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COMPLETELY INDEPENDENT SPANNING TREES IN k-TH POWER OF GRAPHS

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Abstract

Let $T_1, T_2, \ldots, T_k$ be spanning trees of a graph G. For any two vertices u, v of G, if the paths from u to v in these k trees are pairwise openly disjoint, then we say that $T_1, T_2, \ldots, T_k$ are completely independent. Araki showed that the square of a 2-connected graph G on n vertices with $n \geq 4$ has two completely independent spanning trees. In this paper, we prove that the k-th power of a k-connected graph G on n vertices with $n \geq 2k$ has k completely independent spanning trees. In fact, we prove a stronger result: if G is a connected graph on n vertices with $\delta(G) \geq k$ and $n \geq 2k$, then the k-th power G^k of G has k completely independent spanning trees.

Keywords: completely independent spanning tree, power of graphs, spanning trees.

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1. Introduction

The graphs considered in this paper are finite, undirected, and simple (no loops or multiple edges). The *vertex set* and the *edge set* of G are denoted by $V(G)$ and $E(G)$, respectively. For a vertex $v \in V(G)$, the *neighbour set* $N_G(v)$ is the set of vertices adjacent to v, $\deg_G(v) = |N_G(v)|$ is the *degree* of v. For a subgraph H of G, $N_H(v)$ is the set of the neighbour of v which are in H, and $\deg_H(v) = |N_H(v)|$ is the degree of v in H. The set of (close) neighbour of a edge e in G is denoted by $N_G(e)$ ($N_G[e]$). When no confusion can occur, we shall write $N(v)$, $N(e)$, $N[e]$, instead of $N_G(v)$, $N_G(e)$, $N_G[e]$, respectively. We denote by $\delta(G)$ the minimum degree of the vertices of G. For a subset $U \subseteq V(G)$, the subgraph induced by U

is denoted by $G[U]$, which is the graph on U whose edges are precisely the edges of G with both ends in U. We use $G - U$ to denote the subgraph induced by $V(G) \setminus U$, that is, the graph obtained from G by deleting all the vertices of U together with all the edges with at least one end in U. If $U = \{u\}$, then we shall use $G - u$ instead of $G - \{u\}$. For a subset U, V of $V(G)$, we denote $E_G(U, V)$ the set of edges of G with one end in U and the other end in V.

For $u \in V(G)$ and $U \subseteq V(G)$, the distance between u and U, denoted by $\mathrm{dist}_G(u, U)$, is the length of a shortest path from u to a vertex in U. When U consists of a single vertex, we write $\mathrm{dist}_G(u, v)$ instead of $\mathrm{dist}_G(u, \{v\})$. For a positive integer k, the *k-th power* G^k of a graph G is the graph G^k whose vertex set is $V(G)$, two distinct vertices being adjacent in G^k if and only if their distance in G is at most k. If $k = 1$, $G^1 = G$. In particular, the graph G^2 is referred to as the square of G, the graph G^3 as the cube of G. We say G is *k-connected* if $|V(G)| > k$ and $G - X$ is connected for every set $X \subset V(G)$ with $|X| < k$. For simplicity, we denote $[k] = \{1, \ldots, k\}$.

A tree T of G is a *spanning tree* of G if $V(T) = V(G)$. A leaf is a vertex of degree 1. An internal vertex is a vertex of degree at least 2. A *rooted tree* T is a tree with a specified vertex x, called the *root* of T. A *x-tree* T refer to a rooted tree with root x. The *level* of a vertex v of the x-tree T is the length of the path from the root x to v, the *depth* of the x-tree T is the maximum level of a vertex in the tree, denoted by $D(T)$. A graph is called *homeomorphically irreducible* if it contains no vertices of degree 2. A *homeomorphically irreducible tree* is called a HIT, and a *homeomorphically irreducible spanning tree* of a graph is called a HIST of the graph. A *caterpillar* is a tree in which the internal vertices induce a path.

