# COMPLETELY INDEPENDENT SPANNING TREES IN $k$-TH POWER OF GRAPHS 

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#### Abstract

Let $T_{1}, T_{2}, \ldots, T_{k}$ be spanning trees of a graph $G$. For any two vertices $u, v$ of $G$, if the paths from $u$ to $v$ in these $k$ trees are pairwise openly disjoint, then we say that $T_{1}, T_{2}, \ldots, T_{k}$ are completely independent. Araki showed that the square of a 2 -connected graph $G$ on $n$ vertices with $n \geq 4$ has two completely independent spanning trees. In this paper, we prove that the $k$-th power of a $k$-connected graph $G$ on $n$ vertices with $n \geq 2 k$ has $k$ completely independent spanning trees. In fact, we prove a stronger result: if $G$ is a connected graph on $n$ vertices with $\delta(G) \geq k$ and $n \geq 2 k$, then the $k$-th power $G^{k}$ of $G$ has $k$ completely independent spanning trees.


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## 1. Introduction

The graphs considered in this paper are finite, undirected, and simple (no loops or multiple edges). The vertex set and the edge set of $G$ are denoted by $V(G)$ and $E(G)$, respectively. For a vertex $v \in V(G)$, the neighbour set $N_{G}(v)$ is the set of vertices adjacent to $v, \operatorname{deg}_{G}(v)=\left|N_{G}(v)\right|$ is the degree of $v$. For a subgraph $H$ of $G, N_{H}(v)$ is the set of the neighbour of $v$ which are in $H$, and $\operatorname{deg}_{H}(v)=\left|N_{H}(v)\right|$ is the degree of $v$ in $H$. The set of (close) neighbour of a edge $e$ in $G$ is denoted by $N_{G}(e)\left(N_{G}[e]\right)$. When no confusion can occur, we shall write $N(v), N(e), N[e]$, instead of $N_{G}(v), N_{G}(e), N_{G}[e]$, respectively. We denote by $\delta(G)$ the minimum degree of the vertices of $G$. For a subset $U \subseteq V(G)$, the subgraph induced by $U$
is denoted by $G[U]$, which is the graph on $U$ whose edges are precisely the edges of $G$ with both ends in $U$. We use $G-U$ to denote the subgraph induced by $V(G) \backslash U$, that is, the graph obtained from $G$ by deleting all the vertices of $U$ together with all the edges with at least one end in $U$. If $U=\{u\}$, then we shall use $G-u$ instead of $G-\{u\}$. For a subset $U, V$ of $V(G)$, we denote $E_{G}(U, V)$ the set of edges of $G$ with one end in $U$ and the other end in $V$.

For $u \in V(G)$ and $U \subseteq V(G)$, the distance between $u$ and $U$, denoted by $\operatorname{dist}_{G}(u, U)$, is the length of a shortest path from $u$ to a vertex in $U$. When $U$ consists of a single vertex, we write $\operatorname{dist}_{G}(u, v)$ instead of $\operatorname{dist}_{G}(u,\{v\})$. For a positive integer $k$, the $k$-th power $G^{k}$ of a graph $G$ is the graph $G^{k}$ whose vertex set is $V(G)$, two distinct vertices being adjacent in $G^{k}$ if and only if their distance in $G$ is at most $k$. If $k=1, G^{1}=G$. In particular, the graph $G^{2}$ is referred to as the square of $G$, the graph $G^{3}$ as the cube of $G$. We say $G$ is $k$-connected if $|V(G)|>k$ and $G-X$ is connected for every set $X \subset V(G)$ with $|X|<k$. For simplicity, we denote $[k]=\{1, \ldots, k\}$.

