

PERTURBATIONS IN A SIGNED GRAPH AND ITS INDEX

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Abstract

In this paper we consider the behaviour of the largest eigenvalue (also called the index) of signed graphs under small perturbations like adding a vertex, adding an edge or changing the sign of an edge. We also give a partial ordering of signed cacti with common underlying graph by their indices and demonstrate a general method for obtaining lower and upper bounds for the index. Finally, we provide our computational results related to the generation of small signed graphs.

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1. INTRODUCTION

For a simple graph $G = (V(G), E(G))$, let $\sigma : E(G) \rightarrow \{1, -1\}$ be a mapping defined on the edge set of G . Then, $\dot{G} = (G, \sigma)$ is called the *signed graph* derived from G , while G is its underlying graphs and σ is its sign function (or signature). Observe that G and $\dot{G}$ share the same set of vertices (i.e., $V(\dot{G}) = V(G)$) and have equal number of edges (i.e., $|E(\dot{G})| = |E(G)|$). We denote their common order and size by n and m , respectively.

The edge set $E(\dot{G})$ may be divided into two disjoint subsets that contain positive and negative edges, respectively. It may be observed that a signed graph is derived from its underlying graph G by reversing the sign of a fixed set of edges from $E(G)$. Clearly, the $n \times n$ adjacency matrix $A_{\dot{G}}$ of $\dot{G}$ is obtained from the standard $(0, 1)$ -adjacency matrix of G in a natural way, that is by reversing the

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sign of all 1's which correspond to negative edges. The characteristic polynomial $\Phi_{\dot{G}}(x) = \det(xI - A_{\dot{G}})$ is also called the *characteristic polynomial* of $\dot{G}$, while the corresponding eigenvalues are real and form the spectrum of $\dot{G}$. We denote them by $\lambda_1, \lambda_2, \dots, \lambda_n$ and also assume that they are ordered non-increasingly. In particular, the largest eigenvalue is called the *index*.

A signed cycle with n vertices is denoted by $\dot{C}_n$. A signed cycle $\dot{C}$, where C is a cycle in G , is *balanced* if it contains an even number of negative edges; otherwise, it is *unbalanced*. Hence, the sign of a cycle $\dot{C}$ is the product of signs of its edges, i.e., it is equal to $\prod_{e \in E(\dot{C})} \sigma(e)$.

The theory of spectra of (simple) graphs based on the adjacency matrix is a highly developed mathematical discipline. In particular, many results concern the behaviour of eigenvalues, mostly the index, under various graph perturbations. Some of well known facts about the index of connected simple graphs are as follows. It is a simple eigenvalue, the modulus of any other eigenvalue does not exceeds the index, and it strictly increases by adding a non-isolated vertex or an edge between non-adjacent vertices.

On the contrary, the existence of negative edges imposes a number of restrictions in passing to signed graphs. Regarding the index, none of the above facts holds for all signed graphs. For example, it is sufficient to consider signed graphs obtained by reversing the sign of all edges in appropriately chosen connected simple graph. We may also observe that the index of any unbalanced cycle $\dot{C}_n$ is of multiplicity two and that the adding of a negative edge between two endvertices of a simple path gives an unbalanced cycle with less index [9].

Recently, Koledin and Stanić [6] used a technique based on relocating the edges in a signed graph to describe those with fixed number of vertices, positive edges and negative edges that maximize the index. Another approach on the basis of small perturbations can be found in a paper of Belardo and Petecki [2].

The purpose of this paper is to give a deeper analysis of the index of a signed graph. In particular, we obtain additional assumptions under which some of mentioned properties of the index of simple graphs also hold in the case of signed graphs. We also consider the behaviour of the index under reversing the signs of edges and give certain partial orderings of specified signed graphs with respect to their indices. A general method for determining lower or upper bounds on the index is also demonstrated. It is well-known that the number of non-isomorphic simple graphs rapidly increases with their order n . According to the Pólya enumeration theorem [8], this number is asymptotically equivalent to

$$\frac{2^{\binom{n}{2}}}{n!}.$$

If we allow the existence of negative edges, then the class of signed graphs is much broader even if we restrict ourselves to representatives of switching isomorphic classes. (Switching isomorphic signed graphs that naturally arise in the context of their spectra are defined in the next section).

An additional purpose of this paper is to introduce the generation of these representatives having a small order and inspect their cospectrality (the details are given in Section 4).

After a preparatory section, we present our main results.

