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LIST STAR EDGE-COLORING OF SUBCUBIC GRAPHS

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Abstract

A *star edge-coloring* of a graph G is a proper edge coloring such that every 2-colored connected subgraph of G is a path of length at most 3. For a graph G, let the *list star chromatic index* of G, $ch'_{st}(G)$, be the minimum k such that for any k-uniform list assignment L for the set of edges, G has a star edge-coloring from L. Dvořák, Mohar and Šámal asked whether the list star chromatic index of every subcubic graph is at most 7. We prove that it is at most 8. We also prove that if the maximum average degree of a subcubic graph G is less than $\frac{7}{3}$ $\left(\text{respectively, } \frac{5}{2}\right)$, then $ch'_{st}(G) \leq 5$ (respectively, $ch'_{st}(G) \leq 6$).

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1. Introduction

All the graphs we consider are finite and simple. For a graph G, we denote by $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ its vertex set, edge set, minimum degree and maximum degree, respectively.

A *proper vertex* (respectively, *edge*) coloring of G is an assignment of colors to the vertices (respectively, edges) of G such that no two adjacent vertices (respectively, edges) receive the same color. A *star coloring* of G is a proper vertex coloring of G such that the union of any two color classes induces a *star forest* in G, i.e., every component of this union is a star. This notion was first mentioned by Grünbaum [6] in 1973, but attracted more attention only in 2001 after the paper [5] by Fertin, Raspaud and Reed. By now, there are more than 30 publications on this topic. The star coloring even in the class of line graphs seems to be difficult. A convenient language for discussions of star coloring of line graphs is the language of star edge-coloring of all graphs.

A *star edge-coloring* of a graph G is a proper edge-coloring such that every 2-colored connected subgraph of G is a path of length at most 3. In other words, we forbid bicolored 4-cycles and 4-paths in G (by a *k-path* we mean a path with k edges). This notion is intermediate between *acyclic edge-coloring*, when every 2-colored subgraph must be only acyclic, and *strong edge-coloring*, when every 2-colored connected subgraph has at most two edges. The *star chromatic index* of G, denoted by $\chi'_{st}(G)$, is the minimum number of colors needed for a star edge-coloring of G. It was first studied by Liu and Deng [9] in 2008. They proved the following upper bound.

Theorem 1 [9]**.** *For every G with maximum degree $\Delta \geq 7$, $\chi'_{st}(G) \leq \left\lceil 16(\Delta-1)^{\frac{3}{2}} \right\rceil$.*

In [3] and later [2] it is proved:

Theorem 2 [3, 2]**.** *The star chromatic index of any tree with maximum degree Δ is at most $\Delta + \lceil \frac{\Delta-1}{2} \rceil$.*

In a seminal paper [4], Dvořák, Mohar and Šámal showed that even determining the star chromatic index of the complete graph K_n with n vertices is a hard problem. They gave the following bounds:

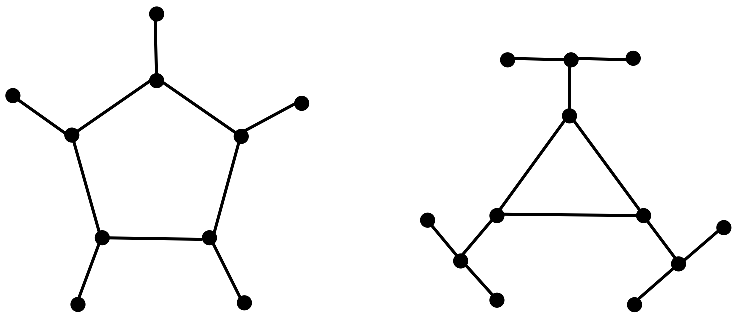
$$2n(1+o(1)) \leq \chi'_{st}(K_n) \leq n\frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log(n)}}}{\log n^{\frac{1}{4}}}.$$

They also studied the star chromatic index of *subcubic graphs*, that is, graphs with maximum degree at most 3. They proved that $\chi'_{st}(G) \leq 7$ for every subcubic graph G, and conjectured that $\chi'_{st}(G) \leq 6$ for every such G.

A natural generalization of star edge-coloring is the list star edge-coloring. An *edge list* L for a graph G is a mapping that assigns a finite set of colors to each edge of G. Given an edge list L for a graph G, we say that G is *L-star edge-colorable* if it has a star edge-coloring c such that $c(e) \in L(e)$ for every edge of G. The *list star chromatic index*, $ch'_{st}(G)$, of a graph G is the minimum k such that for every edge list L for G with $|L(e)| = k$ for every $e \in E(G)$, G is L-star edge-colorable.

Dvořák, Mohar and Šámal [4, Question 3] asked whether $ch'_{st}(G) \leq 7$ for every subcubic G. We prove the following result toward this question.

Theorem 3. *For every subcubic graph G, $ch'_{st}(G) \leq 8$.*



(a) G_1 (b) G_2

Figure 1. Two subcubic graphs with mad=2 and list star chromatic index 5.

We also give sufficient conditions for the list star chromatic index of a subcubic graph to be at most 5 and 6 in terms of the *maximum average degree* $\text{mad}(G) = \max\left\{\frac{2|E(H)|}{|V(H)|},\ H \subseteq G\right\}$. Note that the best possible sufficient condition for 4 colors is $\text{mad}(G) < 2$. If $\text{mad}(G) < 2$ then G is acyclic and by Theorem 2 for $\Delta = 3$, we have $\chi'_{st}(G) \leq 4$. The same proof yields also $ch'_{st}(G) \leq 4$. On the other hand, each of the graphs G_i in Figure 1 has $\text{mad}(G_i) = 2$ and $ch'_{st}(G_i) \geq \chi'_{st}(G_i) = 5$. Our second result is:

Theorem 4. *Let G be a subcubic graph.*

1. *If $mad(G) < \frac{7}{3}$, then $ch'_{st}(G) \leq 5$.*
2. *If $mad(G) < \frac{5}{2}$, then $ch'_{st}(G) \leq 6$.*

As every planar graph with girth g satisfies $\text{mad}(G) < \frac{2g}{g-2}$, Theorem 4 yields the following.

Corollary 1. *Let G be a planar subcubic graph with girth g.*

1. *If $g \geq 14$, then $ch'_{st}(G) \leq 5$.*
2. *If $g \geq 10$, then $ch'_{st}(G) \leq 6$.*

Analogous to Theorem 4 bounds were earlier proved in [7] for the *strong chromatic index*, $\chi'_s(G)$ — the minimum k such that G has a strong edge-coloring with k colors. Recall that a *strong edge-coloring* of a graph G is a proper edge-coloring such that any two edges adjacent to a common edge receive different colors. Since every strong edge-coloring is also a star edge-coloring, the following results give bounds for the star chromatic index. Note that the restrictions on mad in the first two statements of Theorem 5 below are the same as in Theorem 4, but the bounds are different.

Theorem 5 [7]. *Let G be a subcubic graph.*

1. *If $mad(G) < \frac{7}{3}$, then $\chi'_s(G) \leq 6$.*
2. *If $mad(G) < \frac{5}{2}$, then $\chi'_s(G) \leq 7$.*
3. *If $mad(G) < \frac{8}{3}$, then $\chi'_s(G) \leq 8$.*
4. *If $mad(G) < \frac{20}{7}$, then $\chi'_s(G) \leq 9$.*

List versions of two results of the previous theorem (for $\mathrm{mad}(G) < \frac{5}{2}$ and $\mathrm{mad}(G) < \frac{8}{3}$) are proved in [10].