A tree $T$ of $G$ is a spanning tree of $G$ if $V(T)=V(G)$. A leaf is a vertex of degree 1. An internal vertex is a vertex of degree at least 2 . A rooted tree $T$ is a tree with a specified vertex $x$, called the root of $T$. A $x$-tree $T$ refer to a rooted tree with root $x$. The level of a vertex $v$ of the $x$-tree $T$ is the length of the path from the root $x$ to $v$, the depth of the $x$-tree $T$ is the maximum level of a vertex in the tree, denoted by $D(T)$. A graph is called homeomorphically irreducible if it contains no vertices of degree 2. A homeomorphically irreducible tree is called a HIT, and a homeomorphically irreducible spanning tree of a graph is called a HIST of the graph. A caterpillar is a tree in which the internal vertices induce a path.

Let $x, y$ be two vertices of $G$. An $(x, y)$-path is a path with the two ends $x$ and $y$. Two $(x, y)$-paths $P_{1}, P_{2}$ are openly disjoint if they have no common edge and no common vertex except for the two ends $x$ and $y$. Let $T_{1}, T_{2}, \ldots, T_{k}$ be spanning trees in a graph $H$. For any two vertices $u, v$ of $H$, if the paths from $u$ to $v$ in these $k$ trees are pairwise openly disjoint, then we say that $T_{1}, T_{2}, \ldots, T_{k}$ are completely independent spanning trees(CISTs) in $G$. The concept of completely independent spanning trees was proposed by Hasunuma [4]. In [4], Hasunuma gave a characterization for CISTs and proved that the underlying graph of a $k$ connected line digraph always contains $k$ CISTs. It is well known [7, 9] that every $2 k$-edge-connected graph has $k$ edge-disjoint spanning trees. Motivated by this, Hasunuma [5] conjectured that every $2 k$-connected graph has $k$ CISTs. However, Péterfalvi [8] disproved the conjecture by constructing a $k$-connected graph, for each $k \geq 2$, which does not have two CISTs. Recently, Araki [1] provided a new characterization of the existence of $k$ CISTs and showed the following results.

Theorem 1 [1]. Let $G$ be a graph with $n \geq 7$ vertices. If $\delta(G) \geq n / 2$, then $G$ has two completely independent spanning trees.

In [6], Hong et al. give a generalization of Theorem 1.1.
Theorem 2 [1]. If $G$ is a 2-connected graph $G$ on $n$ vertices with $n \geq 4$, then the square $G^{2}$ has two completely independent spanning trees.

It is interesting to note that the above Dirac's conditions and Fleischner's conditions is sufficient for a graph to be Hamiltonian. So, Araki [1] asked that whether other sufficient conditions for a graph to be Hamiltonian also imply the existence of two CISTs. In [3], Fan et al. confirmed that the well-known Ore's condition also implies the existence two CISTs.

In this paper, we generalize Theorem 2. In fact, we prove a stronger result which is Theorem 7.

First, we give the preliminaries of our results as follows.
Let $\left(V_{1}, V_{2}, \ldots, V_{k}\right)$ be a partition of the vertex set $V(G)$ and, for $i \neq j$, $B\left(V_{i}, V_{j}, G\right)$ be a bipartite graph with the edge set $\left\{u v \mid u v \in E(G), u \in V_{i}\right.$ and $\left.v \in V_{j}\right\}$. If the graph $G$ is clear from the context, we may use $B\left(V_{1}, V_{2}\right)$ instead of $B\left(V_{1}, V_{2}, G\right)$. A partition $\left(V_{1}, V_{2}, \ldots, V_{k}\right)$ is called a CIST-partition of $G$ if it satisfies the following two conditions:
(1) for $i=1,2, \ldots, k$, the induced subgraph $G\left[V_{i}\right]$ is connected, and
(2) for any $i \neq j$, the bipartite graph $B\left(V_{i}, V_{j}\right)$ has no tree components, that is, every connected component $H$ of $B\left(V_{i}, V_{j}\right)$ satisfies $|E(H)| \geq|V(H)|$.

The following result obtained by Araki [1] plays a key role in our proof.
Lemma 3 [1]. A connected graph $G$ has $k$ completely independent spanning trees if and only if there is a CIST-partition $\left(V_{1}, \ldots, V_{k}\right)$ of $V(G)$.