Let x, y be two vertices of G. An *(x, y)-path* is a path with the two ends x and y. Two (x, y)-paths P_1, P_2 are *openly disjoint* if they have no common edge and no common vertex except for the two ends x and y. Let $T_1, T_2, \ldots, T_k$ be spanning trees in a graph H. For any two vertices u, v of H, if the paths from u to v in these k trees are pairwise openly disjoint, then we say that $T_1, T_2, \ldots, T_k$ are *completely independent spanning trees*(CISTs) in G. The concept of completely independent spanning trees was proposed by Hasunuma [4]. In [4], Hasunuma gave a characterization for CISTs and proved that the underlying graph of a k-connected line digraph always contains k CISTs. It is well known [7, 9] that every $2k$-edge-connected graph has k edge-disjoint spanning trees. Motivated by this, Hasunuma [5] conjectured that every $2k$-connected graph has k CISTs. However, Péterfalvi [8] disproved the conjecture by constructing a k-connected graph, for each $k \geq 2$, which does not have two CISTs. Recently, Araki [1] provided a new characterization of the existence of k CISTs and showed the following results.

Theorem 1 [1]. *Let G be a graph with $n \geq 7$ vertices. If $\delta(G) \geq n/2$, then G has two completely independent spanning trees.*

In [6], Hong *et al.* give a generalization of Theorem 1.1.

Theorem 2 [1]. *If G is a 2-connected graph G on n vertices with $n \geq 4$, then the square G^2 has two completely independent spanning trees.*

It is interesting to note that the above Dirac's conditions and Fleischner's conditions is sufficient for a graph to be Hamiltonian. So, Araki [1] asked that whether other sufficient conditions for a graph to be Hamiltonian also imply the existence of two CISTs. In [3], Fan *et al.* confirmed that the well-known Ore's condition also implies the existence two CISTs.

In this paper, we generalize Theorem 2. In fact, we prove a stronger result which is Theorem 7.

First, we give the preliminaries of our results as follows.

Let $(V_1, V_2, \ldots, V_k)$ be a partition of the vertex set $V(G)$ and, for $i \neq j$, $B(V_i, V_j, G)$ be a bipartite graph with the edge set $\{uv \,|\, uv \in E(G), u \in V_i$ and $v \in V_j\}$. If the graph G is clear from the context, we may use $B(V_1, V_2)$ instead of $B(V_1, V_2, G)$. A partition $(V_1, V_2, \ldots, V_k)$ is called a CIST-partition of G if it satisfies the following two conditions:

(1) for $i = 1, 2, \ldots, k$, the induced subgraph $G[V_i]$ is connected, and

(2) for any $i \neq j$, the bipartite graph $B(V_i, V_j)$ has no tree components, that is, every connected component H of $B(V_i, V_j)$ satisfies $|E(H)| \geq |V(H)|$.

The following result obtained by Araki [1] plays a key role in our proof.

Lemma 3 [1]. *A connected graph G has k completely independent spanning trees if and only if there is a CIST-partition $(V_1, \ldots, V_k)$ of $V(G)$.*

Lemma 4 [2]. *Let G be a graph with every edge in at least two triangles. Then G contains a HIST.*

Now we give the definition of a good vertex x of H to use in the proof of our result.

Given a graph H and a partition $(V_1, \ldots, V_k)$ of its vertex set, let

$$d_{x_i} = dst_H(x, V_i), x \in V(H), i \in [k].$$

For every $x \in V(H)$, there exists a corresponding sequence $(d_{t_1}, d_{t_2}, \ldots, d_{t_k})$ such that $d_{t_1} \geq d_{t_2} \geq \cdots \geq d_{t_k}$ and $t_1, t_2, \ldots, t_k$ is a permutation of $x_1, x_2, \ldots, x_k$.

We say that a vertex x is *good with respect to H* if $d_{t_j} \leq k - j$ $(j \in [k])$.