2. PREPARATORY

For $U \subset V(\dot{G})$, let $\dot{G}^U$ be the signed graph obtained from $\dot{G}$ by reversing the sign of each edge between a vertex in U and a vertex in $V(\dot{G}) \setminus U$. The signed graph $\dot{G}^U$ is said to be *switching equivalent* to $\dot{G}$. If D is the diagonal matrix of ± 1 's with -1 in the u -th position for each $u \in U$, then we have $A_{\dot{G}^U} = D^{-1}A_{\dot{G}}D$. Consequently, the switching equivalence is an equivalence relation and switching equivalent signed graphs share the same spectrum.

We say that signed graphs $\dot{G}$ and $\dot{H}$ are *switching isomorphic* if $\dot{H}$ is isomorphic to a signed graph that is switching equivalent to $\dot{G}$. In other words, if there exist a diagonal matrix D of ± 1 's and a permutation $(0, 1)$ -matrix P such that $A_{\dot{G}} = D^{-1}(P^{-1}A_{\dot{H}}P)D$. Again, switching isomorphism is an equivalence relation that preserves the eigenvalues. Moreover, up to the vertex labelling, each of switching isomorphic signed graphs can be selected to be a representative of the corresponding switching equivalence class. We shall return to these representatives in Section 4.

We say that signed graphs are *cospectral* if they are not switching isomorphic but share the same spectrum. Such signed graphs are also known as cospectral mates.

Concepts like regularity are directly extended to signed graphs (i.e., a signed graph is regular whenever the same holds for its underlying graph). Most of the standard graph invariants coincide for G and $\dot{G}$. We denote by $\deg(u)$ the degree of a vertex $u \in V(\dot{G})$, but for convenience we also write $\deg_+(u)$ and $\deg_-(u)$ for the positive and negative vertex degree (i.e., the number of positive and negative edges incident with u). Clearly, we have $\deg(u) = \deg_+(u) + \deg_-(u)$.

If considering subgraphs of signed graphs, then their signed functions are the restrictions of the original ones to the corresponding edge subsets. If u (respectively e) is a vertex (respectively edge) of $\dot{G}$, then we write $\dot{G} - u$ (respectively $\dot{G} - e$) to denote the corresponding vertex-deleted (respectively edge-deleted) subgraph.

Recall that the Interlacing Theorem [3, Theorem 0.10] and the Rayleigh

principle [4, p.12] hold for any Hermitian matrix, and therefore they can be applied to the adjacency matrix of any signed graph.

If $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ is an eigenvector associated with the eigenvalue λ of a signed graph $\dot{G}$, then it is usually assumed that the entry x_u corresponds to the vertex u ($1 \leq u \leq n$). If so, then the *eigenvalue equation* reads as follows

$$(1) \quad \lambda x_u = \sum_{v \sim u} \sigma(uv) x_v \quad (1 \leq u \leq n).$$

We recall from [1] the following Schwenk-like formula

$$(2) \quad \Phi_{\dot{G}}(x) = x \Phi_{\dot{G}-u}(x) - \sum_{v \sim u} \Phi_{\dot{G}-u-v}(x) - 2 \sum_{\dot{C} \in \dot{\mathcal{C}}_u} \sigma(\dot{C}) \Phi_{\dot{G}-\dot{C}}(x),$$

where $\dot{\mathcal{C}}_u$ denotes the set of signed cycles passing through u , and $\dot{G} - \dot{C}$ denotes the signed graph obtained from $\dot{G}$ by deleting $\dot{C}$ (we assume that $\Phi_{\dot{G}-\dot{C}}(x) = 1$ if $\dot{G} - \dot{C}$ has no vertices).

We conclude by the following observations related to the switching equivalence classes of signed graphs.

It is well-known that coordinates of an eigenvector associated with the index of a simple and connected graph are non-zero and of the same sign. Here we prove that, in the case of signed graphs, (not the same but) similar holds for any eigenvalue and appropriately selected class representative (depending on the choice of the eigenvalue).

Lemma 1. *Let $\mathcal{E}$ denote a class of switching equivalent signed graphs and let λ be an eigenvalue belonging to the common spectrum. Then $\mathcal{E}$ contains a signed graph for which the eigenvector that corresponds to λ may be chosen in such a way that all its non-zero coordinates are of the same sign.*

