbidden subgraphs have big orders. Thus for $r \geq 4$ we start with determining forbidden subgraphs for $\mathcal{W}_{r}(1)$ with the minimum number of vertices. If $G \in \mathcal{C}\left(\mathcal{W}_{r}(1)\right)$,


Figure 2. All the graphs in $\mathcal{C}\left(\mathcal{W}_{3}(1)\right) \backslash\left\{C_{4}, C_{5}, C_{6}, P_{6}, 2 P_{3}\right\}$.
then $G$ must contain an induced subgraph $P_{r}$ after deletion of any vertex. Thus $r+1$ is the lower bound on the number of vertices of a graph in $\mathcal{C}\left(\mathcal{W}_{r}(1)\right)$. We conclude the following fact.

Proposition 1. If $r$ is an integer, $r \geq 3$, then $C_{r+1}$ is a forbidden subgraph for $\mathcal{W}_{r}(1)$ with the minimum number of vertices.

By Theorem 9 we have that the upper bound on the number of vertices of a graph in $\mathcal{C}\left(\mathcal{W}_{r}(1)\right)$ is $\left\lfloor\frac{(r+2)^{2}}{4}\right\rfloor$. However, for $r=4$ we find no graph that realizes this bound. For any $r \geq 5$ there exists a graph in $\mathcal{C}\left(\mathcal{W}_{r}(1)\right)$ of order $\left\lfloor\frac{(r+2)^{2}}{4}\right\rfloor$. To prove this fact we use the class of graphs that contains all the complements of graphs in $\mathcal{W}_{r}$.

For a given class of graphs $\mathcal{P} \in \mathbf{L}_{\leq}$let us define $\overline{\mathcal{P}}=\{\bar{G}: G \in \mathcal{P}\}$. It is a known fact that if $\mathcal{P} \in \mathbf{L}_{\leq}$, then $\overline{\mathcal{P}}$ is also in $\mathbf{L}_{\leq}$. Moreover, there is a coincidence between forbidden subgraphs for $\mathcal{P}$ and $\overline{\mathcal{P}}$ given by the equality $\mathcal{C}(\overline{\mathcal{P}})=\{\bar{F}$ : $F \in \mathcal{C}(\mathcal{P})\}[2]$. Let $\mathcal{P}_{1}, \mathcal{P}_{2}$ be classes of graphs. By $\mathcal{P}_{1} \circ \mathcal{P}_{2}$ we denote the class of all graphs $G$ whose vertex set can be partitioned into two parts $V_{1}, V_{2}$ (possible empty) such that, for all $i \in\{1,2\}$, if $V_{i}$ is non-empty, then $G\left[V_{i}\right] \in \mathcal{P}_{i}$. In that case $\mathcal{P}_{1} \circ \mathcal{P}_{2}$ is called a product of $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$. In [4] it is proved that $F \in \mathcal{C}\left(\mathcal{P}_{1} \circ \mathcal{P}_{2}\right)$ if and only if $\bar{F} \in \mathcal{C}\left(\overline{\mathcal{P}}_{1} \circ \overline{\mathcal{P}}_{2}\right)$. It is easy to observe that for each class of graphs $\mathcal{P}$ and a positive integer $k$, the class $\mathcal{P}(k)$ is identical with $\mathcal{P} \circ \mathcal{Q}$, where $Q$ consists of all the graphs of order at most $k$. Moreover, for such $\mathcal{Q}$ we have $\overline{\mathcal{Q}}=\mathcal{Q}$. Hence, taking into account the previous consideration, we have the following observation.

Proposition 2. If $\mathcal{P} \in \mathbf{L}_{\leq}$, then
(i) $G \in \mathcal{P}(k)$ if and only if $\bar{G} \in \overline{\mathcal{P}}(k)$, and
(ii) $F \in \mathcal{C}(\mathcal{P}(k))$ if and only if $\bar{F} \in \mathcal{C}(\overline{\mathcal{P}}(k))$, and
(iii) $\bar{G} \in \overline{\mathcal{P}}(k)$ if and only if $\bar{G} \in \overline{\mathcal{P}(k)}$.

Let us consider $\overline{\mathcal{W}_{r}}$. Thus, $\mathcal{C}\left(\overline{\mathcal{W}_{r}}\right)=\left\{\overline{P_{r}}\right\}$ and, by Proposition 2, it follows that $G \in \mathcal{C}\left(\overline{\mathcal{W}}_{r}(1)\right)$ if and only if $\bar{G} \in \mathcal{C}\left(\mathcal{W}_{r}(1)\right)$. As a consequence, the complement of a forbidden subgraph for $\overline{\mathcal{W}_{r}}(1)$ with the maximum number of vertices is a forbidden subgraph for $\mathcal{W}_{r}(1)$ with the maximum number of vertices. Since the vertex connectivity of $\overline{P_{r}}$ is relatively big we will be able to apply Theorem 8. First we give the supporting observation.