The structure of the paper is as follows. In the next section we introduce some notation and prove an analog of Lemma 5.2 in [4] on extensions of partial star edge-colorings. In Section 3 we prove Theorem 3, and in the two last sections we prove parts 1 and 2 of Theorem 4.

2. Preliminaries

For a graph G, let $d_G(v)$ denote the degree of a vertex v in G and $N_G(v)$ denote the set of neighbors of v in G. If G is clear from the content, we may omit the subscript. A vertex of degree k is called a *k-vertex*, and a *k-neighbor* of a vertex v is a k-vertex adjacent to v. An edge xy is *weak* if at least one of x and y is a leaf. A vertex x is *weak* if at least one of the edges incident with x is weak. For brevity, we often will write "*k-se-coloring*" instead of "star edge k-coloring" and "*se-coloring*" instead of "star edge-coloring". A *partial edge-coloring* of a graph G is an edge-coloring of a subgraph G' of G (where G' can equal G).

For a partial edge-coloring ϕ of a graph G and a vertex $v \in V(G)$, $\phi(v)$ denotes the set of colors used on the edges incident with v.

We will heavily use the following lemma.

Lemma 6. *Let ϕ be a partial se-coloring of a graph G and uv be an uncolored edge. If α is a color satisfying at least one of the two properties below, then the coloring ϕ' obtained from ϕ by coloring uv with α also is a partial se-coloring of G.*

(a) *For every* $x \in N[v] \cup N[u]$, $\alpha \notin \phi(x)$*;*

(b) $\phi(u) \cap \phi(v) = \emptyset$, $\alpha \notin \phi(u) \cup \phi(v)$, *and among the edges incident with the neighbors of* v *or* u, *only weak edges may have color* α.

Proof. Suppose (a) or (b) holds, but ϕ' is not a partial se-coloring of G. Then there is a color β and either a path $z_1z_2z_3z_4z_5$ or a cycle $z_1z_2z_3z_4z_1$ containing edge uv whose edges are colored with α and β. By symmetry, we may assume that $u = z_i$ and $v = z_{i+1}$ for $i \in \{1, 2\}$. Then $\phi(z_{i+2}z_{i+3}) = \alpha$. So, (a) cannot hold. Thus (b) holds. If $i = 2$, then we have a contradiction to $\phi(u) \cap \phi(v) = \emptyset$. So $i = 1$. But z_3z_4 is not weak, which violates (b). ■

3. Proof of Theorem 3

Let G be a subcubic graph with the minimum total number of edges and vertices such that there exists a list L for the set of the edges of G with $|L(e)| = 8$ for every $e \in E(G)$ for which G has no L-star-edge-coloring.

Clearly, G is connected.

Lemma 7. G *is* 3*-regular.*

Proof. If G has a 1-vertex u adjacent to some v, then by the minimality of G, graph $G - u$ has an se-coloring ϕ from L. We view it as a partial se-coloring of G. Let W be the set of neighbors of v distinct from u. We extend ϕ by coloring uv with any color $\alpha \in L(uv)$ distinct from the colors of the (at most six) edges incident with the vertices in W. So, $\delta(G) \geq 2$.

Suppose now that G has a 2-vertex v adjacent to u and w. Let $N(u) \subseteq \{v, u_1, u_2\}$ and $N(w) \subseteq \{v, w_1, w_2\}$. By the minimality of G, graph $G - v$ has an L-coloring ϕ of its edges. We view it as a partial se-coloring of G. Let $A(uv) = L(uv) - \phi(u_1) - \phi(u_2)$ and $A(wv) = L(wv) - \phi(w_1) - \phi(w_2)$. By definition, $|A(uv)| \geq 2$ and $|A(vw)| \geq 2$. If there is $\alpha \in A(uv) - \phi(w)$, then by coloring vw with some $\beta \in A(vw) - \alpha$ and uv with α we get an se-coloring of G. Indeed, at each step the conditions of Lemma 6(a) will hold. Otherwise, $d(u) = d(w) = 3$, $d(u_1) = d(u_2) = d(w_1) = d(w_2) = 3$, $uw \notin E(G)$,

$$\begin{aligned} L(uv) &= \{\phi(ww_1), \phi(ww_2)\} \cup \phi(u_1) \cup \phi(u_2), \text{ and} \\ L(vw) &= \{\phi(uu_1), \phi(uu_2)\} \cup \phi(w_1) \cup \phi(w_2). \end{aligned} \tag{1}$$

In particular, for $i = 1, 2$, vertex u_i (respectively, w_i) has two neighbors u_i' and u_i'' (respectively, w_i' and w_i'') distinct from u (respectively, w). We then try to color vw with $\phi(uu_2)$ and uv with either $\phi(u_1u_1')$ or $\phi(u_1u_1'')$. If we do not get an se-coloring of G, then any 2-colored 4-path in G contains edges uv and uu_1, so that each of u_1' and u_1'' is incident with an edge of color $\phi(uu_1)$. It follows that

$|\phi(u_1') \cup \phi(u_1'')| \leq 5$. Similarly, each of u_2' and u_2'' is incident with an edge of color $\phi(uu_2)$, and $|\phi(u_2') \cup \phi(u_2'')| \leq 5$. If there is $\gamma_1 \in L(uu_1) - (\phi(u_1') \cup \phi(u_1'') \cup \phi(u_2))$, then we color uv with $\phi(uu_1)$, vw with $\phi(uu_2)$, and recolor uu_1 with γ_1. By (1) and the definition of γ_1 this would yield an se-coloring of G from L, a contradiction. This means

$$L(uu_1) = \phi(u_1') \cup \phi(u_1'') \cup \phi(u_2). \tag{2}$$

Similarly, $L(uu_2) = \phi(u_2') \cup \phi(u_2'') \cup \phi(u_1)$. In particular, $\phi(uu_2) \in L(uu_1)$ and $\phi(uu_1) \in L(uu_2)$. Then switching the colors of uu_1 and uu_2 we obtain another se-coloring ϕ' of $G - v$. Repeating the above argument for ϕ' in place of ϕ, we get that each of u_1' and u_1'' is incident with an edge of color $\phi'(uu_1) = \phi(uu_2)$. But then $|\phi(u_1') \cup \phi(u_1'')| = 4$, a contradiction to (2). ∎

In the following we will say that two edges are at distance at most 1 if they are adjacent or adjacent to a same edge. Let $C = (v_1, \ldots, v_t)$ be a shortest cycle in G. Since C is shortest, it has no chords. Thus for each $i = 1, \ldots, t$, vertex v_i has a unique neighbor v_i' in $V(G) - V(C)$. Let $G_1 = G - E(C)$. An se-coloring ϕ of G_1 from L is *stable* if for every $i = 1, \ldots, t$, $\phi(v_i v_i')$ differs from $\phi(v_{i-1} v_{i-1}')$, $\phi(v_{i+1} v_{i+1}')$, and from the color of each edge in G_1 at distance at most 1 from $v_i v_i'$ in G_1 (note that G_1 has at most six such edges: two incident with v_i' and at most four others incident with the neighbors of v_i').

