Discussiones Mathematicae
Graph Theory 38 (2018) 801–810
doi:10.7151/dmgt.2038

COMPLETELY INDEPENDENT SPANNING TREES IN k-TH POWER OF GRAPHS

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Abstract

Let $T_1, T_2, \ldots, T_k$ be spanning trees of a graph G. For any two vertices u, v of G, if the paths from u to v in these k trees are pairwise openly disjoint, then we say that $T_1, T_2, \ldots, T_k$ are completely independent. Araki showed that the square of a 2-connected graph G on n vertices with $n \geq 4$ has two completely independent spanning trees. In this paper, we prove that the k-th power of a k-connected graph G on n vertices with $n \geq 2k$ has k completely independent spanning trees. In fact, we prove a stronger result: if G is a connected graph on n vertices with $\delta(G) \geq k$ and $n \geq 2k$, then the k-th power G^k of G has k completely independent spanning trees.

Keywords: completely independent spanning tree, power of graphs, spanning trees.

2010 Mathematics Subject Classification: 05C05.

1. Introduction

The graphs considered in this paper are finite, undirected, and simple (no loops or multiple edges). The *vertex set* and the *edge set* of G are denoted by $V(G)$ and $E(G)$, respectively. For a vertex $v \in V(G)$, the *neighbour set* $N_G(v)$ is the set of vertices adjacent to v, $\deg_G(v) = |N_G(v)|$ is the *degree* of v. For a subgraph H of G, $N_H(v)$ is the set of the neighbour of v which are in H, and $\deg_H(v) = |N_H(v)|$ is the degree of v in H. The set of (close) neighbour of a edge e in G is denoted by $N_G(e)$ ($N_G[e]$). When no confusion can occur, we shall write $N(v)$, $N(e)$, $N[e]$, instead of $N_G(v)$, $N_G(e)$, $N_G[e]$, respectively. We denote by $\delta(G)$ the minimum degree of the vertices of G. For a subset $U \subseteq V(G)$, the subgraph induced by U

is denoted by $G[U]$, which is the graph on U whose edges are precisely the edges of G with both ends in U. We use $G - U$ to denote the subgraph induced by $V(G) \setminus U$, that is, the graph obtained from G by deleting all the vertices of U together with all the edges with at least one end in U. If $U = \{u\}$, then we shall use $G - u$ instead of $G - \{u\}$. For a subset U, V of $V(G)$, we denote $E_G(U, V)$ the set of edges of G with one end in U and the other end in V.

For $u \in V(G)$ and $U \subseteq V(G)$, the distance between u and U, denoted by $\mathrm{dist}_G(u, U)$, is the length of a shortest path from u to a vertex in U. When U consists of a single vertex, we write $\mathrm{dist}_G(u, v)$ instead of $\mathrm{dist}_G(u, \{v\})$. For a positive integer k, the *k-th power* G^k of a graph G is the graph G^k whose vertex set is $V(G)$, two distinct vertices being adjacent in G^k if and only if their distance in G is at most k. If $k = 1$, $G^1 = G$. In particular, the graph G^2 is referred to as the square of G, the graph G^3 as the cube of G. We say G is *k-connected* if $|V(G)| > k$ and $G - X$ is connected for every set $X \subset V(G)$ with $|X| < k$. For simplicity, we denote $[k] = \{1, \ldots, k\}$.

A tree T of G is a *spanning tree* of G if $V(T) = V(G)$. A leaf is a vertex of degree 1. An internal vertex is a vertex of degree at least 2. A *rooted tree* T is a tree with a specified vertex x, called the *root* of T. A *x-tree* T refer to a rooted tree with root x. The *level* of a vertex v of the x-tree T is the length of the path from the root x to v, the *depth* of the x-tree T is the maximum level of a vertex in the tree, denoted by $D(T)$. A graph is called *homeomorphically irreducible* if it contains no vertices of degree 2. A *homeomorphically irreducible tree* is called a HIT, and a *homeomorphically irreducible spanning tree* of a graph is called a HIST of the graph. A *caterpillar* is a tree in which the internal vertices induce a path.