Lemma 4 [2]. Let $G$ be a graph with every edge in at least two triangles. Then $G$ contains a HIST.

Now we give the definition of a good vertex $x$ of $H$ to use in the proof of our result.

Given a graph $H$ and a partition $\left(V_{1}, \ldots, V_{k}\right)$ of its vertex set, let

$$
d_{x_{i}}=d s t_{H}\left(x, V_{i}\right), x \in V(H), i \in[k] .
$$

For every $x \in V(H)$, there exists a corresponding sequence $\left(d_{t_{1}}, d_{t_{2}}, \ldots, d_{t_{k}}\right)$ such that $d_{t_{1}} \geq d_{t_{2}} \geq \cdots \geq d_{t_{k}}$ and $t_{1}, t_{2}, \ldots, t_{k}$ is a permutation of $x_{1}, x_{2}, \ldots, x_{k}$.

We say that a vertex $x$ is good with respect to $H$ if $d_{t_{j}} \leq k-j(j \in[k])$.
Lemma 5. Let $G$ be a connected graph and $H \subseteq G$. Suppose that there are $q$ components $H_{1}, H_{2}, \ldots, H_{q}$ in $G-H$ and $S$ is a subset of $V(H)$ with the following proporty: for every component $H_{s}(s \in[q])$ of $G-H$, there exist $a$ vertex $u \in V\left(H_{s}\right)$ and a vertex $v \in S$ such that $u v \in E(G)$. If $H^{k}$ has a CISTpartition $\left(V_{1}, \ldots, V_{k}\right)$ and every vertex of $S$ is good with respect to $H$, then $G^{k}$ has $k$ completely independent spanning trees.

Proof. Let $\left(V_{1}, \ldots, V_{k}\right)$ be a CIST-partition of the vertex set $V\left(H^{k}\right)$, we try to find a CIST-partition of the vertex set $V\left(G^{k}\right)$ by extending the partition $\left(V_{1}, \ldots, V_{k}\right)$.

Let $H_{1}, H_{2}, \ldots, H_{q}$ be $q$ components of $G-H$. For every component $H_{s}$ $(s \in[q])$, we choose a spanning tree $T_{s}$ and a vertex $u \in V\left(T_{s}\right)$ such that $u v \in$ $E(G)$, where $v \in S$. We may assume that $T_{s}^{\prime}=T_{s} \cup\{v u\}$ and $T_{s}^{\prime}$ is a $v$ tree. Let $d_{x_{i}}=\operatorname{dist}_{H}\left(v, V_{i}\right), i \in[k]$. For the vertex $v$, there exists a sequence $\left(d_{t_{1}}, d_{t_{2}}, \ldots, d_{t_{k}}\right)$ such that $d_{t_{1}} \geq d_{t_{2}} \geq \cdots \geq d_{t_{k}}$ and $t_{1}, t_{2}, \ldots, t_{k}$ is a permutation of $x_{1}, x_{2}, \ldots, x_{k}$. Let $\alpha$ be a one-to-one correspondence from $[k]$ to $[k]$ such that $\operatorname{dist}_{H}\left(v, V_{\alpha(j)}\right)=d_{t_{j}}$.

Let $L_{j}$ be the vertex set of all vertices in $j$-th level of $T_{s}^{\prime}$ for $j \in\left\{1, \ldots, D\left(T_{s}^{\prime}\right)\right\}$. For every $w \in L_{j}$, we assign it to $V_{\alpha(j(\bmod k))}$. For other components, we repeat the above operation, and it follows that we obtain a new partition $\left(V_{1}^{\prime}, V_{2}^{\prime}, \ldots, V_{k}^{\prime}\right)$ of the vertex set $V\left(G^{k}\right)$. It remains to show that $\left(V_{1}^{\prime}, V_{2}^{\prime}, \ldots, V_{k}^{\prime}\right)$ is a CISTpartition of $V\left(G^{k}\right)$.