Lemma 8. *G_1 does not have stable se-colorings from L.*

Proof. Suppose G_1 has a stable se-coloring ϕ from L. For every $i = 1, \ldots, t$, let $L'(v_i v_{i+1}) = L(v_i v_{i+1}) - \{\phi(v_{i-1} v_{i-1}'), \phi(v_i v_i'), \phi(v_{i+1} v_{i+1}'), \phi(v_{i+2} v_{i+2}')\}$ (indices taken modulo t).

Then $|L'(v_i v_{i+1})| \geq 4$ for every $i = 1, \ldots, t$. It is known that every cycle has an se-coloring from any 4-uniform list. (Simply, the square of any cycle of length $t \neq 5$ has a list 4-coloring, and if $t = 5$, then we can color two nonadjacent edges with one color, say α, and all other 3 edges with different colors distinct from α.) So, let ϕ' be an se-coloring of C from L'. We claim that $\phi \cup \phi'$ is an se-coloring of G from L. This follows from the fact that, by the definitions of stable colorings and of L', for every $i = 1, \ldots, t$, $\phi(v_i v_i')$ differs from the colors of all edges at distance at most 1. Thus we can first uncolor all such edges, and then return them their colors one by one, and apply Lemma 6 at every step. So we get an se-coloring of G, a contradiction. ∎

In the rest of the proof we will attempt to construct a stable se-coloring of G_1 from L. For this, fix an se-coloring ψ of $G_2 = G_1 - V(C)$ from L (it exists by the minimality of G). Construct the auxiliary graph H with $V(H) = \{v_i v_i' : i = 1, \ldots, t\}$ by making $v_j v_j'$ adjacent in H to $v_i v_i'$ if $j \in \{i-1, i+1\}$, or $v_j' = v_i'$ or

$v'_j v'_i \in E(G_2)$. Also, every $v_i v'_i \in V(H)$ has list $L_1(v_i v'_i)$ obtained from $L(v_i v'_i)$ by deleting the colors in ψ of the edges incident with v'_i or with its neighbor. Since $|L(v_i v'_i)| = 8$ and at most six edges in G_2 are incident with v'_i or with its neighbor,

$$|L_1(v_i v'_i)| \geq d_H(v_i v'_i) \qquad \text{for every } i = 1, \ldots, t. \tag{3}$$

By definition, if H has a L_1-coloring ψ', then the union $\psi \cup \psi'$ forms a stable se-coloring of G_1 contradicting Lemma 8. Thus H has no L_1-coloring. But by (3), L_1 is a so called *degree list* for H. Since H has Hamiltonian cycle, it is 2-connected. By a well-known result of Borodin [1] (for a short proof, see [8]), for every 2-connected H and a list L_1 satisfying (3), if H has no L_1-coloring, then

(i) $|L_1(v_i v'_i)| = d_H(v_i v'_i)$ for every $i = 1, \ldots, t$;

(ii) all lists are the same; and

(iii) H is a complete graph or an odd cycle.

Since $|V(H)| = t$, we have three cases.

Case 1. $H = K_t$ for $t \geq 5$. If not all v'_i are distinct, say $v'_1 = v'_r$, then since C is a shortest cycle, $r \leq 3$ and $t - r \leq 1$. Thus then $t \leq 4$, which is not the case. So, all v'_i are distinct. But each v'_i is adjacent to at most two other vertices v'_j. Thus to have $H = K_t$ for $t \geq 5$, we need $t = 5$ and $N_G(v'_i) = \{v_i, v'_{i-2}, v'_{i+2}\}$ for all $i = 1, \ldots, 5$. This means, G is the Petersen graph, and ψ colored the edges of the 5-cycle $C_1 = (v'_1, v'_3, v'_5, v'_2, v'_4)$ so that the lists $L_1(v_i v'_i)$ for all $i = 1, \ldots, 5$ become the same. Since $|L(v'_1 v'_3)| = 8$, we can recolor $v'_1 v'_3$ with another color in $L(v'_1 v'_3)$ distinct from the colors of all edges in C_1. Then the list $L_1(v_2 v'_2)$ does not change, but the lists of all other $v_i v'_i$ will change. Thus for the new coloring, condition (ii) will not hold anymore, and we get a stable se-coloring of G_1.

Case 2. $H = K_4$. If not all v'_i are distinct, say $v'_1 = v'_r$, then since C is a shortest cycle, $r = 3$. But then at most 3 colored edges are incident with v'_1 or its neighbor, thus $|L_1(v_1 v'_1)| \geq 5$, a contradiction to (i). So, all v'_i are distinct and $v'_1 v'_3, v'_2 v'_4 \in E(G)$. Since at most 6 colored edges are at distance at most 1 from $v'_1 v'_3$ in G_2, we can recolor it with another color from its list distinct from the colors of these at most 6 edges. If after this recoloring, the list $L_1(v_2 v'_2)$ or $L_1(v_4 v'_4)$ does not change, then (ii) does not hold anymore and we can get a stable se-coloring of G_1. If both, $L_1(v_2 v'_2)$ and $L_1(v_4 v'_4)$ change, then two edges connect $\{v'_1, v'_3\}$ with $\{v'_2, v'_4\}$. Since G is 3-regular, this means that G has only 8 vertices, and so $|L_1(v_i v'_i)| \geq 4$ for each i, contradicting (i).

Case 3. H is a cycle with t vertices, where t is odd. Similarly to Case 2, all v'_i are distinct and not adjacent to each other. Also by (ii), we may assume $L_1(v_i v'_i) = \{\alpha, \beta\}$ for all $i = 1, \ldots, t$. We color $v_i v'_i$ with α for $i = 1, 3, 5, \ldots, t$ and with β for $i = 2, 4, 6, \ldots, t-1$. Then we color $v_1 v_t$ with $\gamma_0 \in L(v_1 v_t) - \psi(v'_1) - \psi(v'_t) - \{\alpha, \beta\}$ and $v_1 v_2$ with $\gamma_1 \in L(v_1 v_2) - \{\alpha, \beta, \gamma_0\}$. Now for $i = 2, \ldots, t-1$, we greedily color

v_iv_{i+1} with a color $\gamma_i \in L(v_iv_{i+1}) - \{\alpha, \beta, \gamma_0, \gamma_1, \gamma_{i-2}, \gamma_{i-1}\}$. Similarly to the end of the proof of Lemma 8, the new coloring is an se-coloring of G, since colors α and β are not used on the edges distinct from $v_1v_1', \ldots, v_tv_t'$ at distance at most 1 from any of them. This proves the theorem.

4. Proof of Theorem 4.1.

Suppose that the theorem is not true. Let H have the fewest edges among the subcubic graphs with $\mathrm{mad}(H) < \frac{7}{3}$ such that for some list L with $|L(e)| = 5$ for each $e \in E(H)$, H has no se-coloring from L. Clearly, H is connected.

