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CONFLICT-FREE CONNECTIONS OF GRAPHS

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Abstract

An edge-colored graph G is conflict-free connected if any two of its vertices are connected by a path, which contains a color used on exactly one of its edges. In this paper the question for the smallest number of colors needed for a coloring of edges of G in order to make it conflict-free connected is investigated. We show that the answer is easy for 2-edge-connected graphs and very difficult for other connected graphs, including trees.

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1. Introduction

We use [23] for terminology and notation not defined here and consider finite and simple graphs only.

An edge-coloring of a graph G is *proper* if any two adjacent edges in this coloring receive different colors. If G is colored with a proper coloring, then we say that G is *properly colored.*

An edge-colored graph G is called *rainbow connected* if any two vertices are connected by a path whose edges have pairwise distinct colors. The concept of rainbow connection in graphs was introduced by Chartrand *et al.* [4]. The *rainbow connection number* of a connected graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. There is an extensive research concerning this parameter, see e.g. [11–14, 16, 17, 21].

As a modification of proper colorings and rainbow colorings of graphs, Andrews *et al.* [2] and independently Borozan *et al.* [3] introduced the concept of proper connection of graphs. An edge-colored graph G is called *properly connected* if any two vertices are connected by a path which is properly colored. The *proper connection number* of a connected graph G, denoted by $pc(G)$, is the smallest number of colors that are needed in order to make G properly connected. One can find many results on proper connection, see e.g. [1, 9, 10, 15, 19].

Motivated by the above mentioned two concepts and by conflict-free colorings of graphs and hypergraphs [6–8, 20] we introduce the concept of conflict-free connection and the concept of proper conflict-free connection.

An edge-colored graph G is called *conflict-free connected* if any two vertices are connected by a path which contains at least one color used on exactly one of its edges. Let us call such a path *conflict-free path*. The *conflict-free connection number* of a connected graph G, denoted by $cfc(G)$, is the smallest number of colors that are needed in order to make G conflict-free connected. The main problem studied in this paper is the following.

Problem 1. For a given connected graph G determine its conflict-free connection number.

An easy observation is that if G has n vertices, then all above mentioned three parameters are bounded from above by $n-1$, since one may color the edges of a given spanning tree of G with distinct colors and color the remaining edges with already used colors.

The rest of this paper is organized as follows. In Section 2 we prove some preliminary results. In Section 3 we study the structure of graphs having conflict-free connection number two. General 1-connected graphs are investigated in Section 4. There it is shown that for precise answers to the above problem it is necessary to know exact values of conflict-free connection numbers of trees. Trees are studied from this point of view in Section 5. The final section, Section 6 is devoted to studying the proper version of Problem 1.

2. Preliminaries

In this section we prove several lemmas which will be useful later. The first one is the following analogue of Whitney's theorem (see [5]).

Lemma 1. *Let u, v be distinct vertices and let $e = xy$ be an edge of a 2-connected graph G. Then there is a $u - v$ path in G containing the edge e.*

Proof. We distinguish two cases.

Case 1. First assume that $\{x, y\} \cap \{u, v\} \neq \emptyset$. Let, w.l.o.g., $x = u$. Because of the 2-connectivity of G there is a $y - v$ path P which avoids the vertex u. Then the path u, e, y, P, v is a required path.

Case 2. Let $\{x, y\} \cap \{u, v\} = \emptyset$. Then, by Corollary 2.40 of Whitney's theorem (see [5], p. 102), G contains two internally disjoint paths, namely $u - x$ path P_1 and $v - x$ path P_2. If there is a $y - u$ path P omitting x such that P and P_2 are vertex disjoint, then the path u, P, y, e, x, P_2, v has the needed property. If the paths P and P_2 have a vertex in common, then let z be the first vertex from $V(P_1) \cup V(P_2)$ when going along P from y. If $z \in V(P_2)$, then denote by Q_1 the subpath of P from y to z and by Q_2 the subpath of P_2 from z to v. Then the path $u, P_1, x, e, y, Q_1, z, Q_2, v$ has the property stated in the lemma. If $z \in V(P_1)$, then denote by R_1 the subpath of P_1 from u to z and by R_2 the subpath of P from z to y. Then the path $u, R_1, z, R_2, y, e, x, P_2, v$ has the stated property. ∎

A *block* of a graph G is a maximal connected subgraph of G that has no cut-vertex. If G itself is connected and has no cut-vertex, then G is a block. An edge is a block if and only if it is a cut-edge. A block consisting of a cut-edge is called *trivial*. Note that any non-trivial block is 2-connected.