Lemma 15. If $r$ is an integer, $r \geq 5$, then $\overline{P_{r}}$ is $\left\lceil\frac{r-1}{2}\right\rceil$-connected.
Proof. Let $G=\overline{P_{r}}$. Observe that the vertices of $G$ can be divided into two sets $W_{1}, W_{2}$ such that subgraphs induced by $W_{i}$ for $i \in\{1,2\}$ are complete graphs and $\left|W_{1}\right|=\left\lceil\frac{r}{2}\right\rceil,\left|W_{2}\right|=\left\lfloor\frac{r}{2}\right\rfloor=\left\lceil\frac{r-1}{2}\right\rceil$. Suppose that there is a vertex-cut-set $S$ of $G$ such that $|S|<\left\lfloor\frac{r}{2}\right\rfloor$. Thus $G-S$ has two disjoint subgraphs $G_{1}$ and $G_{2}$ such that there is no edge joining a vertex of $G_{1}$ with a vertex of $G_{2}$. Furthermore, observe that $V\left(G_{1}\right)=W_{1} \backslash S$ and $V\left(G_{2}\right)=W_{2} \backslash S$ and moreover, $V\left(G_{1}\right) \neq \emptyset$ and $V\left(G_{2}\right) \neq \emptyset$. Let us denote $W_{1}^{\prime}=W_{1} \backslash S$ and $W_{2}^{\prime}=W_{2} \backslash S$. So, by our assumptions, there is no edge joining a vertex of $W_{1}^{\prime}$ with a vertex of $W_{2}^{\prime}$ in $G$. This implies that in $\bar{G}$ each vertex of $W_{1}^{\prime}$ is adjacent to each vertex of $W_{2}^{\prime}$. If $\left|W_{1}^{\prime}\right| \geq 2$ and $\left|W_{2}^{\prime}\right| \geq 2$, then $\bar{G}$ contains $C_{4}$, which contradicts that $\bar{G}=P_{r}$. If one of the sets $W_{1}^{\prime}, W_{2}^{\prime}$ contains exactly one vertex, then since $|S|<\left\lfloor\frac{r}{2}\right\rfloor$, there are at least three vertices in the second set. Thus $\bar{G}$ has a vertex of degree three, which again gives a contradiction with the assumption that $\bar{G}$ is a path.

By Lemma 7 we have the additional fact.
Lemma 16. Let $r$ be an integer, $r \geq 5$. There exists a $\left\{\overline{P_{r}}\right\}$-host-graph of $a$ hypergraph $\mathcal{H}^{*}\left(1, r,\left\lceil\frac{r-1}{2}\right\rceil\right)$ given in Construction 1.

Finally, by Theorem 8, Lemma 16 and Proposition 2, we obtain the conclusion.

Theorem 17. Let $r$ be an integer, $r \geq 5$. The complement of a $\left\{\overline{P_{r}}\right\}$-hostgraph of the hypergraph $\mathcal{H}^{*}\left(1, r,\left\lceil\frac{r-1}{2}\right\rceil\right)$, given in Construction 1, is a forbidden subgraph for $\mathcal{W}_{r}(1)$ with the maximum number of vertices.

In Figure 3 we present the complement of a forbidden subgraph for $\mathcal{W}_{5}(1)$. Theorem 17 says that this graph has the maximum number of vertices among all the graphs in $\mathcal{C}\left(\mathcal{W}_{5}(1)\right)$. Moreover, by Proposition 2, the graph in Figure 3 is in $\mathcal{C}\left(\overline{\mathcal{W}_{5}(1)}\right)$ and also in $\mathcal{C}\left(\overline{\mathcal{W}_{5}}(1)\right)$ and realizes the maximum order among all the graphs in both these families.


Figure 3. The complement of the graph in $\mathcal{C}\left(\mathcal{W}_{5}(1)\right)$ with the maximum order.

## 6. Classes of Graphs That Are Closed Under Substitution

Let $H, G_{1}, \ldots, G_{n}$ be graphs and $v_{1}, \ldots, v_{n}$ be an arbitrary ordering of the set $V(H)$. By $H\left[G_{1}, \ldots, G_{n}\right]$ we denote the graph resulting from $H$ by the simultaneous substitution of each vertex $v_{i}$ with the graph $G_{i}$. Here the substitution of the vertex $v$ with the graph $G$ in the graph $H$ means the removal of $v$ and joining all the vertices of $G$ with all the neighbours of $v$ in $H$. A class $\mathcal{P}$ of graphs is closed under substitution if for any graphs $H, G_{1}, \ldots, G_{n} \in \mathcal{P}$ and every ordering of $V(H)$, the graph $H\left[G_{1}, \ldots, G_{n}\right]$, called a substitution graph, is also in $\mathcal{P}$. By $\mathbf{L}_{<}^{*}$ we denote the class of all non-trivial induced hereditary classes of graphs that are closed under substitution. The smallest of such ones (in the sense of the number of elements) is $\left\{K_{1}\right\}$, among most notable we should list the classes $\mathcal{O}$ of edgeless graphs, $\mathcal{K}$ of complete graphs, the class of perfect graphs and the classes $\mathcal{W}_{r}$, where $r=2$ or $r \geq 4$. Observe that $P_{4}$-free graphs are just cographs. In this section we characterize all forbidden subgraphs for $\mathcal{P}(1)$ where $\mathcal{P} \in \mathbf{L}_{\leq}^{*}$.

A set $W \subseteq V(G)$ is a module in a graph $G$ if for each two vertices $x, y \in W$, $N_{G}(x) \backslash W=N_{G}(y) \backslash W$. The trivial modules in $G$ are $V(G), \emptyset$ and singletons. A graph having only trivial modules is called prime. By PRIME we denote the class of all prime graphs that have at least two vertices.

In 1997 Giakoumakis [14] proved that for each class of graphs $\mathcal{P} \in \mathbf{L}_{\leq}$its closure under substitution $\mathcal{P}^{*}$ consisting of all the graphs in $\mathcal{P}$ and all their substitution graphs can be characterized by $\mathcal{C}\left(\mathcal{P}^{*}\right)$ that consists of all minimal prime extensions of all the graphs in $\mathcal{C}(\mathcal{P})$. It has to be said that $G^{\prime}$ is a minimal prime extension of $G$ if it is a prime induced supergraph of $G$ and it does not contain as a proper induced subgraph any other prime induced supergraph of $G$.