Claim 9. *H has no weak 2-vertices.*

Proof. Suppose H contains a 2-vertex u adjacent to a 1-vertex u_1. Let u_2 be the second neighbor of u. By the minimality of H, graph $H' = H - \{u_1u\}$ has an se-coloring ϕ from L. We can view ϕ as a partial se-coloring of H. Since $|\phi(u_2)| \leq 3$, there is $\alpha \in L(u_1u) - \phi(u_2)$. By Lemma 6(a), if we color u_1u with α, then we get an se-coloring of H from L. ■

Claim 10. *H does not contain a 3-vertex adjacent to two 1-vertices.*

Proof. Suppose that H contains a 3-vertex u with $N(u) = \{u_1, u_2, u_3\}$, where $d(u_1) = d(u_2) = 1$. By the minimality of H, graph $H' = H - \{u_1u\}$ has an se-coloring ϕ from L. As in the proof of Claim 9, we view ϕ as a partial se-coloring of H. Since $|\phi(u_3)| \leq 3$ and $|\phi(u_2)| = 1$, there is $\alpha \in L(u_1u) - \phi(u_2) - \phi(u_3)$. By Lemma 6(a), if we color u_1u with α, then we get an se-coloring of H from L. ■

Let H^* denote the graph obtained from H by deleting all vertices of degree 1. By Claims 9 and 10, $\delta(H^*) \geq 2$.

Claim 11. *H^* has no 3-cycle $C = xvwx$ such that $d_{H^*}(v) = d_{H^*}(w) = 2$.*

Proof. Suppose that H contains a cycle $xvwx$ such that $d_{H^*}(v) = d_{H^*}(w) = 2$. If $z \in \{v, w\}$ has a 1-neighbor in $H - \{v, w\}$, denote this neighbor by z'.

If x has a neighbor in H different from v and w we denote it by t.

Case 1. $H^* = C$. Let ϕ be any coloring of the edges of C from the lists such that all three colors are distinct. By definition, this is a partial se-coloring of H. Now consecutively for each $z \in \{x, v, w\}$, color edge zz' (if it exists) with a color in $L(zz') - \{\phi(xv), \phi(vw), \phi(wx)\}$. By Lemma 6(b), at each step we again will obtain a partial se-coloring of H. So, after the last step we get an se-coloring of H from L, a contradiction.

Case 2. The vertex t exists and $d_H(t) \geq 2$. Let $H_0 = H - \{v, v', w, w'\}$, note that the vertices v' and w' may not exist.

By the minimality of H, graph H_0 has an se-coloring ϕ from L. We view ϕ as a partial se-coloring of H. Color vx with a color $\alpha_1 \in L(vx) - \phi(t)$ and wx with a color $\alpha_2 \in L(wx) - \phi(t) - \alpha_1$. By Lemma 6(a), the new partial edge-coloring ϕ' is an se-coloring. Now color vw with some $\alpha_3 \in L(vw) - \phi'(t)$. Again by Lemma 6(a), the new partial edge-coloring ϕ'' is an se-coloring. Then consecutively for $z \in \{v, w\}$, color edge zz' (if it exists) with a color in $L(zz') - \{\alpha_3\} - \phi(x)$. By Lemma 6(b), at each step we again will obtain a partial se-coloring of H. So, after the last step we get an se-coloring of H from L, a contradiction. ■

Lemma 12. *Graph H^* has no 4-cycle $xuvwx$ such that $d_{H^*}(u) = d_{H^*}(v) = d_{H^*}(w) = 2$. Furthermore, if H^* contains a path $xuvwy$ such that $d_{H^*}(u) = d_{H^*}(v) = d_{H^*}(w) = 2$, then $d_{H^*}(x) = d_{H^*}(y) = 3$. Moreover, if $N_{H^*}(x) = \{u, x_1, x_2\}$ and $N_{H^*}(y) = \{w, y_1, y_2\}$, then $d_{H^*}(x_1) = d_{H^*}(x_2) = d_{H^*}(y_1) = d_{H^*}(y_2) = 3$.*

Proof. Suppose that H contains a path $xuvwy$ or a cycle $xuvwx$ such that $d_{H^*}(u) = d_{H^*}(v) = d_{H^*}(w) = 2$. If u has a 1-neighbor in H, we will denote this neighbor by u'. The vertices v' and w' are defined similarly.

Now we will prove that the vertex v' does not exist. Otherwise, consider $H' = H - v'$. By the minimality of H, graph H' has an se-coloring ϕ from L. We view ϕ as a partial se-coloring of H. By Lemma 6(b), the coloring ϕ' obtained from ϕ by coloring vv' with a color in $L(vv') - \{\phi(xu), \phi(uv), \phi(vw), \phi(wy)\}$ if we have a path (or a color in $L(vv') - \{\phi(xu), \phi(uv), \phi(vw), \phi(wx)\}$ if we have a 4-cycle) is a se-coloring from L of the whole H. This contradicts the choice of H. So

$$d_H(v) = 2. \tag{4}$$

Case 1. H^* contains a cycle $C = xuvwx$ such that $d_{H^*}(u) = d_{H^*}(v) = d_{H^*}(w) = 2$. Let t be the third neighbor of x in H, if it exists.

Case 1.1. $H^* = C$. Let ϕ be any coloring of the edges of C from the lists such that all four colors are distinct. By definition, this is a partial se-coloring of H. Now consecutively for each $z \in \{u, w\}$, color the edge zz' (if it exists) with a color in $L(zz') - \{\phi(xu), \phi(uv), \phi(vw), \phi(wx)\}$. If xt exits, then color the edge xt with a color $L(xt) - \{\phi(xu), \phi(uv), \phi(vw), \phi(wx)\}$. By Lemma 6(b), at each step we again will obtain a partial se-coloring of H. So, after the last step we get an se-coloring of H from L, a contradiction.

Case 1.2. The vertex t exists and $d_H(t) \geq 2$. Let $H_0 = H - \{u, v, w, u', w'\}$. By the minimality of H, graph H_0 has an se-coloring ϕ from L. We view ϕ as a partial se-coloring of H. Color ux with a color $\alpha_1 \in L(ux) - \phi(t)$ and wx with a color $\alpha_2 \in L(wx) - \phi(t) - \alpha_1$. By Lemma 6(a), the new partial edge-coloring ϕ' is an se-coloring. Now color vw with some $\alpha_3 \in L(vw) - \phi'(x)$ and uv with some

$\alpha_4 \in L(uv) - \phi'(x) - \alpha_3$. Again by Lemma 6(a), the new partial edge-coloring ϕ'' is an se-coloring. Then consecutively for $z \in \{u, w\}$, color edge zz' (if it exists) with a color in $L(zz') - \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. By Lemma 6(b), at each step we again will obtain a partial se-coloring of H. So, after the last step we get an se-coloring of H from L, a contradiction.

Case 2. H^* contains a path $P = xuvwy$ such that $d_{H^*}(u) = d_{H^*}(v) = d_{H^*}(w) = 2$. Let $N_H(y) \subseteq \{w, y_1, y_2\}$ (maybe only one of y_1, y_2 exists) and $N_H(x) \subseteq \{u, x_1, x_2\}$. Let $H_1 = H - \{v, u', w'\}$. By the minimality of H, graph H_1 has an se-coloring ϕ from L. We view ϕ as a partial se-coloring of H. Let $A(wv) = L(wv) - \phi(y)$, $A(ww') = L(ww') - \phi(y)$, $A(uv) = L(uv) - \phi(x)$ and $A(uu') = L(uu') - \phi(x)$. Since $|\phi(z)| \leq 3$ for every $z \in V(H)$,

(5) each of $A(wv), A(ww'), A(uv)$ and $A(uu')$ contains at least two colors.