Lemma 5. *Let G be a connected graph and $H \subseteq G$. Suppose that there are q components $H_1, H_2, \ldots, H_q$ in $G - H$ and S is a subset of $V(H)$ with the following proporty: for every component H_s $(s \in [q])$ of $G - H$, there exist a vertex $u \in V(H_s)$ and a vertex $v \in S$ such that $uv \in E(G)$. If H^k has a CIST-partition $(V_1, \ldots, V_k)$ and every vertex of S is good with respect to H, then G^k has k completely independent spanning trees.*

Proof. Let $(V_1, \ldots, V_k)$ be a CIST-partition of the vertex set $V(H^k)$, we try to find a CIST-partition of the vertex set $V(G^k)$ by extending the partition $(V_1, \ldots, V_k)$.

Let $H_1, H_2, \ldots, H_q$ be q components of $G - H$. For every component H_s ($s \in [q]$), we choose a spanning tree T_s and a vertex $u \in V(T_s)$ such that $uv \in E(G)$, where $v \in S$. We may assume that $T_s^{'} = T_s \cup \{vu\}$ and $T_s^{'}$ is a v-tree. Let $d_{x_i} = \mathrm{dist}_H(v, V_i),\ i \in [k]$. For the vertex v, there exists a sequence $(d_{t_1}, d_{t_2}, \ldots, d_{t_k})$ such that $d_{t_1} \geq d_{t_2} \geq \cdots \geq d_{t_k}$ and $t_1, t_2, \ldots, t_k$ is a permutation of $x_1, x_2, \ldots, x_k$. Let α be a one-to-one correspondence from $[k]$ to $[k]$ such that $\mathrm{dist}_H(v, V_{\alpha(j)}) = d_{t_j}$.

Let L_j be the vertex set of all vertices in j-th level of $T_s^{'}$ for $j \in \{1, \ldots, D(T_s^{'})\}$. For every $w \in L_j$, we assign it to $V_{\alpha(j \pmod k)}$. For other components, we repeat the above operation, and it follows that we obtain a new partition $(V_1^{'}, V_2^{'}, \ldots, V_k^{'})$ of the vertex set $V(G^k)$. It remains to show that $(V_1^{'}, V_2^{'}, \ldots, V_k^{'})$ is a CIST-partition of $V(G^k)$.

If $1 \leq j \leq k$, since every vertex of S is good with respect to H and $L_j \subset V_{\alpha(j)}^{'}$ for any $w \in L_j$, we have

$$\mathrm{dist}_G(w, V_{\alpha(j)}) \leq \mathrm{dist}_G(v, V_{\alpha(j)}) + \mathrm{dist}_G(v, w) \leq k - j + j = k.$$

Thus,

$$E_{G^k}(\{w\}, V_{\alpha(j)}) \neq \emptyset.$$

If $k + 1 \leq j \leq D(T_s^{'})$, for any $w_1 \in L_j$, $w_2 \in L_{j-k}$, then

$$L_j, L_{j-k} \subset V_{\alpha(j \pmod k)}^{'}, \ \mathrm{dist}_G(w_1, w_2) \leq k.$$

Thus,

$$w_1 w_2 \in E(G^k).$$

It is easy to see that the induced graph $G^k[V_i^{'}]$ is connected for $i \in [k]$.

Note that $\deg_{B(V_i^{'}, V_j^{'})}(w) \geq 1$ for every vertex $w \in V_i^{'} \setminus V_i$, $j^{'} \neq i^{'}$. Since $B(V_i, V_j)$ has no tree components and the vertex w is adjacent to V_j in G^k by alternative path between $V_i^{'}$ and $V_j^{'}$, we get that $B(V_i^{'}, V_j^{'})$ has no tree component.

Hence, $(V_1^{'}, V_2^{'}, \ldots, V_k^{'})$ is a CIST-partition of $V(G^k)$. By Lemma 3, G^k has k completely independent spanning trees. ■

Lemma 6. *For a homeomorphically irreducible tree (HIT) T with $|V(T)| \geq 2k$, the k-th power T^k of T has k completely independent spanning trees.*

Proof. We first consider the longest path $P = x_0 x_1 \cdots x_{|P|}$ of T.