Since for each class $\mathcal{P} \in \mathbf{L}_{\leq}^{*}$ we have $\mathcal{P}=\mathcal{P}^{*}$ (by the definition of $\mathbf{L}_{\leq}^{*}$ ), the Giakoumakis consideration leads to the following conclusion.

Remark 5. If $\mathcal{P} \in \mathbf{L}_{\leq}$, then $\mathcal{P} \in \mathbf{L}_{\leq}^{*}$ if and only if $\mathcal{C}(\mathcal{P}) \subseteq$ PRIME.

In [4] the following two theorems concerning $\mathcal{C}\left(\mathcal{P}_{1} \circ \mathcal{P}_{2}\right)$ when both $\mathcal{P}_{1}, \mathcal{P}_{2}$ are in $\mathbf{L}_{\leq}^{*}$ have been proven.

Theorem 18 [4]. Let $\mathcal{P}_{1}, \mathcal{P}_{2} \in \mathbf{L}_{\leq}^{*}$ and let $H \in$ PRIME with $V(H)=\left\{v_{1}, \ldots\right.$, $\left.v_{n}\right\}$. If $G=H\left[G_{1}, \ldots, G_{n}\right]$ and $\bar{G} \in \mathcal{C}\left(\mathcal{P}_{1} \circ \mathcal{P}_{2}\right)$, then $H \notin \mathcal{P}_{1}$ or $H \notin \mathcal{P}_{2}$ and there exists a partition $(A, B, C, D)$ of $\{1, \ldots, n\}$ (empty parts are allowed), such that
(i) $G_{i}=K_{1}$ for $i \in A$, and
(ii) $G_{i} \in \mathcal{C}\left(\mathcal{P}_{2}\right) \cap \mathcal{P}_{1}$ for $i \in B$, and
(iii) $G_{i} \in \mathcal{C}\left(\mathcal{P}_{1}\right) \cap \mathcal{P}_{2}$ for $i \in C$, and
(iv) $G_{i} \in \mathcal{C}\left(\mathcal{P}_{1} \cup \mathcal{P}_{2}\right)$ for $i \in D$.

A graph $G$, different from $K_{1}$, is strongly decomposable if in its description $G=H\left[G_{1}, \ldots, G_{n}\right]$ with $H \in$ PRIME, all the graphs $G_{i}$ satisfy $\left|V\left(G_{i}\right)\right| \geq 2$. In the next theorem we will restrict our attention to graphs that are strongly decomposable and are forbidden subgraphs for a product of classes of graphs.

Theorem 19 [4]. Let $\mathcal{P} \in \mathbf{L}_{\leq}^{*} \backslash\left\{\mathcal{O}, \mathcal{K},\left\{K_{1}\right\}\right\}$. A graph $G$ is a forbidden subgraph for $\mathcal{P}_{1} \circ \mathcal{P}_{2}$ and it is strongly decomposable if and only if there exists a representation $H\left[G_{1}, \ldots, G_{n}\right]$ of $G$, with $H \in$ PRIME, $V(H)=\left\{v_{1}, \ldots, v_{n}\right\}$, such that either for $j=1$ and $l=2$ or for $j=2$ and $l=1$ the following three conditions hold:
(i) $H \in \mathcal{C}\left(\mathcal{P}_{j}\right)$, and
(ii) for each $i \in\{1, \ldots, n\}, G_{i} \in \mathcal{C}\left(\mathcal{P}_{l}\right)$, and
(iii) for $M=\left\{i \in\{1, \ldots, n\}: G_{i} \notin \mathcal{P}_{j}\right\}$ and for each $s \in\{1, \ldots, n\} \backslash M$ the subgraph of $H$ induced by $\left\{v_{i}: i \in M \cup\{s\}\right\}$ is in $\mathcal{P}_{l}$; moreover, if $M=\{1, \ldots, n\}$, then $H \in \mathcal{P}_{l}$.

Observe that PRIME includes only two graphs, $K_{2}, \overline{K_{2}}$, with two vertices, no graph on three vertices and only one graph, $P_{4}$, with four vertices. Next $\mathcal{C}(\mathcal{O})=\left\{K_{2}\right\}, \mathcal{C}(\mathcal{K})=\left\{\overline{K_{2}}\right\}, \mathcal{C}\left(\left\{K_{1}\right\}\right)=\left\{K_{2}, \overline{K_{2}}\right\}$. Thus if $\mathcal{P} \in \mathbf{L}_{\leq}^{*} \backslash$ $\left\{\mathcal{O}, \mathcal{K},\left\{K_{1}\right\}\right\}$, then the family $\mathcal{C}(\mathcal{P})$ has to contain at least one graph in PRIME $\backslash$ $\left\{K_{2}, \overline{K_{2}}\right\}$. Since each graph on at least 4 vertices contains as an induced subgraph $K_{2}$ or $\overline{K_{2}}$ and graphs in $\mathcal{C}(\mathcal{P})$ are not comparable with respect to induced subgraph relation, we conclude that $\mathcal{C}(\mathcal{P}) \cap\left\{K_{2}, \overline{K_{2}}\right\}=\emptyset$. Hence we have the following fact.

Remark 6. If $\mathcal{P} \in \mathbf{L}_{\leq}^{*} \backslash\left\{\mathcal{O}, \mathcal{K},\left\{K_{1}\right\}\right\}$, then $\left\{K_{2}, \overline{K_{2}}\right\} \subseteq \mathcal{P}$.
Recall that $\mathcal{P}(1)=\mathcal{P} \circ\left\{K_{1}\right\}$ and $\left\{K_{1}\right\} \in \mathbf{L}_{\leq}^{*}$. Hence, from Theorem 19, we obtain the following immediate consequence.