Case 2.1. Suppose $|A(wv) \cup A(ww')| + |A(uv) \cup A(uu')| \geq 5$. By (5) and symmetry, we may assume $|A(uv) \cup A(uu')| \geq 3$. Color wv with a color $\alpha_1 \in A(wv) - \phi(xu)$ and ww' with a color $\alpha_2 \in A(ww') - \alpha_1$. Since edge uv is not colored, by Lemma 6(a), the new partial edge-coloring ϕ_1 is an se-coloring. By (5) and the fact that $|A(uv) \cup A(uu')| \geq 3$, we can choose distinct $\alpha_3 \in A(uv) - \alpha_1$ and $\alpha_4 \in A(uu') - \alpha_1$. Let ϕ_2 be obtained from ϕ_1 by coloring uv with α_3. We claim that

(6) ϕ_2 is a partial se-coloring of H.

Indeed, suppose there is a color β and either a path $z_1z_2z_3z_4z_5$ or a cycle $z_1z_2z_3z_4z_1$ containing edge uv whose edges are colored with α_3 and β. By symmetry, we may assume that $\{u, v\} = \{z_i, z_{i+1}\}$ for some $i \in \{1, 2\}$. Then $\phi(z_{i+2}z_{i+3}) = \alpha_3$. Since $\alpha_3 \in A(uv) = L(uv) - \phi(x)$, this yields $z_{i+2} = w$ and thus $u = z_i$, $v = z_{i+1}$. Since $\phi_1(vw) = \alpha_1 \neq \phi_1(xu)$, $\beta = \alpha_1$, $i = 1$ and we have no bicolored cycles. Since $i = 1$, $z_4 \neq w'$. So $z_4 = y$ and $z_5 \in \{y_1, y_2\}$. But $\alpha_1 \notin \phi(y)$. This contradiction proves (6).

Now, let ϕ_3 be obtained from ϕ_2 by coloring uu' with α_4. By (6) and Lemma 6(b), ϕ_3 is a partial se-coloring of H. But by (4), ϕ_3 colors all edges of H. This contradiction proves Case 2.1.

If Case 2.1 does not hold, then by (5), we may assume that $A(uv) = A(uu') = \{\alpha_1, \alpha_2\}$ and $A(wv) = A(ww') = \{\beta_1, \beta_2\}$. This means that

(7) $L(uv) = L(uu') = \{\alpha_1, \alpha_2\} \cup \phi(x)$ and $L(wv) = L(ww') = \{\beta_1, \beta_2\} \cup \phi(y)$.

In particular, $d_H(x) = d_H(y) = 3$.

Case 2.2. $\{\alpha_1, \alpha_2\} \cap \{\beta_1, \beta_2\} = \emptyset$. By symmetry, we may assume that $\alpha_1 \neq \phi(wy)$ and $\beta_1 \neq \phi(xu)$. Let ϕ_1 be obtained from ϕ by coloring uv with α_1 and vw

with β_1. By Lemma 6(a), ϕ_1 is a partial se-coloring of H. Then let ϕ_2 be obtained from ϕ_1 by coloring uu' with α_2 and ww' with β_2. Again by Lemma 6(a), ϕ_2 is a partial se-coloring of H. By (4), ϕ_2 colors all edges of H, contradicting the choice of H.

Case 2.3. $|\{\alpha_1, \alpha_2\} \cap \{\beta_1, \beta_2\}| = 1$. By (7), we may assume that $L(wv) = L(ww') = \{1, 2, 3, 4, 5\}$, $\alpha_1 = \beta_1 = 1$, $\beta_2 = 2$, $\phi(wy) = 3$, $\phi(yy_1) = 4$ and $\phi(yy_2) = 5$. By the case, $\alpha_2 \neq 2$. Let ϕ_1 be obtained from ϕ by setting $\phi_1(vw) = 2$ and $\phi_1(uv) = 1$ (in this order). Then we get partial se-colorings after both steps by Lemma 6(a), since $1 \notin \phi(y) \cup \phi(x)$. Let ϕ_2 be obtained from ϕ_1 by setting $\phi_2(uu') = \alpha_2$. If ϕ_2 has a bicolored path $z_1z_2z_3z_4z_5$ with $z_1z_2 = u'u$, then the second edge should be uv, since $\alpha_2 \notin \phi(x)$. But then the third edge must be vw and $\phi_1(vw) = 2$ and $\alpha_2 \neq 2$. Hence no such a bicolored path exists. Thus ϕ_2 is a partial se-coloring of H. So if $3 \notin \phi(y_1)$, then by coloring ww' with 4, we obtain from ϕ_2 an se-coloring of H, a contradiction. Thus $3 \in \phi(y_1)$. Similarly, $3 \in \phi(y_2)$.

Let $\gamma_1, \gamma_2 \in L(wy) - \{3, 4, 5\}$. Return to coloring ϕ. Suppose $\gamma_1 \notin \phi(y_1) \cup \phi(y_2)$. We recolor wy with γ_1, color vw with γ_2, uv with a color $\alpha \in \{1, \alpha_2\} - \gamma_1$, and uu' with $\alpha' \in \{1, \alpha_2\} - \alpha$ (in this order). After each step, by Lemma 6(a), we get a partial se-coloring of H. So the resulting coloring ϕ_3 is a partial se-coloring of H in which only ww' is uncolored. Now after coloring ww' with $\lambda \in \{4, 5\} - \phi_3(uv)$ we get an se-coloring of H from L, a contradiction. Thus by the symmetry between γ_1 and γ_2, $\{\gamma_1, \gamma_2\} \subset \phi(y_1) \cup \phi(y_2)$. In particular, this means $d_H(y_1) = d_H(y_2) = 3$. Let $N_H(y_1) = \{y, y_3, y_4\}$ and $N_H(y_2) = \{y, y_5, y_6\}$. We may assume that $\phi(y_1y_3) = \phi(y_2y_5) = 3$, $\phi(y_1y_4) = \gamma_1$ and $\phi(y_2y_6) = \gamma_2$.

If $4 \notin \phi(y_4)$, consider the se-coloring ϕ_3 from the previous paragraph, but now color ww' with 5. Since $\gamma_1 \notin \phi(y_2)$ and $2 \notin \{\alpha_1, \alpha_2\}$, the only possible bicolored path with 4 edges is $w'wvux$. This means $\phi(xu) = 2$ and $\alpha_2 = \phi_3(uv) = 5$. In this case, recolor vw with 3. Thus $4 \in \phi(y_4)$, and in particular, $d_H(y_4) \geq 2$, so $y_4 \in V(H^*)$. Similarly, $5 \in \phi(y_6)$, and so $y_6 \in V(H^*)$. We claim that also

$$\{y_3, y_5\} \subset V(H^*). \tag{8}$$

Suppose (8) fails, say $d_H(y_5) = 1$. Consider again the partial se-coloring ϕ_2. Recolor y_5y_2 with a $\lambda \in L(y_5y_2) - \{3, 5\} - \phi(y_6)$ (since $5 \in \phi(y_6)$, this set is nonempty) and color ww' with 5. If there is a bicolored 4-path $z_1z_2z_3z_4z_5$ with $z_1 = y_5$ and $z_2 = y_2$, then since $\lambda \notin \phi(y_6)$, $z_3 = y$. Since $\lambda \neq 3$, $z_4 = y_1$ and $\lambda = 4$. But $5 \notin \phi(y_1)$ since $\gamma_1 \notin \{3, 4, 5\}$. So we obtain an se-coloring of H from L, contradicting the choice of H. This proves (8). This together with $y_4, y_6 \in V(H^*)$ shows $d_{H^*}(y) = d_{H^*}(y_1) = d_{H^*}(y_2) = 3$. By symmetry also $d_{H^*}(x) = d_{H^*}(x_1) = d_{H^*}(x_2) = 3$, and so the lemma holds in this case.