Let x, y be two vertices of G. An *(x, y)-path* is a path with the two ends x and y. Two (x, y)-paths P_1, P_2 are *openly disjoint* if they have no common edge and no common vertex except for the two ends x and y. Let $T_1, T_2, \ldots, T_k$ be spanning trees in a graph H. For any two vertices u, v of H, if the paths from u to v in these k trees are pairwise openly disjoint, then we say that $T_1, T_2, \ldots, T_k$ are *completely independent spanning trees*(CISTs) in G. The concept of completely independent spanning trees was proposed by Hasunuma [4]. In [4], Hasunuma gave a characterization for CISTs and proved that the underlying graph of a k-connected line digraph always contains k CISTs. It is well known [7, 9] that every $2k$-edge-connected graph has k edge-disjoint spanning trees. Motivated by this, Hasunuma [5] conjectured that every $2k$-connected graph has k CISTs. However, Péterfalvi [8] disproved the conjecture by constructing a k-connected graph, for each $k \geq 2$, which does not have two CISTs. Recently, Araki [1] provided a new characterization of the existence of k CISTs and showed the following results.

Theorem 1 [1]. *Let G be a graph with $n \geq 7$ vertices. If $\delta(G) \geq n/2$, then G has two completely independent spanning trees.*

In [6], Hong *et al.* give a generalization of Theorem 1.1.

Theorem 2 [1]. *If G is a 2-connected graph G on n vertices with $n \geq 4$, then the square G^2 has two completely independent spanning trees.*

It is interesting to note that the above Dirac's conditions and Fleischner's conditions is sufficient for a graph to be Hamiltonian. So, Araki [1] asked that whether other sufficient conditions for a graph to be Hamiltonian also imply the existence of two CISTs. In [3], Fan *et al.* confirmed that the well-known Ore's condition also implies the existence two CISTs.

In this paper, we generalize Theorem 2. In fact, we prove a stronger result which is Theorem 7.

First, we give the preliminaries of our results as follows.

Let $(V_1, V_2, \ldots, V_k)$ be a partition of the vertex set $V(G)$ and, for $i \neq j$, $B(V_i, V_j, G)$ be a bipartite graph with the edge set $\{uv \mid uv \in E(G), u \in V_i$ and $v \in V_j\}$. If the graph G is clear from the context, we may use $B(V_1, V_2)$ instead of $B(V_1, V_2, G)$. A partition $(V_1, V_2, \ldots, V_k)$ is called a CIST-partition of G if it satisfies the following two conditions:

(1) for $i = 1, 2, \ldots, k$, the induced subgraph $G[V_i]$ is connected, and

(2) for any $i \neq j$, the bipartite graph $B(V_i, V_j)$ has no tree components, that is, every connected component H of $B(V_i, V_j)$ satisfies $|E(H)| \geq |V(H)|$.

The following result obtained by Araki [1] plays a key role in our proof.

Lemma 3 [1]. *A connected graph G has k completely independent spanning trees if and only if there is a CIST-partition $(V_1, \ldots, V_k)$ of $V(G)$.*

Lemma 4 [2]. *Let G be a graph with every edge in at least two triangles. Then G contains a HIST.*

Now we give the definition of a good vertex x of H to use in the proof of our result.

Given a graph H and a partition $(V_1, \ldots, V_k)$ of its vertex set, let

$$d_{x_i} = dst_H(x, V_i), x \in V(H), i \in [k].$$

For every $x \in V(H)$, there exists a corresponding sequence $(d_{t_1}, d_{t_2}, \ldots, d_{t_k})$ such that $d_{t_1} \geq d_{t_2} \geq \cdots \geq d_{t_k}$ and $t_1, t_2, \ldots, t_k$ is a permutation of $x_1, x_2, \ldots, x_k$.

We say that a vertex x is *good with respect to H* if $d_{t_j} \leq k - j$ $(j \in [k])$.