Corollary 5. If $\mathcal{P} \in \mathbf{L}_{\leq}^{*} \backslash\left\{\mathcal{O}, \mathcal{K},\left\{K_{1}\right\}\right\}$, then $G$ is a forbidden subgraph for $\mathcal{P}(1)$ that is strongly decomposable if and only if $G=K_{2}\left[H_{1}, H_{2}\right]$ or $G=\overline{K_{2}}\left[H_{1}, H_{2}\right]=$ $H_{1} \cup H_{2}$ or $G=H_{1}\left[G_{1}, \ldots, G_{n}\right]$, where $H_{1}, H_{2} \in \mathcal{C}(\mathcal{P})$ and $G_{1}, \ldots, G_{n} \in\left\{K_{2}, \overline{K_{2}}\right\}$.

Proof. We apply Theorem 19 together with the notations. If $\mathcal{P}=\mathcal{P}_{j}$ and $\left\{K_{1}\right\}=\mathcal{P}_{l}$, then, by Remark $6, M=\emptyset$ and the graph induced in $H$ by $\left\{v_{i}\right.$ : $i \in M \cup\{s\}\}$ is $K_{1}$. Consequently we obtain that $H_{1}\left[G_{1}, \ldots, G_{n}\right]$ is forbidden for $\mathcal{P} \circ\left\{K_{1}\right\}=\mathcal{P}(1)$. If $\mathcal{P}=\mathcal{P}_{l}$ and $\left\{K_{1}\right\}=\mathcal{P}_{j}$, then $H$ is one of the graphs $K_{2}, \overline{K_{2}}$. By Remark 6 we have $M=\{1,2\}$ and we obtain that $K_{2}\left[H_{1}, H_{2}\right]$ and $H_{1} \cup H_{2}$ are graphs in $\mathcal{C}(\mathcal{P}(1))$. Theorem 19 guarantees no other strongly decomposable graphs in $\mathcal{C}(\mathcal{P}(1))$.

In [5] the author explained that an arbitrary graph can be obtained from a prime graph by the iterative substitution of some of its vertices by prime graphs. This procedure corresponds to the well-known construction (which has been discovered many times and is based on the Gallai Theorem [13]) called a tree decomposition of a graph. For a given graph $G$, all prime graphs applied in this tree-like iterative procedure and all their prime induced subgraphs create the unique family denoted by $Z^{*}(G)$. In the next investigation we use the following fact from this field.

Lemma 20 [5]. Let $G, G^{\prime}$ be graphs. If $G^{\prime} \in$ PRIME, then $G^{\prime} \leq G$ if and only if $G^{\prime} \in Z^{*}(G)$.

Consequently we have the following observation.
Lemma 21. If $\mathcal{P} \in \mathbf{L}_{\leq}^{*}$ and $G$ is a graph, then $G \in \mathcal{P}$ if and only if $Z^{*}(G) \subseteq \mathcal{P}$.
Proof. If $G \in \mathcal{P}$, then all induced subgraphs of $G$ are in $\mathcal{P}$, which means $Z^{*}(G)$ $\subseteq \mathcal{P}$.

Suppose that $Z^{*}(G) \subseteq \mathcal{P}$ and, for a contradiction, $G \notin \mathcal{P}$. Hence there is an induced subgraph of $G$, say $F$, such that $F \in \mathcal{C}(\mathcal{P})$ (obviously $F \notin \mathcal{P}$ ). Remark 5 implies that $F$ is prime, which by Lemma 20 leads to $F \in Z^{*}(G)$, and gives a contradiction.

We use Lemma 21 in proofs of forthcoming results.
Lemma 22. Let $\mathcal{P} \in \mathbf{L}_{\leq}^{*}$ and $H_{1}, H_{2} \in \mathcal{C}(\mathcal{P})$. If $v_{1}, \ldots, v_{n}$ is an arbitrary ordering of the set $V\left(H_{1}\right)$, then $H_{1}\left[H_{2}, K_{1}, \ldots, K_{1}\right]$ is a forbidden subgraph for $\mathcal{P}(1)$.

Proof. Let $G=H_{1}\left[H_{2}, K_{1}, \ldots, K_{1}\right]$ and let $V(G)=\left\{u_{1}, \ldots, u_{l}, v_{2}, \ldots, v_{n}\right\}$, where $v_{1}$ is substituted with vertices $u_{1}, \ldots, u_{l}$ of $H_{2}$. Hence for each $i \in\{1, \ldots, l\}$ the vertices $u_{i}, v_{2}, \ldots, v_{n}$ induce $H_{1}$ in $G$.

First we observe that $G-v \notin \mathcal{P}$ for any vertex $v \in V(G)$. Indeed, if $v=v_{i}$ for some $i \in\{2, \ldots, n\}$, then $H_{2}$ is an induced subgraph of $G-v$. If $v=u_{i}$ for some $i \in\{1, \ldots, l\}$, then $H_{1}$ is an induced subgraph of $G-v$.

Now we argue that for each $v \in V(G)$ there is $x \in V(G) \backslash\{v\}$ such that $G-\{v, x\} \in \mathcal{P}$. If $v \in\left\{v_{2}, \ldots, v_{n}\right\}$, then we choose as $x$ one of the vertices $u_{1}, \ldots, u_{l}$. If $v \in\left\{u_{1}, \ldots, u_{l}\right\}$, then we choose as $x$ one of the vertices $v_{2}, \ldots, v_{n}$. In both cases $Z^{*}(G-\{v, x\})$ contains only proper prime induced subgraphs of $H_{1}$ and $H_{2}$, which means $Z^{*}(G-\{v, x\}) \subseteq \mathcal{P}$ and, by Lemma 21, implies $G$ $\{v, x\} \in \mathcal{P}$.