Case 2.4. $\{\alpha_1, \alpha_2\} = \{\beta_1, \beta_2\}$. By (7), we may assume that $L(wv) = L(ww') = \{1, 2, 3, 4, 5\}$, $\alpha_1 = \beta_1 = 1$, $\alpha_2 = \beta_2 = 2$, $\phi(wy) = 3$, $\phi(yy_1) = 4$

and $\phi(yy_2) = 5$. Consider the partial se coloring ϕ_1 defined in Case 2.3. Let ϕ_4 be obtained from ϕ_1 by coloring uu' with 2. If there is a bicolored 4-path $z_1z_2z_3z_4z_5$ with $z_1 = u'$ and $z_2 = u$, then since $2 \notin \phi(x)$, $z_3 = v$ and so $z_4 = w$. But $\phi(wy) = 3 \neq 1$. Thus ϕ_4 is a partial se-coloring of H. Repeating the argument of the end of the first paragraph of Case 2.3, we conclude that $3 \in \phi(y_1)$ and $3 \in \phi(y_2)$.

Let $\gamma_1, \gamma_2 \in L(wy) - \{3, 4, 5\}$. Return to coloring ϕ. Suppose $\gamma_1 \notin \phi(y_1) \cup \phi(y_2)$. We uncolor wy, color vw with $\lambda \in \{4, 5\} - \phi(xu)$, ww' with $\lambda' \in \{4, 5\} - \lambda$, uv with a color $\alpha \in \{1, 2\} - \gamma_1$, uu' with $\alpha' \in \{1, 2\} - \alpha$ and finally wy with γ_1 (in this order). After each step, by Lemma 6(a), we get a partial se-coloring of H. So the resulting coloring ϕ_5 is an se-coloring of H, a contradiction. Thus by the symmetry between γ_1 and γ_2, $\{\gamma_1, \gamma_2\} \subset \phi(y_1) \cup \phi(y_2)$. In particular, this means $d_H(y_1) = d_H(y_2) = 3$. Let $N_H(y_1) = \{y, y_3, y_4\}$ and $N_H(y_2) = \{y, y_5, y_6\}$. We may assume that $\phi(y_1y_3) = \phi(y_2y_5) = 3$, $\phi(y_1y_4) = \gamma_1$ and $\phi(y_2y_6) = \gamma_2$.

If $4 \notin \phi(y_4)$, consider the se-coloring ϕ_5 from the previous paragraph, in which recolor the edge $e \in \{wv, ww'\}$ of color 4 with 3. We will get an se-coloring of H from L, unless $e = wv$ and $\phi(xu) = 3$. But in this case, we recolor wv with 5 and ww' with 3 (i.e., switch the colors of wv and ww'). Thus $4 \in \phi(y_4)$. Similarly, $5 \in \phi(y_6)$. As in Case 2.3, we claim that also (8) holds and the proof word by word repeats such proof in Case 2.3. So we again get $d_{H^*}(y) = d_{H^*}(y_1) = d_{H^*}(y_2) = 3$ and by symmetry $d_{H^*}(x) = d_{H^*}(x_1) = d_{H^*}(x_2) = 3$. This proves the lemma. ■

Lemma 13. *H^* does not contain a 3-vertex adjacent to three 2-vertices such that at least two of these vertices have 2-neighbors in H^*.*

Proof. Suppose that H^* contains a 3-vertex u adjacent to 2-vertices x, y, z such that y has a 2-neighbor y_1 and z has a 2-neighbor z_1. By Claim 11, $y_1, z_1 \notin \{x, y, z\}$. By Lemma 12, $y_1 \neq z_1$. Let w (respectively, t) denote the second neighbor in H^* of y_1 (respectively, z_1). For each $r \in \{x, y, y_1, z, z_1\}$, if r has a (unique) 1-neighbor in H, then we denote this neighbor by r' (see Figure 2). Let v be the neighbor of x different from x' and u.

Let $H_1 = H - \{u, x', y, y', z, z', y_1', z_1'\}$. By the minimality of H, graph H_1 has an se-coloring ϕ from L. We view ϕ as a partial se-coloring of H. Let $A(xu) = L(xu) - \phi(v)$, $A(xx') = L(xx') - \phi(v)$, $A(yy_1) = L(yy_1) - \phi(w)$, $A(y_1y_1') = L(y_1y_1') - \phi(w)$, $A(zz_1) = L(zz_1) - \phi(t)$ and $A(z_1z_1') = L(z_1z_1') - \phi(t)$. Similarly to (5), we have

$$\text{(9)} \qquad \begin{aligned} &\text{each of } A(xu), A(xx'), A(yy_1), A(y_1y_1'), A(zz_1) \\ &\text{and } A(z_1z_1') \text{ contains at least two colors.} \end{aligned}$$

Case 1. There is $\alpha \in A(yy_1) \cap A(zz_1)$. Color yy_1 and zz_1 with α, then color xu with a color $\beta \in A(xu) - \alpha$, then y_1y_1' with $\alpha_1 \in A(y_1y_1') - \alpha$, z_1z_1'

with $\alpha_2 \in A(z_1z_1') - \alpha$ and xx' with $\beta' \in A(xx') - \beta$. Since edges uz and uy are not colored, by Lemma 6(a), the new partial edge-coloring ϕ_1 of H is an se-coloring. Then we color uy with $\gamma_1 \in L(uy) - \{\alpha, \beta, \phi(xv)\}$ and uz with $\gamma_2 \in L(uz) - \{\alpha, \beta, \phi(xv), \gamma_1\}$. Let ϕ_2 be the new coloring. If Lemma 6(b) does not apply to $\phi_2(zu)$, then $\phi_2(zu) = \phi_1(tz_1)$. But the color α of z_1z is not in $\phi_2(u) \cup \phi_2(t)$ by definition. So there is no bicolored 4-path in ϕ_2 containing uz. Similarly, there is no bicolored 4-path in ϕ_2 containing uy. Thus, ϕ_2 is a partial se-coloring of H. Finally, color yy' with $\lambda_1 \in L(yy') - \{\alpha, \beta, \phi_2(uy), \phi_2(uz)\}$ and zz' with $\lambda_2 \in L(zz') - \{\alpha, \beta, \phi_2(uy), \phi_2(uz)\}$. Let ϕ_3 be the new coloring. As above, if Lemma 6(b) does not apply to $\phi_3(zz')$, then $\phi_3(zz') = \phi_1(tz_1)$. But the color α of z_1z is not in $\phi_3(t)$ by definition. So there is no bicolored 4-path in ϕ_3 containing zz'. Similarly, there is no bicolored 4-path in ϕ_3 containing yy'. Thus, ϕ_3 is an se-coloring of H, a contradiction.

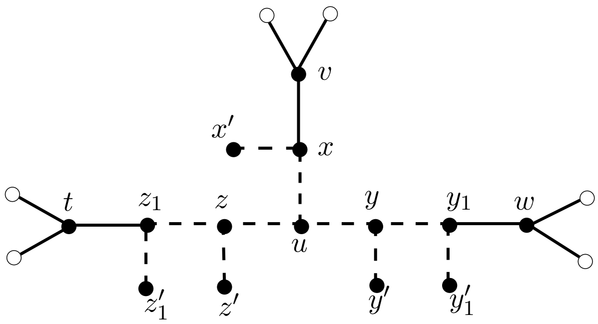


Figure 2. Forbidden configuration of Lemma 13 in H.