Lemma 2. *Let G be a connected graph. Then from each of its non-trivial blocks an edge can be chosen so that the set of all such chosen edges forms a matching.*

Proof. The proof is by induction on the number of blocks of G. If G has exactly one block, then the lemma trivially holds.

Let the lemma hold for every connected graph with $b \geq 1$ blocks. Let G have $b + 1$ blocks. Consider a leaf-block B with the (unique) cut-vertex v. If B is trivial, i.e $B = vu$, then G has the same required matching as $G' = G - u$. Now assume that B is not trivial. The graph $G' = G - (B - v)$ has fewer blocks than G, therefore, by induction hypothesis, it has a required set M' of independent edges. Choosing one edge of B not incident with v and adding it to M' we get a required matching of G. ∎

It is easy to see that for any star $K_{1,r}$ on $r + 1$ vertices we have $cfc(K_{1,r}) = r$, $r \geq 2$.

Theorem 3. *If P_n is a path on n edges, then $cfc(P_n) = \lceil \log_2(n + 1) \rceil$.*

Proof. First we prove that $cfc(P_n) \leq \lceil \log_2(n + 1) \rceil$. Let $P_n = e_1, e_2, \ldots, e_n$ be a path on n edges. Color the edge e_i with color $x + 1$, where 2^x is the largest power of 2 that divides i. Clearly, the largest color in such a coloring is $\lceil \log_2(n+1) \rceil$. Every subpath Q of P_n is conflict-free, because the maximum color of the edges of Q appears only once on Q.

Now we show that $cfc(P_n) \geq \lceil \log_2(n+1) \rceil$. We prove that any path with conflict-free connection number k has at most $2^k - 1$ edges. We use induction on k. The statement is evidently true for $k = 1$ and $k = 2$. Let P_n be a path with $cfc(P_n) = k$. Then there is an edge e_i with a unique color. Delete this edge from P_n. The resulting paths $P_{i-1} = e_1, e_2, \ldots, e_{i-1}$ and $P_{n-i} = e_{i+1}, e_{i+2}, \ldots, e_n$ have conflict-free connection number at most $k-1$ (their edges are colored with $k-1$ colors). Therefore, by the induction hypothesis, P_{i-1} and P_{n-i} have at most $2^{k-1} - 1$ edges. Consequently, P_n has at most $2 \cdot (2^{k-1} - 1) + 1 = 2^k - 1$ edges. ■

3. Graphs with Conflict-Free Connection Number Two

Lemma 4. *If G is a 2-connected and non-complete graph, then $cfc(G) = 2$.*

Proof. Since G contains non-adjacent edges it holds $cfc(G) \geq 2$.

Let e be an edge of G. Color e with color 2 and all other edges of G with color 1. By Lemma 1, for every two distinct vertices u and v there is, in G, a $u - v$ path containing the edge e. Clearly, this $u - v$ path is conflict-free. ■

Let $C(G)$ be the subgraph of G induced on the set of cut-edges of G. Note that $C(G)$ can be empty. The following lemma provides a necessary condition for graphs G with cut-edges to have $cfc(G) = 2$.

Lemma 5. *If $cfc(G) = 2$ for a graph G with cut-edges, then $C(G)$ is a linear forest whose each component has at most three edges.*

Proof. $C(G)$ is a forest since no cut-edge is incident with a cycle. Its maximum degree is at most 2, because no two edges with the same color can be adjacent in $C(G)$. Hence, $C(G)$ is a linear forest. Theorem 3 implies that each path with at least four edges requires at least three colors in a conflict-free coloring, therefore each component of $C(G)$ has at most three edges. ■

Theorem 6. *If G is a connected graph and $C(G)$ is a linear forest whose each component is of order 2, then $cfc(G) = 2$.*

Proof. Since the edges of $C(G)$ form a matching, each vertex of degree at least two is incident with a non-trivial block. By Lemma 2, we can choose from each non-trivial block one edge so that all chosen edges create a matching S. Next, we color the edges from S with color 2 and all remaining edges of G with color 1.

Now we need to show that any two distinct vertices x and y are connected by any conflict-free $x-y$ path, i.e., an $x-y$ path which contains exactly one edge colored with color 1 or 2. We distinguish several cases.

Case 1. Let x and y belong to the same block. If this block is trivial, then x and y are adjacent, and we are done. If this block B is non-trivial, then by Lemma 1, there is an $x-y$ path in B containing the edge of B colored with color 2. Clearly, this $x-y$ path is conflict-free.

Case 2. Let x and y be in different blocks. Consider a shortest $x-y$ path in G. This path goes through blocks, say $B_1, \ldots, B_r$, $r \geq 2$, in this order, where $x \in V(B_1)$ and $y \in V(B_r)$. Let v_i be the common vertex of blocks B_i and B_{i+1}, $1 \leq i \leq r-1$.