If $1 \leq j \leq k$, since every vertex of $S$ is good with respect to $H$ and $L_{j} \subset V_{\alpha(j)}^{\prime}$ for any $w \in L_{j}$, we have

$$
\operatorname{dist}_{G}\left(w, V_{\alpha(j)}\right) \leq \operatorname{dist}_{G}\left(v, V_{\alpha(j)}\right)+\operatorname{dist}_{G}(v, w) \leq k-j+j=k
$$

Thus,

$$
E_{G^{k}}\left(\{w\}, V_{\alpha(j)}\right) \neq \emptyset
$$

If $k+1 \leq j \leq D\left(T_{s}^{\prime}\right)$, for any $w_{1} \in L_{j}, w_{2} \in L_{j-k}$, then

$$
L_{j}, L_{j-k} \subset V_{\alpha(j(\bmod k))}^{\prime}, \operatorname{dist}_{G}\left(w_{1}, w_{2}\right) \leq k
$$

Thus,

$$
w_{1} w_{2} \in E\left(G^{k}\right)
$$

It is easy to see that the induced graph $G^{k}\left[V_{i}^{\prime}\right]$ is connected for $i \in[k]$.
Note that $\operatorname{deg}_{B\left(V_{i}^{\prime}, V_{j}^{\prime}\right)}(w) \geq 1$ for every vertex $w \in V_{i}^{\prime} \backslash V_{i}, j^{\prime} \neq i^{\prime}$. Since $B\left(V_{i}, V_{j}\right)$ has no tree components and the vertex $w$ is adjacent to $V_{j}$ in $G^{k}$ by alternative path between $V_{i}^{\prime}$ and $V_{j}^{\prime}$, we get that $B\left(V_{i}^{\prime}, V_{j}^{\prime}\right)$ has no tree component.

Hence, $\left(V_{1}^{\prime}, V_{2}^{\prime}, \ldots, V_{k}^{\prime}\right)$ is a CIST-partition of $V\left(G^{k}\right)$. By Lemma $3, G^{k}$ has $k$ completely independent spanning trees.

Lemma 6. For a homeomorphically irreducible tree (HIT) $T$ with $|V(T)| \geq 2 k$, the $k$-th power $T^{k}$ of $T$ has $k$ completely independent spanning trees.

Proof. We first consider the longest path $P=x_{0} x_{1} \cdots x_{|P|}$ of $T$.
Case 1. $|P|<k$. If $|P|<k$, then $T^{k}$ is a complete graph. Also, $|V(T)| \geq 2 k$ and any partition $\left(V_{1}, V_{2}, \ldots, V_{k}\right)$ of $V(T)$ with $\left|V_{i}\right| \geq 2$ is a CIST-partition of $V\left(T^{k}\right)$. Hence, by Lemma $3, T^{k}$ has $k$ completely independent spanning trees.

Case 2. $|P| \geq k$. Denote $P_{0}=x_{0} x_{1} \cdots x_{k}$. Since $T$ is a homeomorphically irreducible tree (HIT), we choose a caterpillar $T_{0}$ such that its internal vertices are $V\left(P_{0}-\left\{x_{0}, x_{k}\right\}\right)$ and its leaf vertices are $N\left(x_{1}\right) \cup \cdots \cup N\left(x_{k-1}\right) \backslash V\left(P_{0}-\left\{x_{0}, x_{k}\right\}\right)$. We regard $T_{0}$ as a rooted tree which is rooted at $x_{0}$ in the following proof.