Case 2. $A(yy_1) \cap A(zz_1) = \emptyset$. Color xu with a color $\beta \in A(xu)$, then color yy_1 with a color $\alpha_1 \in A(yy_1) - \beta$, then zz_1 with a color $\alpha_2 \in A(zz_1) - \beta$, then y_1y_1' with $\alpha_1' \in A(y_1y_1') - \alpha_1$, z_1z_1' with $\alpha_2' \in A(z_1z_1') - \alpha_2$ and xx' with $\beta' \in A(xx') - \beta$. Similarly to Case 1, by Lemma 6(a), the new partial edge coloring ϕ_1 of H is an se-coloring. Then we color uy with $\gamma_1 \in L(uy) - \{\alpha_1, \alpha_2, \beta, \phi(xv)\}$ and uz with $\gamma_2 \in L(uz) - \{\alpha_1, \alpha_2, \beta, \gamma_1\}$. Let ϕ_2 be the new coloring. If Lemma 6(b) does not apply to $\phi_2(zu)$, then $\phi_2(zu) \in \{\phi_1(tz_1), \phi_1(xv)\}$. But the color α_2 of z_1z is not in $\phi_2(u) \cup \phi_2(t)$, and the color β of xu is not in $\phi_2(v) \cup \phi_2(z)$, by definition. So there is no bicolored 4-path in ϕ_2 containing uz. Similarly, there is no bicolored 4-path in ϕ_2 containing uy. Thus, ϕ_2 is a partial se-coloring of H. Finally, color yy' with $\lambda_1 \in L(yy') - \{\alpha_1, \beta, \phi_2(uy), \phi_2(uz)\}$ and zz' with $\lambda_2 \in L(zz') - \{\alpha_2, \beta, \phi_2(uy), \phi_2(uz)\}$. Let ϕ_3 be the new coloring. As above, if Lemma 6(b) does not apply to $\phi_3(zz')$, then $\phi_3(zz') = \phi_1(tz_1)$. But the color α_2 of z_1z is not in $\phi_2(t)$ by definition. So there is no bicolored 4-path in ϕ_3 containing zz'. Similarly, there is no bicolored 4-path in ϕ_3 containing yy'. Thus, ϕ_3 is an se-coloring of H, a contradiction. ■

We will now show that $|E(H^*)| \geq \frac{7}{6}|V(H^*)|$, which will contradict the fact that $\mathrm{mad}(H) < \frac{7}{3}$. For this, we will use the discharging method. First, recall that by Claims 9 and 10, $\delta(H^*) \geq 2$. Also, by Lemma 12, for each path uvw in H^*,

$$\text{if } d_{H^*}(u) = d_{H^*}(v) = d_{H^*}(w) = 2, \quad \text{then } u \text{ and } w \text{ have distinct 3-neighbors in } H^*. \tag{10}$$

For each vertex v of H^*, we define the *charge* of v as $\omega(v) = d(v) - \frac{7}{3}$. So

$$\sum_{v \in V(H^*)} \omega(v) = \sum_{v \in V(H^*)} d_{H^*}(v) - \frac{7}{3}|V(H^*)| = 2|E(H^*)| - \frac{7}{3}|V(H^*)|. \tag{11}$$

During the discharging process, we will modify ω to a new charge ω^* so that the total sum of charges will not change. On the other hand, we will show that $\omega^*(v) \geq 0$ for all $v \in V(H^*)$. By (11), this will yield $|E(H^*)| \geq \frac{7}{6}|V(H^*)|$ contradicting $\mathrm{mad}(H) < \frac{7}{3}$.

The discharging rules are as follows:

(R1) Every 2-vertex in H^* adjacent to two 3-vertices receives $\frac{1}{6}$ from each of the two neighbors.

(R2) Every 2-vertex in H^* adjacent to exactly one 3-vertex receives $\frac{1}{3}$ from this 3-vertex.

(R3) Every 2-vertex in H^* adjacent to two 2-vertices, say x and y, receives $\frac{1}{6}$ from the other neighbor of x and $\frac{1}{6}$ from the other neighbor of y. Note that by (10), these "other neighbors" are distinct 3-vertices in H^*.

By (R1)–(R3) none of the 2-vertices in H^* gives away any charge, and each of them receives charge exactly $\frac{1}{3}$ from other vertices. Thus $\omega^*(v) = 0$ for each 2-vertex v.

Now, let v be a 3-vertex in H^*. If v has no 2-neighbors, then it keeps its charge $\frac{2}{3}$. If v has exactly one 2-neighbor, then by (R1)–(R3), it gives away at most $\frac{1}{3} + \frac{1}{6}$ and is left with charge at least $\frac{2}{3} - \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$. If v has exactly two 2-neighbors, then by Lemma 12, Rule (R3) does not apply to v. Thus in this case v gives away at most $\frac{1}{3} + \frac{1}{3}$ and is left with charge at least 0. Finally, suppose v has three 2-neighbors. Again by Lemma 12, Rule (R3) does not apply to v. Moreover, by Lemma 13, at most one 2-neighbor of v has also a 2-neighbor. This means that (R2) applies to v at most once. So, v is left with charge at least $\frac{2}{3} - \frac{1}{3} - \frac{1}{6} - \frac{1}{6} = 0$. This completes the proof of Theorem 4.1.

5. Proof of Theorem 4.2.

Suppose that the theorem is not true. Let H have the fewest edges among the subcubic graphs with $\mathrm{mad}(H) < \frac{5}{2}$ such that for some list L with $|L(e)| = 6$ for each $e \in E(H)$, H has no se-coloring from L. Clearly, H is connected.

Claims 9 and 10 hold for the graph H since they hold for such graph no matter what is the mad. Then we have the following.

Claim 14. *H has no weak 2-vertices.*

Claim 15. *H does not contain a 3-vertex adjacent to two 1-vertices.*

So, as in the previous section, the graph H^* obtained from H by deleting all vertices of degree 1 has minimum degree at least two.

Lemma 16. *H^* does not contain a 2-vertex adjacent to a 2-vertex.*

Proof. Suppose that H contains a path $xuvy$ or a cycle $xuvx$ such that $d_{H^*}(u) = d_{H^*}(v) = 2$. If u (respectively, v) has a 1-neighbor in H, denote this neighbor by u' (respectively, by v'), otherwise it does not exist.

Case 1. H^* contains a cycle $C = xuvx$ such that $d_{H^*}(u) = d_{H^*}(v) = 2$. Let w be the third neighbor of x in H, if it exists. If $H^* = C$, then H has at most 6 edges, and we can greedily color them with all colors distinct. So, $H^* \neq C$, and thus the vertex w exists and $d_H(w) \geq 2$. Let $H_0 = H - \{u, u', v, v'\}$. By the minimality of H, graph H_0 has an se-coloring ϕ from L. We view ϕ as a partial se-coloring of H. Color ux with a color $\alpha_1 \in L(ux) - \phi(w)$, then vx with a color $\alpha_2 \in L(vx) - \phi(w) - \alpha_1$, and then color uv with a color $\alpha_3 \in L(uv) - \phi(w) - \alpha_1 - \alpha_2$. By Lemma 6(a), the new partial edge-coloring ϕ' of H is an se-coloring. Now consecutively for $z \in \{u, v\}$, color edge zz' (if it exists) with a color in $L(zz') - \phi'(x) - \alpha_3$. By Lemma 6(b), at each step we again will obtain a partial se-coloring of H. So, after the last step we get an se-coloring of H from L, a contradiction.