Proof. It is sufficient to prove that all coordinates of are non-negative. Let $\dot{G} \in \mathcal{E}$, and let $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ be a vector satisfying $A_{\dot{G}}\mathbf{x} = \lambda\mathbf{x}$. If $\mathbf{x}$ is non-negative, then $\dot{G}$ is the desired signed graph. Otherwise, let D stand for the diagonal matrix of ± 1 's with -1 exactly in positions (i, i) for which $x_i < 0$. Observe that $D\mathbf{x}$ is non-negative. In addition, there is a signed graph $H \in \mathcal{E}$ with the adjacency matrix $A_{\dot{H}} = D^{-1}A_{\dot{G}}D$. Since $D^{-1} = D$ it also holds $D^{-1}A_{\dot{H}}D = A_{\dot{G}}$, and so we have the following chain of implications:

$$D^{-1}A_{\dot{H}}D\mathbf{x} = \lambda\mathbf{x} \Rightarrow A_{\dot{H}}D\mathbf{x} = D^{-1}\lambda\mathbf{x} \Rightarrow A_{\dot{H}}D\mathbf{x} = \lambda D\mathbf{x}.$$

Thus, $\dot{H}$ is the desired signed graph with $D\mathbf{x}$ in the role of the corresponding eigenvector. ■

Here is a structural property of a representative.

Lemma 2. *If $\mathcal{E}(G)$ denotes the class of switching equivalent signed graphs whose underlying graph is G , then*

- (i) *for every vertex $u \in V(G)$, $\mathcal{E}(G)$ contains a signed graph in which $\deg(u) = \deg_+(u)$ holds;*
- (ii) *$\mathcal{E}(G)$ contains a signed graph such that $\deg_+(u) \geq \deg_-(u)$ holds for all its vertices.*

Proof. (i) If $uv_1, uv_2, \dots, uv_k$ are the negative edges incident with u (in $\dot{G} \in \mathcal{E}(G)$), then reversing the sign of all edges incident with the vertices $v_1, v_2, \dots, v_k$ gives the desired signed graph.

(ii) The desired signed graph is obtained by selecting arbitrary signed graph $\dot{G} \in \mathcal{E}(G)$ and repeating the following procedure. If there is no a vertex $u \in V(\dot{G})$ for which $\deg_+(u) < \deg_-(u)$, we are done. Otherwise, take any such vertex and reverse the sign of all edges incident. Since this procedure is followed by a strict decreasing in the total number of negative edges, after finite repetition we arrive at the result. ■

3. INDEX

We first prove the following theorem.

Theorem 3. *Let $\dot{G}$ be a signed graph with two non-adjacent vertices u and v , let $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ be an eigenvector associated with its index, and let $\dot{G}_{uv}$ be a signed graph obtained from $\dot{G}$ by adding the edge uv . If $\dot{G}_{uv}$ is connected and at least one of the entries x_u, x_v is non-zero, then for at least one of choices $\sigma(uv) = 1$ or $\sigma(uv) = -1$ we have $\lambda_1(\dot{G}) < \lambda_1(\dot{G}_{uv})$.*

Proof. We may assume that $\|\mathbf{x}\|=1$, and then using the Rayleigh principle, we get

$$(3) \quad \lambda_1(\dot{G}_{uv}) \geq \mathbf{x}^T A_{\dot{G}_{uv}} \mathbf{x} = \lambda_1(\dot{G}) + 2\sigma(uv)x_u x_v.$$

If $x_u x_v > 0$ (respectively $x_u x_v < 0$), then the result follows by choosing positive (respectively negative) signature for uv . If, say $x_u = 0$ (so, $x_v \neq 0$), and equality holds in (3) then $\mathbf{x}$ corresponds to $\lambda_1(\dot{G}_{uv})$, but then the eigenvalue equation (1) cannot hold for u in both signed graphs, and we are done. ■

Observe that the previous theorem covers a special case concerning a disconnected signed graph with an isolated vertex. The next corollary follows immediately.

Corollary 4. *Given a connected signed graph $\dot{G}$, let $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ be an eigenvector associated with its index and $\dot{G}_u$ a signed graph obtained by joining*

an isolated vertex u to the vertices $v_1, v_2, \dots, v_k$ of $\dot{G}$. If at least one of the entries x_{v_i} ($1 \leq i \leq k$) is non-zero, then for appropriately chosen signatures for uv_i ($1 \leq i \leq k$) we have $\lambda_1(\dot{G}) < \lambda_1(\dot{G}_u)$.

Proof. The proof is obtained by consecutive application of the previous theorem to $\dot{G}$ and u forming together one (disconnected) signed graph. ■

Remark 5. The signed graph $\dot{G}$ considered in Theorem 3 and Corollary 4 is switching equivalent to a signed graph for which all signatures mentioned in these statements may be taken to be positive.

We may also consider the multiplicity of the index.

Theorem 6. Let $\dot{G}_u$ be a signed graph obtained by joining an isolated vertex u to a subset of vertices of a signed graph $\dot{G}$. If $\lambda_1(\dot{G}) < \lambda_1(\dot{G}_u)$, then $\lambda_1(\dot{G}_u)$ is a simple eigenvalue.