Case 1. $|P| < k$. If $|P| < k$, then T^k is a complete graph. Also, $|V(T)| \geq 2k$ and any partition $(V_1, V_2, \ldots, V_k)$ of $V(T)$ with $|V_i| \geq 2$ is a CIST-partition of $V(T^k)$. Hence, by Lemma 3, T^k has k completely independent spanning trees.

Case 2. $|P| \geq k$. Denote $P_0 = x_0x_1 \cdots x_k$. Since T is a homeomorphically irreducible tree (HIT), we choose a caterpillar T_0 such that its internal vertices are $V(P_0 - \{x_0, x_k\})$ and its leaf vertices are $N(x_1) \cup \cdots \cup N(x_{k-1}) \backslash V(P_0 - \{x_0, x_k\})$. We regard T_0 as a rooted tree which is rooted at x_0 in the following proof.

Let L_i be the set of all vertices with the same level of x_i in T_0, where $i \in \{0, 1, \ldots, k\}$. Note that

$$L_0 = \{x_0\},\ L_1 = \{x_1\},\ |L_i| \geq 2,\ i \in \{2, \ldots, k\}.$$

We have $|T_0| \geq 2k$ and the distance between x and y is at most k for every pair $x, y \in V(T_0)$. Thus, $T^k[T_0]$ is a complete graph and any partition $(V_1, V_2, \ldots, V_k)$ of the vertex set $V(T^k[T_0])$ with $|V_i| \geq 2$ is a CIST-partition. Specially, we choose a partition of $V(T^k[T_0])$ as $(L_0 \cup L_1, L_2, \ldots, L_k)$.

Since P is a longest path in T and T_0 is a caterpillar, $E_T(L_0 \cup L_1 \cup L_2 \cup V(P_0 - x_k), T - T_0)$ is empty. Let $S = V(T_0 - (L_0 \cup L_1 \cup L_2 \cup V(P_0 - x_k)))$, it follows that $xx_{i-1} \in E(T)$ for any $x \in S \cap L_i$ $(i \geq 3)$ and

$$\begin{aligned}
&\mathrm{dist}_T(x, L_i) = 0,\\
&\mathrm{dist}_T(x, L_j) = i - j,\ 1 \leq j < i,\\
&\mathrm{dist}_T(x, L_j) = j - i + 2,\ i + 1 \leq j \leq k.
\end{aligned}$$

In addition, there exists a corresponding sequence $(d_{t_1}, d_{t_2}, \ldots, d_{t_k})$ such that $d_{t_1} \geq d_{t_2} \geq \cdots \geq d_{t_k}$ and $t_1, t_2, \ldots, t_k$ is a permutation of $\mathrm{dist}_T(x, L_0 \cup L_1)$, $\mathrm{dist}_T(x, L_2), \ldots, \mathrm{dist}_T(x, L_k)$. So, $d_{t_j} \leq k - j$ $(j \in [k])$. Hence, every vertex of S is good with respect to T_0. By Lemma 5, T^k has k completely independent spanning trees. ∎

Theorem 7. *If G is a connected graph on n vertices with $n \geq 2k$ and $\delta(G) \geq k$, then the k-th power G^k of G has k completely independent spanning trees.*

Proof. If $k = 1$, then the theorem holds trivially. Therefore, we may assume that $k \geq 2$. Now, suppose that $k = 2$. Since $\delta(G) \geq 2$, we have that G has a cycle C_m.

If $|V(C_m)| \geq 2k = 4$, then let $C_m = x_1x_2 \cdots x_m$ and let a partition of the vertex set $V(C_m)$ be as follows:

$$V_i = \{x_j \,|\, j \equiv i \pmod 2,\ 1 \leq j \leq m\},\ i \in [2].$$

Since $m \geq 4$, we have $|V_i| \geq 2$. It is easy to see that the induced graph $G_m^2[V_i]$ is connected for $i \in [2]$. Actually, $G_m^2[V_i]$ is either a path or a cycle. If $x_j \in V_i$, then $|V_{3-i} \cap \{v \,|\, \mathrm{dist}_{C_m}(x_j, v) \leq 2\}| \geq 2$. Thus, $\deg_{B(V_i, V_{3-i})}(x_j) \geq 2$ and $B(V_i, V_{3-i})$ has no tree component. Hence, (V_1, V_2) is a CIST-partition of C_m^2.