Lemma 23. Let $\mathcal{P} \in \mathbf{L}_{\leq}^{*}, H_{1}, H_{2} \in \mathcal{C}(\mathcal{P})$ and $X \in$ PRIME. If $v_{1}, \ldots, v_{n}$ is an ordering of the set $V(X)$ such that $X\left[\left\{v_{2}, \ldots, v_{n}\right\}\right]=H_{1}$ and $X-v_{i} \in \mathcal{P}$ for each $i \in\{2, \ldots, n\}$, then $X\left[H_{2}, K_{1}, \ldots, K_{1}\right]$ is a forbidden subgraph for $\mathcal{P}(1)$.

Proof. Let $G=X\left[H_{2}, K_{1}, \ldots, K_{1}\right]$ and let $V(G)=\left\{u_{1}, \ldots, u_{l}, v_{2}, \ldots, v_{n}\right\}$, where $v_{1}$ is substituted with vertices $u_{1}, \ldots, u_{l}$ of $H_{2}$. Thus $G$ contains two disjoint subgraphs $H_{1}, H_{2}$ induced by vertices $v_{2}, \ldots, v_{n}$ and $u_{1}, \ldots, u_{l}$, respectively. Hence $G \notin \mathcal{P}(1)$.

Now we argue that each pair of vertices $u_{i}, v_{j}$, with $i \in\{1, \ldots, l\}$ and $j \in$ $\{2, \ldots, n\}$ satisfies the condition $G-\left\{u_{i}, v_{j}\right\} \in \mathcal{P}$. Indeed, $Z^{*}\left(G-\left\{u_{i}, v_{j}\right\}\right)$ contains only prime graphs that are induced subgraphs of $H_{2}-u_{i}$ and $X-v_{j}$. Both these graphs are in $\mathcal{P}$, which implies $Z^{*}\left(G-\left\{u_{i}, v_{j}\right\}\right) \subseteq \mathcal{P}$. Lemma 21 yields $G-\left\{u_{i}, v_{j}\right\} \in \mathcal{P}$, as we desired.

Now we are ready to prove that $G-v \in \mathcal{P}(1)$ for each $v \in V(G)$, which means that for each vertex $v \in V(G)$ there is $x \in V(G) \backslash\{v\}$ such that $G-\{x, v\} \in \mathcal{P}$. If $v=u_{i}$ for some $i \in\{1, \ldots, l\}$, then we put $x=v_{j}$ for an arbitrary $j \in\{2, \ldots, n\}$, and if $v=v_{j}$ for some $j \in\{2, \ldots, n\}$, then we put $x=u_{i}$ for an arbitrary $i \in\{1, \ldots, l\}$. The earlier consideration confirms that $G-\{x, v\} \in \mathcal{P}$ in both cases.

Theorem 24. Let $\mathcal{P} \in \mathbf{L}_{\leq}^{*} \backslash\left\{\mathcal{O}, \mathcal{K},\left\{K_{1}\right\}\right\}$. A graph $G$ is a forbidden subgraph for $\mathcal{P}(1)$ if and only if $G$ has one of the following forms:
(i) $G=G_{1}\left[H_{1}, H_{2}\right]$, or
(ii) $G=H_{1}\left[G_{1}, \ldots, G_{\left.\mid V\left(H_{1}\right)\right]}\right.$, or
(iii) $G=H_{1}\left[H_{2}, K_{1}, \ldots, K_{1}\right]$, or
(iv) $G=X\left[H_{2}, K_{1}, \ldots, K_{1}\right]$, or
(v) $G=Y\left[G_{1}, \ldots, G_{s}, K_{1}, \ldots, K_{1}\right]$,
where $H_{1}, H_{2} \in \mathcal{C}(\mathcal{P})$ and $G_{i} \in\left\{K_{2}, \overline{K_{2}}\right\}$ for all permissible $i$; further $X, Y \in$ PRIME and, assuming that $V(X)=\left\{v_{1}, \ldots, v_{n_{1}}\right\}$ and $V(Y)=\left\{u_{1}, \ldots, u_{n_{2}}\right\}$, the following conditions are fulfilled:

- $X\left[\left\{v_{2}, \ldots, v_{n_{1}}\right\}\right] \in \mathcal{C}(\mathcal{P})$, and
- for each $i \in\left\{2, \ldots, n_{1}\right\}, X-v_{i} \in \mathcal{P}$, and
- $n_{2} \geq s+2$, and
- for each $i \in\{1, \ldots, s\}, Y-u_{i} \in \mathcal{P}$, and
- for each $i \in\left\{s+1, \ldots, n_{2}\right\}, Y-u_{i} \notin \mathcal{P}$ and there exists $j \in\left\{s+1, \ldots, n_{2}\right\} \backslash\{i\}$ satisfying $Y-\left\{u_{i}, u_{j}\right\} \in \mathcal{P}$.

Proof. Lemmas 22, 23 and Corollary 5 show that graphs having forms (i), (ii), (iii) or (iv) are forbidden subgraphs for $\mathcal{P}(1)$. Recall that a graph $G$ belongs to $\mathcal{C}(\mathcal{P}(1))$ if the graph resulting by the removal of any vertex of $G$ does not belong to $\mathcal{P}$ and for each vertex $v \in V(G)$ there exists another vertex $x \in V(G)$ such that $G-\{v, x\} \in \mathcal{P}$. Observe that if a graph has the form (v), then it satisfies these conditions. Namely, if $v$ is one of the vertices of $G_{i}$ with $i \in\{1, \ldots, s\}$, then we choose another vertex of $G_{i}$ as $x$. If $v$ is one of the vertices $u_{i}$ with $i \geq s+1$, then the role of $x$ is played by $u_{j}$ given by the assumptions of the theorem. In both cases the conclusion follows by the construction of $G$.