Lemma 5. *Let G be a connected graph and $H \subseteq G$. Suppose that there are q components $H_1, H_2, \ldots, H_q$ in $G - H$ and S is a subset of $V(H)$ with the following proporty: for every component H_s $(s \in [q])$ of $G - H$, there exist a vertex $u \in V(H_s)$ and a vertex $v \in S$ such that $uv \in E(G)$. If H^k has a CIST-partition $(V_1, \ldots, V_k)$ and every vertex of S is good with respect to H, then G^k has k completely independent spanning trees.*

Proof. Let $(V_1, \dots, V_k)$ be a CIST-partition of the vertex set $V(H^k)$, we try to find a CIST-partition of the vertex set $V(G^k)$ by extending the partition $(V_1, \dots, V_k)$.

Let $H_1, H_2, \dots, H_q$ be q components of $G - H$. For every component H_s ($s \in [q]$), we choose a spanning tree T_s and a vertex $u \in V(T_s)$ such that $uv \in E(G)$, where $v \in S$. We may assume that $T_s^{'} = T_s \cup \{vu\}$ and $T_s^{'}$ is a v-tree. Let $d_{x_i} = \text{dist}_H(v, V_i)$, $i \in [k]$. For the vertex v, there exists a sequence $(d_{t_1}, d_{t_2}, \dots, d_{t_k})$ such that $d_{t_1} \geq d_{t_2} \geq \cdots \geq d_{t_k}$ and $t_1, t_2, \dots, t_k$ is a permutation of $x_1, x_2, \dots, x_k$. Let α be a one-to-one correspondence from $[k]$ to $[k]$ such that $\text{dist}_H(v, V_{\alpha(j)}) = d_{t_j}$.

Let L_j be the vertex set of all vertices in j-th level of $T_s^{'}$ for $j \in \{1, \dots, D(T_s^{'})\}$. For every $w \in L_j$, we assign it to $V_{\alpha(j \pmod k)}$. For other components, we repeat the above operation, and it follows that we obtain a new partition $(V_1^{'}, V_2^{'}, \dots, V_k^{'})$ of the vertex set $V(G^k)$. It remains to show that $(V_1^{'}, V_2^{'}, \dots, V_k^{'})$ is a CIST-partition of $V(G^k)$.

If $1 \leq j \leq k$, since every vertex of S is good with respect to H and $L_j \subset V_{\alpha(j)}^{'}$ for any $w \in L_j$, we have

$$\text{dist}_G(w, V_{\alpha(j)}) \leq \text{dist}_G(v, V_{\alpha(j)}) + \text{dist}_G(v, w) \leq k - j + j = k.$$

Thus,

$$E_{G^k}(\{w\}, V_{\alpha(j)}) \neq \emptyset.$$

If $k + 1 \leq j \leq D(T_s^{'})$, for any $w_1 \in L_j$, $w_2 \in L_{j-k}$, then

$$L_j, L_{j-k} \subset V_{\alpha(j \pmod k)}^{'}, \ \text{dist}_G(w_1, w_2) \leq k.$$

Thus,

$$w_1 w_2 \in E(G^k).$$

It is easy to see that the induced graph $G^k[V_i^{'}]$ is connected for $i \in [k]$.

Note that $\deg_{B(V_i^{'}, V_j^{'})}(w) \geq 1$ for every vertex $w \in V_i^{'} \setminus V_i$, $j^{'} \neq i^{'}$. Since $B(V_i, V_j)$ has no tree components and the vertex w is adjacent to V_j in G^k by alternative path between $V_i^{'}$ and $V_j^{'}$, we get that $B(V_i^{'}, V_j^{'})$ has no tree component.

Hence, $(V_1^{'}, V_2^{'}, \dots, V_k^{'})$ is a CIST-partition of $V(G^k)$. By Lemma 3, G^k has k completely independent spanning trees. ■

Lemma 6. *For a homeomorphically irreducible tree (HIT) T with $|V(T)| \geq 2k$, the k-th power T^k of T has k completely independent spanning trees.*

Proof. We first consider the longest path $P = x_0 x_1 \cdots x_{|P|}$ of T.