Proof. The proof is a direct consequence of the Interlacing Theorem. Namely, if $\lambda_1(\dot{G}_u) = \lambda_2(\dot{G}_u)$, then it must hold $\lambda_1(\dot{G}_u) = \lambda_1(\dot{G})$. A contradiction. ■

We denote by $\dot{G} \cdot e$ the signed graph obtained by reversing the sign of the edge e of $\dot{G}$.

Theorem 7. If e is an edge that does not belong to any unbalanced cycle of a connected signed graph $\dot{G}$, then $\lambda_1(\dot{G}) \geq \lambda_1(\dot{G} \cdot e)$.

Proof. We write $\dot{G}'$ for $\dot{G} \cdot e$. If u is a vertex incident with e , then we have

$$\Phi_{\dot{G}}(x) = x\Phi_{\dot{G}-u}(x) - \sum_{v \sim u} \Phi_{\dot{G}-u-v}(x) - 2 \sum_{\dot{C} \in \dot{\mathcal{C}}_u} \sigma(\dot{C})\Phi_{\dot{G}-\dot{C}}(x)$$

and

$$\Phi_{\dot{G}'}(x) = x\Phi_{\dot{G}'-u}(x) - \sum_{v \sim u} \Phi_{\dot{G}'-u-v}(x) - 2 \sum_{\dot{C} \in \dot{\mathcal{C}}_u} \sigma(\dot{C})\Phi_{\dot{G}'-\dot{C}}(x).$$

Let $\dot{\mathcal{C}}_e \subseteq \dot{\mathcal{C}}_u$ be the subset containing only the cycles traversing along e . Using the above formulas, we get

$$(4) \quad \Phi_{\dot{G}'}(x) - \Phi_{\dot{G}}(x) = 4 \sum_{\dot{C} \in \dot{\mathcal{C}}_e} \Phi_{\dot{G}-\dot{C}}(x).$$

By setting $x = \lambda_1(\dot{G})$ and using the Interlacing Theorem, we get $\Phi_{\dot{G}'}(\lambda_1(\dot{G})) \geq 0$, which gives the assertion. ■

Observe that the set $\dot{\mathcal{C}}_e$ from the previous proof may be empty. So, to obtain a strict inequality we need an additional assumption.

Corollary 8. *If the edge e from the previous theorem belongs to at least one balanced cycle, then $\lambda_1(\dot{G}) > \lambda_1(\dot{G} \cdot e)$.*

Proof. Assume that there is a signed graph $\dot{G}$ of a minimal order for which our claim fails to hold. By the previous theorem, we have $\lambda_1(\dot{G}) \geq \lambda_1(\dot{G}')$ (where $\dot{G}'$ stands for $\dot{G} \cdot e$), and so according to our assumption it must be $\lambda_1(\dot{G}) = \lambda_1(\dot{G}')$.

If $\Phi_{\dot{G}-\dot{C}}(\lambda_1(\dot{G})) > 0$ holds for at least one cycle that contains e , we are done (by (4)). So, assume in further that $\Phi_{\dot{G}-\dot{C}}(\lambda_1(\dot{G})) = 0$ holds for all of them. (In other words, $\dot{G}$ and $\dot{G} - \dot{C}$ share the same index.)

If there are at least two balanced cycles containing e , say $\dot{C}'$ and $\dot{C}''$, then there are two vertices u and v such that $u \in V(\dot{C}') \setminus V(\dot{C}'')$ and $v \in V(\dot{C}'') \setminus V(\dot{C}')$. The removal of one of these vertices, say u , results in a signed component $\dot{H}$ that contains e and satisfies $\lambda_1(\dot{G}) = \lambda_1(\dot{H})$. On the contrary, the minimality of $\dot{G}$ and the Interlacing Theorem respectively yield the following inequalities $\lambda_1(\dot{H}) > \lambda_1(\dot{H} \cdot e) \geq \lambda_1(\dot{G} - \dot{C}') = \lambda_1(\dot{G})$. A contradiction.

Let there be exactly one balanced cycle, say $\dot{C}$, containing e . Since e does not belong to any unbalanced cycle, we conclude that no edge of $\dot{C}$ belongs to any other cycle.