Case 2. H^* contains a path $P = xuvy$ such that $d_{H^*}(u) = d_{H^*}(v) = 2$. Let $N_H(y) \subseteq \{v, y_1, y_2\}$ (maybe only one of y_1, y_2 exists) and $N_H(x) \subseteq \{u, x_1, x_2\}$. Let $H_1 = H - \{u', v'\} - uv$. Similarly to Case 1, H_1 has an se-coloring ψ from L. We view ψ as a partial se-coloring of H. First, we try to extend ψ to uv. If there is $\alpha_1 \in L(uv) - \psi(x) - \psi(y)$, then we color uv, which by Lemma 6(a), would yield a new partial se-coloring of H. Otherwise, $L(uv) \subseteq \psi(x) \cup \psi(y)$, which yields that $\psi(x)$ and $\psi(y)$ are disjoint sets of size 3 each. So, we may assume that

$$\begin{aligned} &L(uv) = \{1, \dots, 6\}, \text{ where } \psi(xu) = 1, \psi(xx_1) = 2, \\ &\psi(xx_2) = 3, \psi(yy_1) = 4, \psi(yy_2) = 5, \psi(vy) = 6. \end{aligned} \tag{12}$$

In particular, $d_H(x) = d_H(y) = 3$. For $i = \{1, 2\}$, let $N_H(y_i) = \{y, z_i, t_i\}$ (see Figure 3). If coloring uv with 4 does not create a bicolored 4-path, we do this. Otherwise, this is a path of colors 4 and 6, and so $6 \in \psi(y_1)$. Similarly, after trying to color uv with 5, we conclude that $6 \in \psi(y_2)$ and so $|\psi(y_1) \cup \psi(y_2)| \leq 5$.

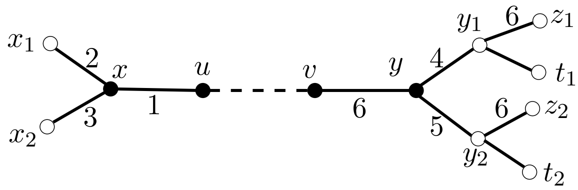


Figure 3. Two adjacent 2-vertices in H^*.

So we may recolor vy with $\alpha_2 \in L(vy) - (\psi(y_1) \cup \psi(y_2))$ and color uv with 6. By the definition of α_2 and the fact that all colors $1, \ldots, 6$ are distinct, the new edge-coloring ψ' is a partial se-coloring of H from L.

Now we simply color uu' (if exists) with $\alpha_3 \in L(uu') - \psi'(x) - \psi'(v)$ and vv' (if exists) with $\alpha_4 \in L(vv') - \psi'(u) - \psi'(y)$ (note that we allow $\alpha_4 = \alpha_3$). By Lemma 6(b), this yields an se-coloring of H from L, a contradiction. ■

Lemma 17. *H^* does not contain a 3-vertex adjacent to three 2-vertices.*

Proof. Suppose that H^* contains a 3-vertex v adjacent to three 2-vertices x_1, x_2 and x_3 whose second neighbors in H^* are y_1, y_2 and y_3, respectively. By Lemma 16, $d_{H^*}(y_i) = 3$ for all $i \in \{1, 2, 3\}$. So for $i \in \{1, 2, 3\}$, let $N_H(y_i) = \{x_i, u_i, w_i\}$ (some of these vertices y_i may coincide). Also, for $i \in \{1, 2, 3\}$, let x'_i denote the neighbor of degree 1 of x_i in H, if exists (see Figure 4).

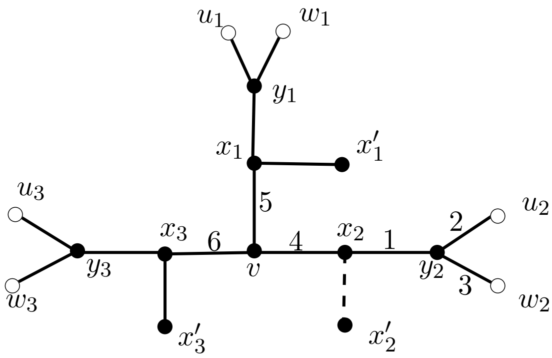


Figure 4. Forbidden configuration of Lemma 17 in H

By the minimality of H, graph $H_0 = H - \{v, x'_1, x'_2, x'_3\}$ has an se-coloring ϕ from L. We view ϕ as a partial se-coloring of H. If for some $i \in \{1, 2, 3\}$, color $\phi(x_iy_i)$ is present in both, $\phi(u_i)$ and $\phi(w_i)$, then we can recolor x_iy_i with a color in $L(x_iy_i) - (\phi(u_i) \cup \phi(w_i))$. Thus by the symmetry between u_i and w_i, we may assume that

$$\phi(x_iy_i) \notin \phi(u_i) \quad \text{for all } i \in \{1, 2, 3\}. \tag{13}$$

We will extend ϕ to the whole H in two steps.

Step 1. We extend ϕ to the edges incident with v. We color vx_1 with $\beta_1 \in L(vx_1) - \phi(y_1) - \phi(y_2x_2) - \phi(y_3x_3)$, then color vx_2 with $\beta_2 \in L(vx_2) - \phi(y_2) - \phi(y_3x_3) - \beta_1$, and then vx_3 with $\beta_3 \in L(vx_3) - \phi(y_3) - \beta_1 - \beta_2$. We claim that the resulting coloring ϕ' is a partial se-coloring of H. Indeed, if not, then for some $i \in \{1, 2, 3\}$, edge vx_i is in a bicolored path or cycle P with 4 edges. Since $\beta_i \notin \phi(y_i)$, the second edge of the color β_i in P must be x_jy_j for some $j \neq i$. Then edge vx_j is also in P. By the symmetry between i and j, we conclude that x_iy_i is in P and may assume $i < j$. But then by the definition of β_i, it differs from $\phi(x_jy_j)$, a contradiction.

Step 2. We extend ϕ' to those of $x_ix'_i$ that exist. For each such i, we color $x_ix'_i$ with a color $\gamma_i \in L(x_ix'_i) - \phi'(v) - \{\phi'(x_iy_i), \phi'(y_iw_i)\}$. If the resulting coloring ϕ'' is not an se-coloring of H, then for some $i \in \{1, 2, 3\}$ there is a bicolored 4-path P starting from x'_i. Since $\gamma_i \notin \phi'(v)$, the second edge of color γ_i in P is incident with y_i. Since γ_i was chosen distinct from $\phi'(y_iw_i)$, this second edge is y_iu_i. But this contradicts (13). ∎

For $j \in \{1, 2, 3\}$, let V_j denote the set of vertices of degree j in H^*. As it was mentioned above, by Claims 14 and 15, $V_1 = \emptyset$. By Lemma 16, every $v \in V_2$ has two neighbors in V_3, and by Lemma 17, every $v \in V_3$ has at most two neighbors in V_2. It follows that $|V_3| \geq |V_2|$, which yields $\text{mad}(H^*) \geq 5/2$. This proves Theorem 4.2.

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