Let H_i $(i \in [q])$ be the connected components of graphs $G - C_m$. Since G is connected, we have that there exists an edge which connects H_i to C_m for every

$i \in [q]$. Also, we have $\operatorname{dist}_G(x, V_i) = 0$ for every vertex $x \in V_i$ ($i \in [2]$) and $\operatorname{dist}_G(x, V_{3-i}) = 1$. Hence, for the set $S = V(C_m)$, every vertex of S is good with respect to C_m. By Lemma 5, G^2 has 2 completely independent spanning trees.

Now we assume that $|C| < 4$ for any cycle C of G.

We choose a cycle $C = x_1x_2x_3$. Note that $n \geq 4$, therefore there exists a vertex $y \in V(G-C)$ such that $yx_i \in E(G)$. Without loss of generality, we assume that $yx_1 \in E(G)$. Let $H = C \cup \{yx_1\}$ and let a partition of the vertex set $V(H^2)$ be as follows:

$$V_1 = \{x_1, x_2\}, V_2 = \{x_3, y\}.$$

Since $n \geq 4$, we have $|V_i| \geq 2$. It is easy to see that (V_1, V_2) is a CIST-partition of the vertex set $V(H^2)$. As we have stated in the previous case, for the set $S = V(H)$, every vertex of S is good with respect to H. By Lemma 5, G^2 has 2 completely independent spanning trees.

Thus, we only consider the case $k \geq 3$.

Case 1. There exists an edge xy such that $|N(x) - N[y]| \geq k - 1$ and $|N(y) - N[x]| \geq k - 1$.

Let

$$\begin{aligned}
&xy = x_1y_1, H = G[N[x_1] \cup N[y_1]] - N(x_1) \cap N(y_1),\\
&N(x_1) = \left\{y_1, x_2, x_3, \ldots, x_{\deg_G(x_1)}\right\},\\
&N(y_1) = \left\{x_1, y_2, y_3, \ldots, y_{\deg_G(y_1)}\right\}.
\end{aligned}$$

Since $\delta(G) \geq k \geq 3$ and $\operatorname{dist}(u, v) \leq 3$ for any two vertices u, v, combine with $|V(H)| \geq 2k$, H^k is a complete graph. Thus, any partition $(V_1, V_2, \ldots, V_k)$ of the vertex set $V(H^k)$ with $|V_i| \geq 2$ is a CIST-partition. Specially, we choose a partition of the vertex set $V(H^k)$ as follows:

$$\begin{aligned}
&V_1 = \{x_1, y_1\},\\
&V_i = \{x_j \mid j \equiv i \,(\bmod\ (k-1)),\ 2 \leq j \leq \deg_G(x_1)\}\\
&\quad \cup \{y_j \mid j \equiv i \,(\bmod\ (k-1)),\ 2 \leq j \leq \deg_G(y_1)\},\ i \in \{2, \ldots, k\}.
\end{aligned}$$

For every $x \in V_i$ ($i \geq 2$), we have

$$\operatorname{dist}_G(x, V_i) = 0,\ \operatorname{dist}_G(x, V_1) = 1,\ \operatorname{dist}_G(x, V_j) = 2,\ j \neq i.$$

Hence, for the set $S = V(H)$, every vertex of S is good with respect to H. By Lemma 5, G^k has k completely independent spanning trees.

Case 2. There exists an edge xy such that $|N(x) - N[y]| \geq k - 2$ and $|N(x) - N[y]| \geq k - 2$ and which does not satisfy Case 1.