Case 1. $|P| < k$. If $|P| < k$, then T^k is a complete graph. Also, $|V(T)| \geq 2k$ and any partition $(V_1, V_2, \dots, V_k)$ of $V(T)$ with $|V_i| \geq 2$ is a CIST-partition of $V(T^k)$. Hence, by Lemma 3, T^k has k completely independent spanning trees.

Case 2. $|P| \geq k$. Denote $P_0 = x_0x_1 \cdots x_k$. Since T is a homeomorphically irreducible tree (HIT), we choose a caterpillar T_0 such that its internal vertices are $V(P_0 - \{x_0, x_k\})$ and its leaf vertices are $N(x_1) \cup \cdots \cup N(x_{k-1}) \backslash V(P_0 - \{x_0, x_k\})$. We regard T_0 as a rooted tree which is rooted at x_0 in the following proof.

Let L_i be the set of all vertices with the same level of x_i in T_0, where $i \in \{0, 1, \ldots, k\}$. Note that

$$L_0 = \{x_0\},\ L_1 = \{x_1\},\ |L_i| \geq 2,\ i \in \{2, \ldots, k\}.$$

We have $|T_0| \geq 2k$ and the distance between x and y is at most k for every pair $x, y \in V(T_0)$. Thus, $T^k[T_0]$ is a complete graph and any partition $(V_1, V_2, \ldots, V_k)$ of the vertex set $V(T^k[T_0])$ with $|V_i| \geq 2$ is a CIST-partition. Specially, we choose a partition of $V(T^k[T_0])$ as $(L_0 \cup L_1, L_2, \ldots, L_k)$.

Since P is a longest path in T and T_0 is a caterpillar, $E_T(L_0 \cup L_1 \cup L_2 \cup V(P_0 - x_k), T - T_0)$ is empty. Let $S = V(T_0 - (L_0 \cup L_1 \cup L_2 \cup V(P_0 - x_k)))$, it follows that $xx_{i-1} \in E(T)$ for any $x \in S \cap L_i$ $(i \geq 3)$ and

$$\begin{aligned}
&\mathrm{dist}_T(x, L_i) = 0,\\
&\mathrm{dist}_T(x, L_j) = i - j,\ 1 \leq j < i,\\
&\mathrm{dist}_T(x, L_j) = j - i + 2,\ i + 1 \leq j \leq k.
\end{aligned}$$

In addition, there exists a corresponding sequence $(d_{t_1}, d_{t_2}, \ldots, d_{t_k})$ such that $d_{t_1} \geq d_{t_2} \geq \cdots \geq d_{t_k}$ and $t_1, t_2, \ldots, t_k$ is a permutation of $\mathrm{dist}_T(x, L_0 \cup L_1)$, $\mathrm{dist}_T(x, L_2), \ldots, \mathrm{dist}_T(x, L_k)$. So, $d_{t_j} \leq k - j$ $(j \in [k])$. Hence, every vertex of S is good with respect to T_0. By Lemma 5, T^k has k completely independent spanning trees. ■

Theorem 7. *If G is a connected graph on n vertices with $n \geq 2k$ and $\delta(G) \geq k$, then the k-th power G^k of G has k completely independent spanning trees.*

Proof. If $k = 1$, then the theorem holds trivially. Therefore, we may assume that $k \geq 2$. Now, suppose that $k = 2$. Since $\delta(G) \geq 2$, we have that G has a cycle C_m.

If $|V(C_m)| \geq 2k = 4$, then let $C_m = x_1x_2 \cdots x_m$ and let a partition of the vertex set $V(C_m)$ be as follows:

$$V_i = \{x_j \,|\, j \equiv i \pmod 2,\ 1 \leq j \leq m\},\ i \in [2].$$

Since $m \geq 4$, we have $|V_i| \geq 2$. It is easy to see that the induced graph $G_m^2[V_i]$ is connected for $i \in [2]$. Actually, $G_m^2[V_i]$ is either a path or a cycle. If $x_j \in V_i$, then $|V_{3-i} \cap \{v \,|\, \mathrm{dist}_{C_m}(x_j, v) \leq 2\}| \geq 2$. Thus, $\deg_{B(V_i, V_{3-i})}(x_j) \geq 2$ and $B(V_i, V_{3-i})$ has no tree component. Hence, (V_1, V_2) is a CIST-partition of C_m^2.