Let $L_{i}$ be the set of all vertices with the same level of $x_{i}$ in $T_{0}$, where $i \in$ $\{0,1, \ldots, k\}$. Note that

$$
L_{0}=\left\{x_{0}\right\}, L_{1}=\left\{x_{1}\right\},\left|L_{i}\right| \geq 2, i \in\{2, \ldots, k\}
$$

We have $\left|T_{0}\right| \geq 2 k$ and the distance between $x$ and $y$ is at most $k$ for every pair $x, y \in V\left(T_{0}\right)$. Thus, $T^{k}\left[T_{0}\right]$ is a complete graph and any partition $\left(V_{1}, V_{2}, \ldots, V_{k}\right)$ of the vertex set $V\left(T^{k}\left[T_{0}\right]\right)$ with $\left|V_{i}\right| \geq 2$ is a CIST-partition. Specially, we choose a partition of $V\left(T^{k}\left[T_{0}\right]\right)$ as $\left(L_{0} \cup L_{1}, L_{2}, \ldots, L_{k}\right)$.

Since $P$ is a longest path in $T$ and $T_{0}$ is a caterpillar, $E_{T}\left(L_{0} \cup L_{1} \cup L_{2} \cup\right.$ $\left.V\left(P_{0}-x_{k}\right), T-T_{0}\right)$ is empty. Let $S=V\left(T_{0}-\left(L_{0} \cup L_{1} \cup L_{2} \cup V\left(P_{0}-x_{k}\right)\right)\right)$, it follows that $x x_{i-1} \in E(T)$ for any $x \in S \cap L_{i}(i \geq 3)$ and

$$
\begin{aligned}
\operatorname{dist}_{T}\left(x, L_{i}\right) & =0 \\
\operatorname{dist}_{T}\left(x, L_{j}\right) & =i-j, 1 \leq j<i \\
\operatorname{dist}_{T}\left(x, L_{j}\right) & =j-i+2, i+1 \leq j \leq k
\end{aligned}
$$

In addition, there exists a corresponding sequence $\left(d_{t_{1}}, d_{t_{2}}, \ldots, d_{t_{k}}\right)$ such that $d_{t_{1}} \geq d_{t_{2}} \geq \cdots \geq d_{t_{k}}$ and $t_{1}, t_{2}, \ldots, t_{k}$ is a permutation of $\operatorname{dist}_{T}\left(x, L_{0} \cup L_{1}\right)$, $\operatorname{dist}_{T}\left(x, L_{2}\right), \ldots, \operatorname{dist}_{T}\left(x, L_{k}\right)$. So, $d_{t_{j}} \leq k-j(j \in[k])$. Hence, every vertex of $S$ is good with respect to $T_{0}$. By Lemma $5, T^{k}$ has $k$ completely independent spanning trees.

Theorem 7. If $G$ is a connected graph on $n$ vertices with $n \geq 2 k$ and $\delta(G) \geq k$, then the $k$-th power $G^{k}$ of $G$ has $k$ completely independent spanning trees.

Proof. If $k=1$, then the theorem holds trivially. Therefore, we may assume that $k \geq 2$. Now, suppose that $k=2$. Since $\delta(G) \geq 2$, we have that $G$ has a cycle $C_{m}$.

If $\left|V\left(C_{m}\right)\right| \geq 2 k=4$, then let $C_{m}=x_{1} x_{2} \cdots x_{m}$ and let a partition of the vertex set $V\left(C_{m}\right)$ be as follows:

$$
V_{i}=\left\{x_{j} \mid j \equiv i(\bmod 2), 1 \leq j \leq m\right\}, i \in[2]
$$

Since $m \geq 4$, we have $\left|V_{i}\right| \geq 2$. It is easy to see that the induced graph $G_{m}^{2}\left[V_{i}\right]$ is connected for $i \in[2]$. Actually, $G_{m}^{2}\left[V_{i}\right]$ is either a path or a cycle. If $x_{j} \in V_{i}$, then $\left|V_{3-i} \cap\left\{v \mid \operatorname{dist}_{C_{m}}\left(x_{j}, v\right) \leq 2\right\}\right| \geq 2$. Thus, $\operatorname{deg}_{B\left(V_{i}, V_{3-i}\right)}\left(x_{j}\right) \geq 2$ and $B\left(V_{i}, V_{3-i}\right)$ has no tree component. Hence, $\left(V_{1}, V_{2}\right)$ is a CIST-partition of $C_{m}^{2}$.