Case 2.1. Let B_1 be a trivial block. Then B_2 is a non-trivial block by the assumption on $C(G)$ and $v_1 \neq y$. If $r = 2$, then the admired $x-y$ path is a concatenation of the edge xv_1 and a $v_1 - y$ path going through the edge colored with 2 in B_2. If $r \geq 3$, then the admired $x-y$ path is a concatenation of the edge xv_1, a $v_1 - v_2$ path going through the edge colored with 2 in B_2, a $v_{i-1} - v_i$ path in B_i omitting the edge colored with 2 in B_i for $3 \leq i \leq r-1$, and a $v_{r-1} - y$ path omitting the edge assigned 2 in B_r.

Case 2.2. Let B_1 be a non-trivial block. Then $x \neq v_1$. The conflict-free $x-y$ path is a concatenation of an $x - v_1$ path in B_1 going through the edge assigned 2, a $v_{i-1} - v_i$ path in B_i omitting the edge colored with 2 in B_i for $2 \leq i \leq r-1$, and a $v_{r-1} - y$ path omitting the edge assigned 2 in B_r. ■

Lemma 5 gives a necessary condition for a connected graph having conflict-free connection number two. The following theorem points out that this condition is not sufficient. To formulate it we need a new notion. The *t-corona* of a graph H, denoted by $Cor_t(H)$, is a graph obtained from H by adding t pendant edges to each vertex of H.

Theorem 7. *If C_n denotes the n-cycle, $n \geq 4$, and G is its 2-corona, then $C(G)$ is a linear forest whose components are paths on two edges and $cfc(G) = 3$.*

Proof. Let $C_n = v_1, v_2, \ldots, v_n$ be the n-cycle. Denote by x_i and y_i the ends of pendant edges of G added to the vertex v_i of C_n. Suppose that the conflict-free connection number of G is two. Since there is only one $x_i - y_i$ path in G, the edges x_iv_i and y_iv_i must have different colors, say 1 and 2, respectively. Without loss of generality, we can assume that the edge v_1v_2 has color 1. The graph G contains only two $x_1 - x_2$ paths, moreover, the path x_1, v_1, v_2, x_2 is monochromatic. This implies that only one edge of C_n has color 2, say v_iv_{i+1}. Consequently, there is no conflict-free $y_i - y_{i+1}$ path in G, a contradiction.

It is easy to see that the following 3-edge-coloring c makes G conflict-free connected: $c(x_iv_i) = 1$ and $c(y_iv_i) = 2$ for $1 \leq i \leq n$; $c(v_nv_1) = 3$ and $c(v_iv_{i+1}) = 2$ for $1 \leq i \leq n-1$. ∎

4. General 1-Connected Graphs with Cut-Edges

Let G be a connected graph and $h(G) = \max\{cfc(K) : K \text{ is a component of } C(G)\}$.

Theorem 8. *If G is a connected graph with cut-edges, then $h(G) \leq cfc(G) \leq 1 + h(G)$. Moreover, these bounds are tight.*

Proof. The inequality $h(G) \leq cfc(G)$ holds. So it suffices to show that $cfc(G) \leq h(G) + 1$.

Let us start with coloring the edges of G. First we color every component K of $C(G)$ with at most $h(G)$ colors, say $1, 2, \ldots, h(G)$, so that any two vertices of K are connected by a conflict-free path.

Then according to Lemma 2 we choose in any non-trivial block of G an edge so that the set S of such chosen edges forms a matching. We color the edges from S with color $h(G) + 1$ and the uncolored edges of G with color 1.

Next, we have to show that for any two distinct vertices x and y there is a conflict free $x - y$ path. If the vertices x and y are from the same component K of $C(G)$, then such a path exists. If they are in the same non-trivial block, then by Lemma 1, there is an $x - y$ path through an edge colored with color $h(G) + 1$. If none of the above situations appears, then x and y are either from distinct components of $C(G)$, or distinct non-trivial blocks, or one is from a component of $C(G)$ and the other from a non-trivial block.