Let H_i $(i \in [q])$ be the connected components of graphs $G - C_m$. Since G is connected, we have that there exists an edge which connects H_i to C_m for every

$i \in [q]$. Also, we have $\mathrm{dist}_G(x, V_i) = 0$ for every vertex $x \in V_i$ ($i \in [2]$) and $\mathrm{dist}_G(x, V_{3-i}) = 1$. Hence, for the set $S = V(C_m)$, every vertex of S is good with respect to C_m. By Lemma 5, G^2 has 2 completely independent spanning trees.

Now we assume that $|C| < 4$ for any cycle C of G.

We choose a cycle $C = x_1x_2x_3$. Note that $n \geq 4$, therefore there exists a vertex $y \in V(G-C)$ such that $yx_i \in E(G)$. Without loss of generality, we assume that $yx_1 \in E(G)$. Let $H = C \cup \{yx_1\}$ and let a partition of the vertex set $V(H^2)$ be as follows:

$$V_1 = \{x_1, x_2\}, V_2 = \{x_3, y\}.$$

Since $n \geq 4$, we have $|V_i| \geq 2$. It is easy to see that (V_1, V_2) is a CIST-partition of the vertex set $V(H^2)$. As we have stated in the previous case, for the set $S = V(H)$, every vertex of S is good with respect to H. By Lemma 5, G^2 has 2 completely independent spanning trees.

Thus, we only consider the case $k \geq 3$.

Case 1. There exists an edge xy such that $|N(x) - N[y]| \geq k-1$ and $|N(y) - N[x]| \geq k-1$.

Let

$$\begin{aligned} &xy = x_1y_1, H = G[N[x_1] \cup N[y_1]] - N(x_1) \cap N(y_1),\\ &N(x_1) = \left\{y_1, x_2, x_3, \ldots, x_{\deg_G(x_1)}\right\},\\ &N(y_1) = \left\{x_1, y_2, y_3, \ldots, y_{\deg_G(y_1)}\right\}. \end{aligned}$$

Since $\delta(G) \geq k \geq 3$ and $\mathrm{dist}(u, v) \leq 3$ for any two vertices u, v, combine with $|V(H)| \geq 2k$, H^k is a complete graph. Thus, any partition $(V_1, V_2, \ldots, V_k)$ of the vertex set $V(H^k)$ with $|V_i| \geq 2$ is a CIST-partition. Specially, we choose a partition of the vertex set $V(H^k)$ as follows:

$$\begin{aligned} &V_1 = \{x_1, y_1\},\\ &V_i = \{x_j \mid j \equiv i \,(\mathrm{mod}\ (k-1)),\ 2 \leq j \leq \deg_G(x_1)\}\\ &\quad \cup \{y_j \mid j \equiv i \,(\mathrm{mod}\ (k-1)),\ 2 \leq j \leq \deg_G(y_1)\},\ i \in \{2, \ldots, k\}. \end{aligned}$$

For every $x \in V_i$ ($i \geq 2$), we have

$$\mathrm{dist}_G(x, V_i) = 0,\ \mathrm{dist}_G(x, V_1) = 1,\ \mathrm{dist}_G(x, V_j) = 2,\ j \neq i.$$

Hence, for the set $S = V(H)$, every vertex of S is good with respect to H. By Lemma 5, G^k has k completely independent spanning trees.

Case 2. There exists an edge xy such that $|N(x) - N[y]| \geq k-2$ and $|N(x) - N[y]| \geq k-2$ and which does not satisfy Case 1.

Let

$$H = G[N[xy]].$$

Since $\delta(G) \geq k$, $|V(H)| \geq 2k-1$.

Case 2.1. If $d(x) > k$ or $d(y) > k$, then xy satisfies the Case 1 which is a contradiction.