Let $H_{i}(i \in[q])$ be the connected components of graphs $G-C_{m}$. Since $G$ is connected, we have that there exists an edge which connects $H_{i}$ to $C_{m}$ for every
$i \in[q]$. Also, we have $\operatorname{dist}_{G}\left(x, V_{i}\right)=0$ for every vertex $x \in V_{i}(i \in[2])$ and $\operatorname{dist}_{G}\left(x, V_{3-i}\right)=1$. Hence, for the set $S=V\left(C_{m}\right)$, every vertex of $S$ is good with respect to $C_{m}$. By Lemma $5, G^{2}$ has 2 completely independent spanning trees.

Now we assume that $|C|<4$ for any cycle $C$ of $G$.
We choose a cycle $C=x_{1} x_{2} x_{3}$. Note that $n \geq 4$, therefore there exists a vertex $y \in V(G-C)$ such that $y x_{i} \in E(G)$. Without loss of generality, we assume that $y x_{1} \in E(G)$. Let $H=C \cup\left\{y x_{1}\right\}$ and let a partition of the vertex set $V\left(H^{2}\right)$ be as follows:

$$
V_{1}=\left\{x_{1}, x_{2}\right\}, V_{2}=\left\{x_{3}, y\right\}
$$

Since $n \geq 4$, we have $\left|V_{i}\right| \geq 2$. It is easy to see that $\left(V_{1}, V_{2}\right)$ is a CISTpartition of the vertex set $V\left(H^{2}\right)$. As we have stated in the previous case, for the set $S=V(H)$, every vertex of $S$ is good with respect to $H$. By Lemma 5 , $G^{2}$ has 2 completely independent spanning trees.

Thus, we only consider the case $k \geq 3$.
Case 1. There exists an edge $x y$ such that $|N(x)-N[y]| \geq k-1$ and $|N(y)-N[x]| \geq k-1$.

Let

$$
\begin{aligned}
& x y=x_{1} y_{1}, H=G\left[N\left[x_{1}\right] \cup N\left[y_{1}\right]\right]-N\left(x_{1}\right) \cap N\left(y_{1}\right), \\
& N\left(x_{1}\right)=\left\{y_{1}, x_{2}, x_{3}, \ldots, x_{\operatorname{deg}_{G}\left(x_{1}\right)}\right\}, \\
& N\left(y_{1}\right)=\left\{x_{1}, y_{2}, y_{3}, \ldots, y_{\operatorname{deg}_{G}\left(y_{1}\right)}\right\} .
\end{aligned}
$$

Since $\delta(G) \geq k \geq 3$ and $\operatorname{dist}(u, v) \leq 3$ for any two vertices $u$, $v$, combine with $|V(H)| \geq 2 k, H^{k}$ is a complete graph. Thus, any partition $\left(V_{1}, V_{2}, \ldots, V_{k}\right)$ of the vertex set $V\left(H^{k}\right)$ with $\left|V_{i}\right| \geq 2$ is a CIST-partition. Specially, we choose a partition of the vertex set $V\left(H^{k}\right)$ as follows:

$$
\begin{aligned}
V_{1} & =\left\{x_{1}, y_{1}\right\} \\
V_{i} & =\left\{x_{j} \mid j \equiv i(\bmod (k-1)), 2 \leq j \leq \operatorname{deg}_{G}\left(x_{1}\right)\right\} \\
& \cup\left\{y_{j} \mid j \equiv i(\bmod (k-1)), 2 \leq j \leq \operatorname{deg}_{G}\left(y_{1}\right)\right\}, i \in\{2, \ldots, k\} .
\end{aligned}
$$

For every $x \in V_{i}(i \geq 2)$, we have

$$
\operatorname{dist}_{G}\left(x, V_{i}\right)=0, \operatorname{dist}_{G}\left(x, V_{1}\right)=1, \operatorname{dist}_{G}\left(x, V_{j}\right)=2, j \neq i
$$

Hence, for the set $S=V(H)$, every vertex of $S$ is good with respect to $H$. By Lemma $5, G^{k}$ has $k$ completely independent spanning trees.