Consider a shortest $x - y$ path P. Let $v_1, \ldots, v_{r-1}$ be all cut-vertices of G lying on P, in this order. The path P goes through blocks $B_1, \ldots, B_r$ indicated by the vertices x and v_1, v_1 and $v_2, \ldots, v_{r-1}$ and y, respectively. Some of them may be trivial but at least one is non-trivial. If B_1 (respectively B_i, for some $i \in \{2, \ldots, r-1\}$, or B_r) is the first one which is non-trivial, then in it we choose a conflict free $x - v_1$ path ($v_{i-1} - v_i$ path, or $v_{r-1} - y$ path) through the edge of B_1 (B_i or B_r) colored with color $1 + h(G)$. Then in the remaining blocks B_j, $j \in \{1, \ldots, r\} - \{1\}$ ($j \in \{1, \ldots, r\} - \{i\}$ or $j \in \{1, \ldots, r\} - \{r\}$) we choose a monochromatic $v_{j-1} - v_j$ path. The admired conflict-free $x - y$ path is then concatenated of these so chosen one conflict-free and the remaining monochromatic paths. Clearly, the resulting $x - y$ path contains exactly one edge colored with the largest color $1 + h(G)$.

Now we show that for every positive integer k there is a graph G with $h(G) = k$ and $cfc(G) = 1 + h(G)$. Let P be a path with $cfc(P) = k$. Let G be a graph obtained from an arbitrary 2-connected graph H by adding two copies of the path P to two distinct vertices of H; one to each. Clearly, $h(G) = k$. Let u and v be the leaves of G. Any $u - v$ path in G contains all edges of the added paths, therefore no $u-v$ path has a conflict-free coloring with $h(G)$ colors. Consequently, $cfc(G) \geq 1 + h(G)$. ∎

5. Trees

A *k-edge ranking* of a connected graph G is a labeling of its edges with labels $1, \ldots, k$ such that every path between two edges with the same label i contains an edge with label $j > i$. A graph G is said to be *k-edge rankable* if it has a k-edge ranking. The minimum k for which G is k-edge rankable is denoted by $rank(G)$.

Lemma 9. *If G is a connected graph, then $cfc(G) \leq rank(G)$.*

Proof. Consider an edge ranking of G. Let x and y be two vertices of G and let P be an $x - y$ path in G. Let k be the maximum label used on P. If P contains only one edge of label k, then P is conflict-free. So suppose that P contains at least two such edges. Then, by the definition of edge-ranking, P contains an edge of label greater than k, a contradiction. ∎

The main result in this section is the following.

Theorem 10. *If T is an n-vertex tree of maximum degree $\Delta(T) \geq 3$ and diameter $d(T)$, then*

$$\max\{\Delta(T), \log_2 d(T)\} \leq cfc(T) \leq \frac{(\Delta(T) - 2) \cdot \log_2 n}{\log_2 \Delta(T) - 1}.$$

Proof. The lower bound immediately follows from Theorem 3. By Lemma 9, the parameter $cfc(T)$ is bounded by $rank(T)$, which is not greater than the mentioned upper bound, see [18]. ∎

6. Proper Conflict-Free Connection of Graphs

If we require a graph to be simultaneously properly colored and conflict-free connected, then we get a definition of the *proper conflict-free connection.* The *proper conflict-free connection number* of a connected graph G, denoted by $pcfc(G)$, is the smallest number of colors that are needed in order to make G properly conflict-free connected.

Recall that the *edge chromatic number* (or, equivalently, *chromatic index*) of a graph G, denoted by $\chi'(G)$, is the minimum number of colors that are needed to make the graph G properly colored. Clearly, $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ denotes the maximum degree of G. Vizing [22] proved that $\chi'(G) \leq \Delta(G) + 1$.

Observe that for any tree T there is $pcfc(T) = cfc(T)$. For 2-connected graphs we have the following.

Theorem 11. *If G is a 2-connected graph, then*

$$\Delta(G) \leq \chi'(G) \leq pcfc(G) \leq \chi'(G) + 1 \leq \Delta(G) + 2.$$

Proof. The first two inequalities in the theorem are obvious. To prove the third inequality consider the proof of Lemma 4. By this lemma, G has a 2-edge coloring such that only one edge of G, say e, has color 2 and there exists a conflict-free path between any two vertices.

Consider the graph $G' = G - e$. The graph G' has a proper edge-coloring with at most $\chi'(G)$ colors, since its supergraph G has such a coloring. Let these colors be $3, 4, \ldots, \chi'(G), \chi'(G)+1$ and 1. If we color the edge e with color 2, then we obtain a proper edge-coloring of G in which any two vertices are connected by a conflict-free path. This gives the third inequality.

The fourth inequality follows from the above mentioned Vizing's theorem. ∎

Combining the techniques from the proofs of Theorems 8 and 11 one can get the following.

Theorem 12. *Let G be a connected graph with $\Delta^*(G) = \Delta(G - E(C(G)))$ and $h(G) = \max\{cfc(K) : K$ is a component of $C(G)\}$. Then*

$$pcfc(G) \leq \Delta^*(G) + h(G) + 2.$$

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