Case 2.2. If $d(x) = k, d(y) = k$, then $|V(H)| = 2k-1$. Since $n \geq 2k$, there exists a vertex $w \in V(G-H)$ such that w is adjacent to a vertex h of $H \backslash \{x, y\}$. In other words, $h \in N(x) \backslash \{y\}$ or $N(y) \backslash \{x\}$. We suppose that $H_0 = G[V(H) \cup w]$.

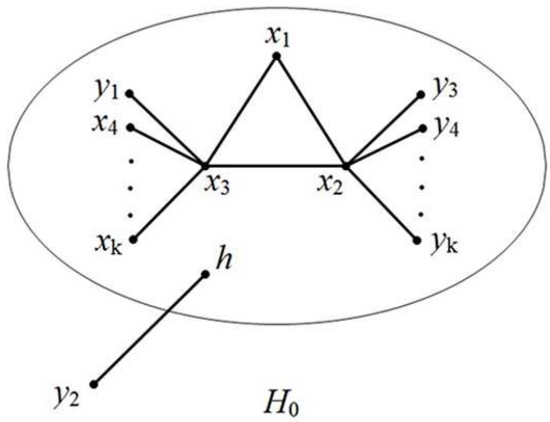

Figure 1.

(1) We first consider the case $k \geq 4$. If $h \in N(x) \backslash \{y\}$ (or $h \in N(y) \backslash \{x\}$), then we label H_0 as in Figure 1. Let

$$\begin{aligned} &x = x_3,\ y = x_2 \text{ (or } x = x_2,\ y = x_3),\\ &N(x) \cap N(y) = \{x_1\},\ w = y_2,\\ &N(x_3) = \{x_2, x_1, y_1, x_4, x_5, \ldots, x_k\},\\ &N(x_2) = \{x_3, x_1, y_3, y_4, y_5, \ldots, y_k\}. \end{aligned}$$

Since $\delta(G) \geq k \geq 4$, H_0^k is a complete graph. Thus, any partition $(V_1, V_2, \ldots, V_k)$ of the vertex set $V(H_0^k)$ with $|V_i| \geq 2$ is a CIST-partition. Specially, we choose a partition of the vertex set $V(H_0^k)$ as follows:

$$V_i = \{x_i, y_i\},\ i \in [k].$$

If $h = x_\ell$ ($\ell = 1, 4, 5, \ldots, k$) or $h = y_\ell$ ($\ell = 1$), then for $j \in [k]$ we have

$$\text{dist}_G(x_1, V_1) = 0,\ \text{dist}_G(x_1, V_2) = 1,\ \text{dist}_G(x_1, V_j) \leq 2\ (j \neq 1, 2),$$

$$\text{dist}_G(y_1, V_1) = 0,\ \text{dist}_G(y_1, V_3) = 1,\ \text{dist}_G(y_1, V_j) \leq 2\ (j \neq 1, 3),$$

$$\mathrm{dist}_G(y_2,V_2)=0,\ \mathrm{dist}_G(y_2,V_\ell)=1,\ \mathrm{dist}_G(y_2,V_3)\le 2,$$

$$\mathrm{dist}_G(y_2,V_j)\le 3\ (j\ne 2,3,\ell),$$

$$\mathrm{dist}_G(y_i,V_i)=0,\ \mathrm{dist}_G(y_i,V_2)=1,\ \mathrm{dist}_G(y_i,V_j)\le 2, 3\le i\le k\ (j\ne 2,i),$$

$$\mathrm{dist}_G(x_i,V_i)=0,\ \mathrm{dist}_G(x_i,V_3)=1,\ \mathrm{dist}_G(x_i,V_j)\le 2, 4\le i\le k\ (j\ne 3,i).$$

Again, for the set $S=V(H-\{x_2,x_3\})$, every vertex of S is good with respect to H. By Lemma 5, G^k has k completely independent spanning trees.

(2) Now suppose that $k=3$.

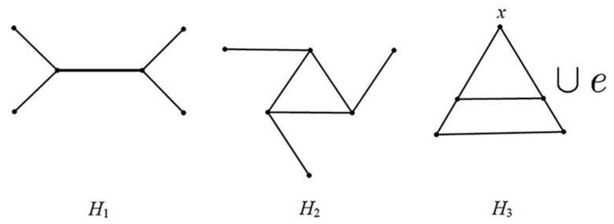

Figure 2.