Case 2. There exists an edge $x y$ such that $|N(x)-N[y]| \geq k-2$ and $|N(x)-N[y]| \geq k-2$ and which does not satisfy Case 1.

Let

$$
H=G[N[x y]]
$$

Since $\delta(G) \geq k,|V(H)| \geq 2 k-1$.
Case 2.1. If $d(x)>k$ or $d(y)>k$, then $x y$ satisfies the Case 1 which is a contradiction.

Case 2.2. If $d(x)=k, d(y)=k$, then $|V(H)|=2 k-1$. Since $n \geq 2 k$, there exists a vertex $w \in V(G-H)$ such that $w$ is adjacent to a vertex $h$ of $H \backslash\{x, y\}$. In other words, $h \in N(x) \backslash\{y\}$ or $N(y) \backslash\{x\}$. We suppose that $H_{0}=G[V(H) \cup w]$.


Figure 1.
(1) We first consider the case $k \geq 4$. If $h \in N(x) \backslash\{y\}$ (or $h \in N(y) \backslash\{x\}$ ), then we label $H_{0}$ as in Figure 1. Let

$$
\begin{aligned}
& x=x_{3}, y=x_{2}\left(\text { or } x=x_{2}, y=x_{3}\right), \\
& N(x) \cap N(y)=\left\{x_{1}\right\}, w=y_{2}, \\
& N\left(x_{3}\right)=\left\{x_{2}, x_{1}, y_{1}, x_{4}, x_{5}, \ldots, x_{k}\right\}, \\
& N\left(x_{2}\right)=\left\{x_{3}, x_{1}, y_{3}, y_{4}, y_{5}, \ldots, y_{k}\right\} .
\end{aligned}
$$

Since $\delta(G) \geq k \geq 4, H_{0}^{k}$ is a complete graph. Thus, any partition $\left(V_{1}, V_{2}, \ldots\right.$, $\left.V_{k}\right)$ of the vertex set $V\left(H_{0}^{k}\right)$ with $\left|V_{i}\right| \geq 2$ is a CIST-partition. Specially, we choose a partition of the vertex set $V\left(H_{0}^{k}\right)$ as follows:

$$
V_{i}=\left\{x_{i}, y_{i}\right\}, i \in[k] .
$$

If $h=x_{\ell}(\ell=1,4,5, \ldots, k)$ or $h=y_{\ell}(\ell=1)$, then for $j \in[k]$ we have

$$
\begin{aligned}
& \operatorname{dist}_{G}\left(x_{1}, V_{1}\right)=0, \operatorname{dist}_{G}\left(x_{1}, V_{2}\right)=1, \operatorname{dist}_{G}\left(x_{1}, V_{j}\right) \leq 2(j \neq 1,2), \\
& \operatorname{dist}_{G}\left(y_{1}, V_{1}\right)=0, \operatorname{dist}_{G}\left(y_{1}, V_{3}\right)=1, \operatorname{dist}_{G}\left(y_{1}, V_{j}\right) \leq 2(j \neq 1,3),
\end{aligned}
$$

$\operatorname{dist}_{G}\left(y_{2}, V_{2}\right)=0, \operatorname{dist}_{G}\left(y_{2}, V_{\ell}\right)=1, \operatorname{dist}_{G}\left(y_{2}, V_{3}\right) \leq 2$,
$\operatorname{dist}_{G}\left(y_{2}, V_{j}\right) \leq 3(j \neq 2,3, \ell)$,
$\operatorname{dist}_{G}\left(y_{i}, V_{i}\right)=0, \operatorname{dist}_{G}\left(y_{i}, V_{2}\right)=1, \operatorname{dist}_{G}\left(y_{i}, V_{j}\right) \leq 2,3 \leq i \leq k(j \neq 2, i)$,
$\operatorname{dist}_{G}\left(x_{i}, V_{i}\right)=0, \operatorname{dist}_{G}\left(x_{i}, V_{3}\right)=1, \operatorname{dist}_{G}\left(x_{i}, V_{j}\right) \leq 2,4 \leq i \leq k(j \neq 3, i)$.
Again, for the set $S=V\left(H-\left\{x_{2}, x_{3}\right\}\right)$, every vertex of $S$ is good with respect to $H$. By Lemma $5, G^{k}$ has $k$ completely independent spanning trees.
(2) Now suppose that $k=3$.