Corollary 5 characterizes all strongly decomposable graphs in $\mathcal{C}(\mathcal{P}(1))$. It means that to finish the proof it is enough to show that if $G$ is not strongly decomposable and forbidden for $\mathcal{P}(1)$, then $G$ has either the form (iii) or (iv) or (v). The mentioned earlier observation that graphs in $\mathcal{C}(\mathcal{P}(1))$ are pairwise incomparable with respect to the induced subgraph relation allows us to to simplify analysis. Namely, it is enough to show that such $G$ contains as an induced subgraph a graph of one of the forms (i), (ii), (iii), (iv), (v). As a consequence, we observe that $G$ has to be of the corresponding form.

Assume that $G$ is not strongly decomposable. By Theorem 18, Remark 5 and the iterative construction of graphs via prime graphs, we can assume that $G$ has a form $W\left[U_{1}, \ldots, U_{l}, K_{1}, \ldots, K_{1}\right]$, where $W, U_{1}, \ldots, U_{l} \in$ PRIME and $V(W)=\left\{w_{1}, \ldots, w_{l}, w_{l+1}, \ldots, w_{n}\right\}$ with $n \geq l+1$ (we adopt the convention that $l=0$ is equivalent to $\left.G=W\left[K_{1}, \ldots, K_{1}\right]=W\right)$. Moreover, graphs $U_{1}, \ldots, U_{l}$ are forbidden subgraphs for $\mathcal{P}$ or are elements of the set $\left\{K_{2}, \overline{K_{2}}\right\}$.

Suppose that two of the graphs $U_{1}, \ldots, U_{l}$, say $U_{i}, U_{j}$, are forbidden subgraphs for $\mathcal{P}$. Hence $K_{2}\left[U_{i}, U_{j}\right]$ or $\overline{K_{2}}\left[U_{i}, U_{j}\right]$ is an induced subgraph of $G$ depending on whether or not $w_{i}, w_{j}$ are adjacent in $W$. In both cases it leads to the conclusion that $G$ contains an induced subgraph of the form (i).

In the next part of the proof we assume that at most one among graphs $U_{1}, \ldots, U_{l}$ is in $\mathcal{C}(\mathcal{P})$ and, without loss of generality, only $U_{1}$ can be such a graph. Following this assumption $W \notin \mathcal{P}$. If not, then $Z^{*}(G-v) \subseteq \mathcal{P}$, where $v$ is an arbitrary vertex of $U_{1}$ and next, by Remark $6, G-v \in \mathcal{P}$ giving $G \in \mathcal{P}(1)$, which is impossible. Thus $W \notin \mathcal{P}$.

Now we consider the case $U_{1} \in \mathcal{C}(\mathcal{P})$. It means that if $l \geq 2$, then $U_{2}, \ldots, U_{l} \in$ $\left\{K_{2}, \overline{K_{2}}\right\}$. If there is $W^{\prime} \leq W$ such that $W^{\prime} \in \mathcal{C}(\mathcal{P})$ with $w_{1} \in V\left(W^{\prime}\right)$, then $G$ contains an induced subgraph of the form (iii). Otherwise, since $W \notin \mathcal{P}$
there is $W^{\prime} \leq W$ such that $W^{\prime} \in \mathcal{C}(\mathcal{P})$ but $w_{1} \notin V\left(W^{\prime}\right)$ and moreover, for $W^{\prime \prime}=W\left[\left\{w_{1}\right\} \cup V\left(W^{\prime}\right)\right]$ we have $W^{\prime \prime}-x \in \mathcal{P}$ for each $x \in V\left(W^{\prime}\right)$. Observe that $W^{\prime \prime}\left[U_{1}, K_{1}, \ldots, K_{1}\right] \leq G$ and $W^{\prime \prime}\left[U_{1}, K_{1}, \ldots, K_{1}\right]$ is of the form (iv), which completes the proof in this case.

Suppose that $U_{1} \notin \mathcal{C}(\mathcal{P})$. Hence $G=W\left[U_{1}, \ldots, U_{l}, K_{1}, \ldots, K_{1}\right]$, where $U_{1}$, $\ldots, U_{l} \in\left\{K_{2}, \overline{K_{2}}\right\}$. Assume that $V(G)=\left\{w_{1}^{1}, w_{1}^{2}, \ldots, w_{l}^{1}, w_{l}^{2}, w_{l+1}, \ldots, w_{n}\right\}$, where for $i \in\{1, \ldots, l\} w_{i}$ is substituted with vertices $w_{i}^{1}, w_{i}^{2}$ of either $K_{2}$ or $\overline{K_{2}}$. Next we show that $W-w_{i} \notin \mathcal{P}$ for $i \in\{l+1, \ldots, n\}$. For a contradiction, let $W-w_{i} \in \mathcal{P}$ for some $i$ from the range. Hence, because $K_{2}, \overline{K_{2}} \in \mathcal{P}$, by Remark 6 , we have $Z^{*}\left(G-w_{i}\right) \subseteq \mathcal{P}$. It implies, by Lemma 21, that $G \in \mathcal{P}(1)$ and gives a contradiction. Therefore $W-w_{i} \notin \mathcal{P}$ for $i \in\{l+1, \ldots, n\}$. By the definition of $\mathcal{C}(\mathcal{P}(1))$ we know that there exists a vertex $v \in V(G) \backslash\left\{w_{i}\right\}$ such that $G-\left\{w_{i}, v\right\} \in \mathcal{P}$. We ask whether or not $v$ could be $w_{t}^{j}$ for some $t \in\{1, \ldots, l\}$ and $j \in\{1,2\}$. Without loss of generality, let $v=w_{t}^{2}$ for some $t$ from the range. Thus $G\left[\left\{w_{1}^{1}, \ldots, w_{l}^{1}, w_{l+1}, \ldots, w_{i-1}, w_{i+1}, \ldots, w_{n}\right\}\right]=W-w_{i}$. We observed previously that $W-w_{i} \notin \mathcal{P}$, which means that $G-\left\{w_{i}, w_{t}^{2}\right\} \notin \mathcal{P}$ and excludes this possibility. Thus $v$ must be $w_{j}$ for some $j \in\{l+1, \ldots, n\} \backslash\{i\}$ and moreover, it implies $n \geq l+2$. Finally, we show that if $l \geq 1$, then $W-w_{i} \in \mathcal{P}$ for each $i \in\{1, \ldots, l\}$. If not, then $W-w_{i} \notin \mathcal{P}$ for some $i \in\{1, \ldots, l\}$. It implies $G-\left\{w_{i}^{1}, w_{i}^{2}\right\} \notin \mathcal{P}$. By the definition of graphs in $\mathcal{C}(\mathcal{P}(1))$ we know that there exists $v \in V(G) \backslash\left\{w_{i}^{2}\right\}$ such that $G-\left\{v, w_{i}^{2}\right\} \in \mathcal{P}$. Obviously $v \neq w_{i}^{1}$. Moreover, $W-w_{t} \leq G-\left\{w_{t}, w_{i}^{2}\right\}$ for each $t \in\{l+1, \ldots, n\}$ and $W \leq G-\left\{w_{t}^{j}, w_{i}^{2}\right\}$ for each $t \in\{1, \ldots, l\} \backslash\{i\}$ and $j \in\{1,2\}$. Because $W-w_{t} \notin \mathcal{P}$ for $t \in\{l+1, \ldots, n\}$ and $W \notin \mathcal{P}$, we obtain a contradiction. Hence we conclude that $W-w_{i} \in \mathcal{P}$ for each $i \in\{1, \ldots, l\}$. Thus, adopting $l=s$ and $n=n_{2}, G$ satisfies all the conditions that define the form (v) in this case.