Claim 8. *If a connected graph G contains a subgraph which is isomorphic to one of the H_1,H_2, H_3 in Figure 2 (where $H_3=H_3'\cup e$ and e has exactly one end in $V(H_3')$), then G^3 has 3 completely independent spanning trees.*

Proof. If $H_1\subseteq G$, then the proof follows by Case 1. If $H_2\subseteq G$, then H_2 is isomorphic to $H_0=G[V(H)\cup y_2]$ in Case 2.2(1), where $H=G[W]$, $W=\{x_1,x_2,x_3,y_1,y_3\}$ and y_2 is adjacent to a vertex x_1 of H. The proof follows by Case 2.2(1). Suppose $H_3\subseteq G$, if e is adjacent to x, then there exists a subgraph of H_3 which is isomorphic to H_2. Otherwise, e is adjacent to $V(H_3')\setminus x$, then we obtain a subgraph of H_3 which is isomorphic to H_1. The claim is true for H_1,H_2,H_3. □

Now we begin to prove the case $k=3$ and we relabel H as in left of Figure 3.

Since $k=3$, we have $h\ne y,z$. If $h=x$, then $H\cup\{wh\}$ contains a subgraph which is isomorphic to H_2, it is true by Claim 8. Thus, $h=u$ or $h=v$. By symmetry, we may assume that $h=u$ and $H_0=G[V(H)\cup w]$, as in right of Figure 3.

If $\deg_{G-H_0}(u)\ge 1$, then G contains a subgraph which is isomorphic to H_1, it is true by Claim 8. Thus, $N(u)\subseteq V(H_0)$.

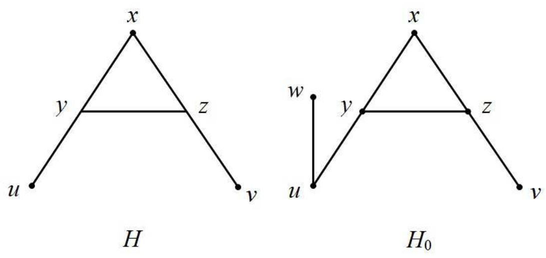

Figure 3.

If $uv \in E(G)$, then there exists a subgraph of G which is isomorphic to H_3, it is true by Claim 8. Otherwise, $uv \notin E(G)$ and $ux \in E(G)$.

If $xv \in E(G)$, then we obtain a subgraph of G which is isomorphic to H_1, it is true by Claim 8. Otherwise, $xv \notin E(G)$. Since $\delta \geq 3$, we have $\deg_{G-\{x,y,u\}}(v) \geq 2$. Thus, we also can obtain a subgraph of G which is isomorphic to H_1, it is true by Claim 8.

Hence, G^3 has 3 completely independent spanning trees.

Case 3. Every edge e of G is contained in at least two triangles. By Lemma 4, G contains a HIST T. Also, by Lemma 6, the k-th power T^k of T has k completely independent spanning trees.

Hence, the k-th power G^k has k completely independent spanning trees. The proof of Theorem 7 is completed. ■

An immediate consequence of Theorem 7 is the following corollary.

Corollary 9. *If G is a k-connected graph on n vertices with $n \geq 2k$, then the k-th power G^k of G has k completely independent spanning trees.*

Acknowledgements

This research was supported by National Natural Science Foundation of China (No. 11701257 and No. 11301254) and the Key Project in Universities of Henan Province (No. 18A110025 and No. 16B110009) and Natural Science Foundation of Henan Province under(Grant No. 172102410069). This work was supported by the Youth Backbone Teacher Foundation of Henan's University (Grant No. 2015GGJS-115) and Innovation Scientists Technicians Troop Construction Projects of Henan Province (Grant No. C20150027). The author would like to express thanks to the referee for his valuable corrections and suggestions of the manuscript that greatly improve the format and correctness and the quality of our paper.

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Received 28 April 2016
Revised 30 January 2017
Accepted 7 February 2017