Figure 2.
Claim 8. If a connected graph $G$ contains a subgraph which is isomorphic to one of the $H_{1}, H_{2}, H_{3}$ in Figure $2\left(\right.$ where $H_{3}=H_{3}^{\prime} \cup e$ and $e$ has exactly one end in $V\left(H_{3}^{\prime}\right)$ ), then $G^{3}$ has 3 completely independent spanning trees.

Proof. If $H_{1} \subseteq G$, then the proof follows by Case 1. If $H_{2} \subseteq G$, then $H_{2}$ is isomorphic to $H_{0}=G\left[V(H) \cup y_{2}\right]$ in Case 2.2(1), where $H=G[W], W=$ $\left\{x_{1}, x_{2}, x_{3}, y_{1}, y_{3}\right\}$ and $y_{2}$ is adjacent to a vertex $x_{1}$ of $H$. The proof follows by Case 2.2(1). Suppose $H_{3} \subseteq G$, if $e$ is adjacent to $x$, then there exists a subgraph of $H_{3}$ which is isomorphic to $H_{2}$. Otherwise, $e$ is adjacent to $V\left(H_{3}^{\prime}\right) \backslash x$, then we obtain a subgraph of $H_{3}$ which is isomorphic to $H_{1}$. The claim is true for $H_{1}, H_{2}, H_{3}$.

Now we begin to prove the case $k=3$ and we relabel $H$ as in left of Figure 3.

Since $k=3$, we have $h \neq y, z$. If $h=x$, then $H \cup\{w h\}$ contains a subgraph which is isomorphic to $H_{2}$, it is true by Claim 8. Thus, $h=u$ or $h=v$. By symmetry, we may assume that $h=u$ and $H_{0}=G[V(H) \cup w]$, as in right of Figure 3.

If $\operatorname{deg}_{G-H_{0}}(u) \geq 1$, then $G$ contains a subgraph which is isomorphic to $H_{1}$, it is true by Claim 8. Thus, $N(u) \subseteq V\left(H_{0}\right)$.


Figure 3.

If $u v \in E(G)$, then there exists a subgraph of $G$ which is isomorphic to $H_{3}$, it is true by Claim 8. Otherwise, $u v \notin E(G)$ and $u x \in E(G)$.

If $x v \in E(G)$, then we obtain a subgraph of $G$ which is isomorphic to $H_{1}$, it is true by Claim 8. Otherwise, $x v \notin E(G)$. Since $\delta \geq 3$, we have $\operatorname{deg}_{G-\{x, y, u\}}(v) \geq$ 2. Thus, we also can obtain a subgraph of $G$ which is isomorphic to $H_{1}$, it is true by Claim 8.

Hence, $G^{3}$ has 3 completely independent spanning trees.
Case 3. Every edge $e$ of $G$ is contained in at least two triangles. By Lemma 4, $G$ contains a HIST $T$. Also, by Lemma 6 , the $k$-th power $T^{k}$ of $T$ has $k$ completely independent spanning trees.

Hence, the $k$-th power $G^{k}$ has $k$ completely independent spanning trees. The proof of Theorem 7 is completed.

An immediate consequence of Theorem 7 is the following corollary.
Corollary 9. If $G$ is a $k$-connected graph on $n$ vertices with $n \geq 2 k$, then the $k$-th power $G^{k}$ of $G$ has $k$ completely independent spanning trees.

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