In Figures $4,5(\mathrm{~d}), 5(\mathrm{e})$, and 6 we present all possible graphs in $\mathcal{C}\left(\mathcal{W}_{4}(1)\right)$ that have forms pointed out in Theorem 24(i), 24(iii) and Theorem 24(iv). Some examples of graphs in $\mathcal{C}\left(\mathcal{W}_{4}(1)\right)$ having the construction given by Theorem 24(ii) are shown in Figures 5(a), 5(b), 5(c). Figure 7 illustrates Theorem 24(v). It refers to cases $s=0, s=1, s=2$, represented by $Y$ being $C_{5}, \overline{P_{5}}, P_{6}$, respectively. It should be mentioned here that the graph in Figure 3 has the form given by Theorem 24(v) with $s=0$.


Figure 4. All the graphs in $\mathcal{C}\left(\mathcal{W}_{4}(1)\right)$ of the form given in Theorem 24(i).

(a)

(b)


$$
P_{4}\left[\bar{K}_{2}, \bar{K}_{2}, \bar{K}_{2}, \bar{K}_{2}\right]
$$

(c)


$$
P_{4}\left[\bar{K}_{2}, K_{2}, \bar{K}_{2}, \bar{K}_{2}\right]
$$

$$
P_{4}\left[K_{2}, K_{2}, \bar{K}_{2}, K_{2}\right]
$$

(d)


$$
P_{4}\left[K_{1}, P_{4}, K_{1}, K_{1}\right]
$$



$$
P_{4}\left[P_{4}, K_{1}, K_{1}, K_{1}\right]
$$

Figure 5 . Some examples of graphs in $\mathcal{C}\left(\mathcal{W}_{4}(1)\right)$ of the form given in Theorem 24(ii) ((a), (b), (c)) and all the graphs in $\mathcal{C}\left(\mathcal{W}_{4}(1)\right)$ of the form given in Theorem 24(iii)) ((d), (e)).


Figure 6. The unique graph in $\mathcal{C}\left(\mathcal{W}_{4}(1)\right)$ of the form given in Theorem 24(iv).

## 7. Concluding Remarks

In this final section we would like to present relations between the concept of a $\mathcal{P}(k)$-apex graph and a concept of an $(H, k)$-stable graph. According to [12, 16], let $H$ be a fixed graph, a graph $G$ is $(H, k)$-stable whenever the deletion of any set of $k$ edges of $G$ results in a graph that still contains a subgraph isomorphic to $H$.

An $(H, k)$-stable graph $G$ is minimal if for every $A \subseteq E(G),|A|=k$, there is $e \in E(G) \backslash A$ such that $(G-A)-e$ does not contain a subgraph isomorphic to $H$. Let us denote by $\operatorname{Stab}(H, k)$ the set of all minimal ( $H, k$ )-stable graphs.

Proposition 3. Let $k$ be an integer and $H$ be a graph such that $|V(H)| \geq 4$. Next let $\mathcal{Q}$ be the class of all graphs that do not contain $L(H)$ (the line graph of $H)$ as an induced subgraph. If $G \in \operatorname{Stab}(H, k)$, then $L(G) \in \mathcal{C}(\mathcal{Q}(k))$.