Let

$$H = G[N[xy]].$$

Since $\delta(G) \geq k$, $|V(H)| \geq 2k-1$.

Case 2.1. If $d(x) > k$ or $d(y) > k$, then xy satisfies the Case 1 which is a contradiction.

Case 2.2. If $d(x) = k, d(y) = k$, then $|V(H)| = 2k-1$. Since $n \geq 2k$, there exists a vertex $w \in V(G-H)$ such that w is adjacent to a vertex h of $H\backslash\{x,y\}$. In other words, $h \in N(x)\backslash\{y\}$ or $N(y)\backslash\{x\}$. We suppose that $H_0 = G[V(H) \cup w]$.

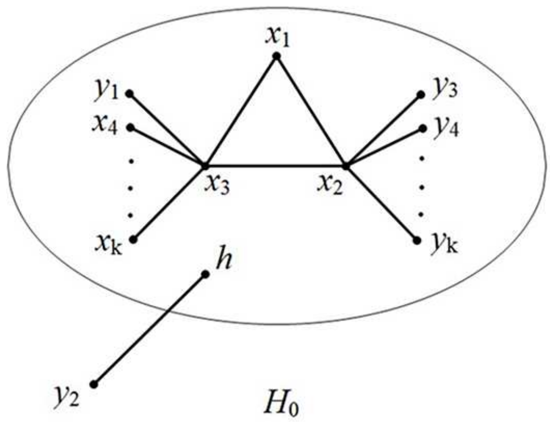

Figure 1.

(1) We first consider the case $k \geq 4$. If $h \in N(x)\backslash\{y\}$ (or $h \in N(y)\backslash\{x\}$), then we label H_0 as in Figure 1. Let

$$\begin{aligned}
&x = x_3,\ y = x_2 \text{ (or } x = x_2,\ y = x_3\text{)},\\
&N(x) \cap N(y) = \{x_1\},\ w = y_2,\\
&N(x_3) = \{x_2, x_1, y_1, x_4, x_5, \ldots, x_k\},\\
&N(x_2) = \{x_3, x_1, y_3, y_4, y_5, \ldots, y_k\}.
\end{aligned}$$

Since $\delta(G) \geq k \geq 4$, H_0^k is a complete graph. Thus, any partition $(V_1, V_2, \ldots, V_k)$ of the vertex set $V(H_0^k)$ with $|V_i| \geq 2$ is a CIST-partition. Specially, we choose a partition of the vertex set $V(H_0^k)$ as follows:

$$V_i = \{x_i, y_i\},\ i \in [k].$$

If $h = x_\ell$ ($\ell = 1, 4, 5, \ldots, k$) or $h = y_\ell$ ($\ell = 1$), then for $j \in [k]$ we have

$$\mathrm{dist}_G(x_1, V_1) = 0,\ \mathrm{dist}_G(x_1, V_2) = 1,\ \mathrm{dist}_G(x_1, V_j) \leq 2\ (j \neq 1, 2),$$

$$\mathrm{dist}_G(y_1, V_1) = 0,\ \mathrm{dist}_G(y_1, V_3) = 1,\ \mathrm{dist}_G(y_1, V_j) \leq 2\ \ (j \neq 1, 3),$$

$$\mathrm{dist}_G(y_2, V_2) = 0,\ \mathrm{dist}_G(y_2, V_\ell) = 1,\ \mathrm{dist}_G(y_2, V_3) \leq 2,$$

$$\mathrm{dist}_G(y_2, V_j) \leq 3\ (j \neq 2, 3, \ell),$$

$$\mathrm{dist}_G(y_i, V_i) = 0,\ \mathrm{dist}_G(y_i, V_2) = 1,\ \mathrm{dist}_G(y_i, V_j) \leq 2, 3 \leq i \leq k\ (j \neq 2, i),$$

$$\mathrm{dist}_G(x_i, V_i) = 0,\ \mathrm{dist}_G(x_i, V_3) = 1,\ \mathrm{dist}_G(x_i, V_j) \leq 2, 4 \leq i \leq k\ (j \neq 3, i).$$

Again, for the set $S = V(H - \{x_2, x_3\})$, every vertex of S is good with respect to H. By Lemma 5, G^k has k completely independent spanning trees.

(2) Now suppose that $k = 3$.

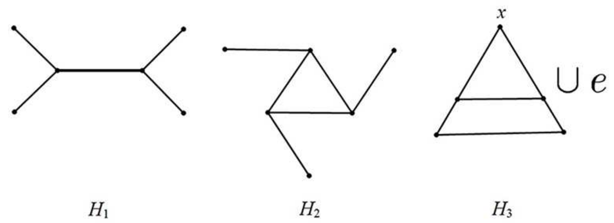

Figure 2.

Claim 8. *If a connected graph G contains a subgraph which is isomorphic to one of the H_1,H_2, H_3 in Figure 2 (where $H_3 = H_3' \cup e$ and e has exactly one end in $V(H_3')$), then G^3 has 3 completely independent spanning trees.*

Proof. If $H_1 \subseteq G$, then the proof follows by Case 1. If $H_2 \subseteq G$, then H_2 is isomorphic to $H_0 = G[V(H) \cup y_2]$ in Case 2.2(1), where $H = G[W]$, $W = \{x_1, x_2, x_3, y_1, y_3\}$ and y_2 is adjacent to a vertex x_1 of H. The proof follows by Case 2.2(1). Suppose $H_3 \subseteq G$, if e is adjacent to x, then there exists a subgraph of H_3 which is isomorphic to H_2. Otherwise, e is adjacent to $V(H_3') \setminus x$, then we obtain a subgraph of H_3 which is isomorphic to H_1. The claim is true for H_1, H_2, H_3. □

Now we begin to prove the case $k = 3$ and we relabel H as in left of Figure 3.

Since $k = 3$, we have $h \neq y, z$. If $h = x$, then $H \cup \{wh\}$ contains a subgraph which is isomorphic to H_2, it is true by Claim 8. Thus, $h = u$ or $h = v$. By symmetry, we may assume that $h = u$ and $H_0 = G[V(H) \cup w]$, as in right of Figure 3.

If $\deg_{G-H_0}(u) \geq 1$, then G contains a subgraph which is isomorphic to H_1, it is true by Claim 8. Thus, $N(u) \subseteq V(H_0)$.

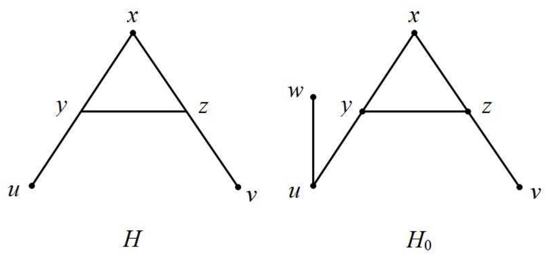

Figure 3.

If $uv \in E(G)$, then there exists a subgraph of G which is isomorphic to H_3, it is true by Claim 8. Otherwise, $uv \notin E(G)$ and $ux \in E(G)$.

If $xv \in E(G)$, then we obtain a subgraph of G which is isomorphic to H_1, it is true by Claim 8. Otherwise, $xv \notin E(G)$. Since $\delta \geq 3$, we have $\deg_{G-\{x,y,u\}}(v) \geq 2$. Thus, we also can obtain a subgraph of G which is isomorphic to H_1, it is true by Claim 8.

Hence, G^3 has 3 completely independent spanning trees.

Case 3. Every edge e of G is contained in at least two triangles. By Lemma 4, G contains a HIST T. Also, by Lemma 6, the k-th power T^k of T has k completely independent spanning trees.

Hence, the k-th power G^k has k completely independent spanning trees. The proof of Theorem 7 is completed. ■

An immediate consequence of Theorem 7 is the following corollary.

Corollary 9. *If G is a k-connected graph on n vertices with $n \geq 2k$, then the k-th power G^k of G has k completely independent spanning trees.*

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