Proof. On the contrary, suppose that $L(G) \notin \mathcal{C}(\mathcal{Q}(k))$. Consider now two cases.
(a) $Y=C_{5} ; s=0$ :

(b) $Y=P_{6} ; s=2$ :


$$
G_{1}=P_{6}\left[\bar{K}_{2}, \bar{K}_{2}, K_{1}, K_{1}, K_{1}, K_{1}\right]
$$



$$
G_{2}=P_{6}\left[K_{2}, K_{2}, K_{1}, K_{1}, K_{1}, K_{1}\right]
$$

$$
G_{3}=P_{6}\left[K_{2}, \bar{K}_{2}, K_{1}, K_{1}, K_{1}, K_{1}\right]
$$


(c) $Y=\overline{P_{5}} ; s=3$ :



$$
G_{1}=\overline{P_{5}}\left[\bar{K}_{2}, K_{2}, K_{2}, K_{1}, K_{1}\right]
$$


$G_{2}=\overline{P_{5}}\left[K_{2}, \bar{K}_{2}, \bar{K}_{2}, K_{1}, K_{1}\right]$

Figure 7. Some examples of graphs in $\mathcal{C}\left(\mathcal{W}_{4}(1)\right)$ of the form given in Theorem 24(v).

Case 1. $L(G) \in \mathcal{Q}(k)$. It follows that there is a set $B \subseteq V(L(G)),|B| \leq k$ such that $L(G)-B \in \mathcal{Q}$. The graph $L(G)-B$ is a line graph of some graph $G^{\prime}$. Thus $L(G)-B=L\left(G^{\prime}\right) \nsupseteq L(H)$. From Whitney’s Theorem [22] and assumptions it follows that $G^{\prime} \nsupseteq H$. The graph $G^{\prime}$ is obtained by removing at most $k$ edges from the graph $G$ which correspond in a unique way to the vertices of the set $B$. This contradicts our assumption that $G \in \operatorname{Stab}(H, k)$.

Case 2. $L(G) \geq F \in \mathcal{C}(\mathcal{Q}(k))$. If $L(G)=F$, then the conclusion is obvious. Suppose that $L(G) \neq F$. Thus $F$ is a line graph of some graph $G^{\prime}$ which is a proper spanning subgraph of $G$. Let $e \in E(G) \backslash E\left(G^{\prime}\right)$. From the assumption $G \in \operatorname{Stab}(H, k)$ it follows that for the edge $e$ there is a set $B^{\prime} \subseteq E(G) \backslash\{e\}$, $\left|B^{\prime}\right|=k$ such that $(G-e)-B^{\prime}$ has no subgraph $H$. Obviously, $\left|B^{\prime} \cap E\left(G^{\prime}\right)\right| \leq k$. Since $G^{\prime} \subseteq G-e$, then $G^{\prime}-B^{\prime}$ has no subgraph $H$. This fact implies that there is
a set $A^{\prime} \subseteq V(F),\left|A^{\prime}\right|=k$ such that $F-A^{\prime} \in \mathcal{Q}$. This contradicts our assumption that $F \in \mathcal{C}(\mathcal{Q}(k))$ and the proof is complete.

In [16] the minimum size of $\left(P_{4}, k\right)$-stable graphs was determined. In Section 5 of this paper we deal with the minimum and maximum order of graphs in $\mathcal{C}\left(\mathcal{W}_{r}(k)\right)$. Since $L\left(P_{r+1}\right)=P_{r}$ we have the following observation.

Corollary 6. Let $k, r$ be integers, $r \geq 3$. If $G \in \operatorname{Stab}\left(P_{r+1}, k\right)$, then $L(G) \in$ $\mathcal{C}\left(\mathcal{W}_{r}(k)\right)$.

Let us define a vertex version of the $H$-stability. Let $H$ be a graph and $k$ be a positive integer. A graph $G$ of order at least $k$ is said to be $(H, k)$-vertex stable if for any set $S$ of $k$ vertices the subgraph $G-S$ contains an induced subgraph isomorphic to $H$. An $(H, k)$-vertex stable graph $G$ is minimal if for every $W \subseteq V(G),|W|=k$, there is $v \in V(G) \backslash W$ such that $(G-W)-v$ does not contain $H$. Let us denote by $\operatorname{Stab}_{V}(H, k)$ the set of all minimal $(H, k)$-vertex stable graphs. Observe the following fact.

Proposition 4. If $k$ is an integer and $H$ is a connected graph, then $\operatorname{Stab}_{V}(H, k)=$ $\mathcal{C}(\mathcal{P}(k))$, where $\mathcal{P}$ is the class of all graphs that do not contain $H$ as an induced subgraph.

Proof. If $G \in \mathcal{C}(\mathcal{P}(k))$, then $G-v \in \mathcal{P}(k)$ and $G-v \notin \mathcal{P}(k-1)$ for every $v \in V(G)$. In the case when $G-v \in \mathcal{P}(k-1)$ for an vertex $v$, then there is a set $A \subseteq V(G),|A|=k-1$ such that $(G-v)-A \in \mathcal{P}$. This contradicts our assumption that $G \in \mathcal{C}(\mathcal{P}(k))$. It implies that for every set $A \subseteq V(G),|A|=k$ we have $G-A \geq H$, i.e., $G \in \operatorname{Stab}_{V}(H, k)$. Thus, $\mathcal{C}(\mathcal{P}(k)) \subseteq \operatorname{Stab}_{V}(H, k)$.

Now let $G \in \operatorname{Stab}_{V}(H, k)$. Then for every $A \subseteq V(G),|A|=k$, there is $v \in V(G) \backslash A$ such that $(G-A)-v$ does not contain $H$ as an induced subgraph. It follows that for every $v \in V(G)$ there is a set $A \subseteq V(G),|A|=k$ such that $(G-v)-A \in \mathcal{P}$, i.e., $G \in \mathcal{C}(\mathcal{P}(k))$. Hence $\operatorname{Stab}_{V}(H, k) \subseteq \mathcal{C}(\mathcal{P}(k))$.

Yet another version of an $(H, k)$-stable graph was studied in a series of papers $[3,6-8,10,11]$ where the $(H, k)$-vertex stability was considered taking into account, instead of induced subgraphs, subgraphs of $G$ isomorphic to $H$. In case of $H=$ $K_{q}$, both concepts coincide.

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