Assume first that $\dot{C}$ contains exactly one vertex, say v , of degree at least 3 (in $\dot{G}$) and let v' and v'' be its neighbours in $\dot{C}$. Since $\lambda_1(\dot{G}) = \lambda_1(\dot{G} - \dot{C})$, we have $\lambda_1(\dot{G}) = \lambda_1(\dot{G} - vv') = \lambda_1(\dot{G} - vv'') = \lambda_1(\dot{G} - vv' - vv'')$. On the contrary, since $\dot{G} - vv' - vv''$ consists of two components one of them being a path, we conclude that inserting an edge of any signature between v and v' (in $\dot{G} - vv' - vv''$) strictly increases the index (by Theorem 3, since the coordinate of an eigenvector to $\lambda_1(\dot{G} - vv' - vv'')$ that corresponds to an endvertex of a path is non-zero). A contradiction.

If $\dot{C}$ contains more than one vertex of degree at least 3, then $\dot{G} - \dot{C}$ is disconnected and $\lambda_1(\dot{G} - \dot{C}) = \lambda_1(\dot{G})$, which means that there exists a component, say $\dot{H}$, such that $\lambda_1(\dot{H}) = \lambda_1(\dot{G} - \dot{C})$. Now, by removing all vertices of $V(\dot{G})$ that are outside $V(\dot{H}) \cup V(\dot{C})$, we get a signed graph in which $\dot{C}$ contains only one vertex of degree at least 3, and then application of the previous part of the proof leads to the final contradiction. ■

We say that a cycle in a signed graph is *independent* of a fixed set of cycles if it contains an edge that does not belong to any cycle of that set. According to this terminology, in the previous corollary we dealt with a balanced cycle that is independent of the set of all unbalanced ones.

A signed cactus is a connected signed graph in which any two cycles have at most one common vertex. We proceed with another characterization of these signed graphs.

Lemma 9. *A connected signed graph is a signed cactus if and only if every its cycle is independent of the remaining ones.*

Proof. The first implication follows directly from definition.

Assume that a connected signed graph whose every cycle is independent of the remaining ones is not a signed cactus. Then its two cycles share a (non-empty) set of edges, and therefore each edge belonging to any of them also belongs to another cycle formed by combining the edges of these two, and we are done. ■

Now, Corollary 8 and Lemma 9 provide a partial ordering of signed cacti by their indices.

Corollary 10. *Given a simple cactus G having k cycles enumerated by $1, 2, \dots, k$, let $\dot{G}^{(i)}$ ($1 \leq i \leq k$) be a signed cactus derived from G in which the cycles $1, 2, \dots, i$ are unbalanced and remaining cycles balanced. Then,*

$$\lambda_1(G) > \lambda_1(\dot{G}^{(1)}) > \lambda_1(\dot{G}^{(2)}) > \dots > \lambda_1(\dot{G}^{(k)}).$$

Proof. The proof follows directly from the last two statements. ■

At the end of this section we demonstrate a method for obtaining lower or upper bounds on the index of a signed graph $\dot{G}$. Namely, the adjacency matrix $A_{\dot{G}}$ may be decomposed into the sum

$$A_{\dot{G}} = P + N,$$

where P and N correspond to positive and negative edges of $\dot{G}$, respectively. If we use introduced notation for the eigenvalues, then by applying the Courant-Weyl inequalities [10, Theorem 1.3], we get

$$\lambda_1(P) + \lambda_n(N) \leq \lambda_1(\dot{G}) \leq \lambda_1(P) + \lambda_1(N).$$

Equivalently,

$$\lambda_1(P) - \lambda_1(-N) \leq \lambda_1(\dot{G}) \leq \lambda_1(P) - \lambda_n(-N).$$

Observe that $-N$ is the adjacency matrix of a simple graph which together with that of P is making up the partition of edges of the underlying graph G . If we denote these graphs by G_P and G_{-N} , then a lower bound for $\lambda_1(G_P)$ and an upper bound for $\lambda_1(G_{-N})$ sum up to a lower bound for $\lambda_1(\dot{G})$, and similarly for the second inequality.

In this way we have transferred the problem to the field of simple graphs, and so we may use any of known bounds for the corresponding eigenvalues (more details can be found in [10]). For example, we have

$$(5) \quad \lambda_1(\dot{G}) \geq \sqrt{d_+^2} - \max_{u \in V(\dot{G})} m_-(u),$$

where $\overline{d_+^2}$ stands for the average square of positive vertex degrees and $m_-(u)$ is the the average negative vertex degree in the set of neighbours of u . Namely, using the Hofmeister inequality (cf. [10, p.32]), we get $\lambda_1(G_P) \geq \sqrt{\frac{1}{n} \sum_{u=1}^n \deg(u)^2}$ which gives the first term. The second term is obtained from the well-known upper bound $\lambda_1(G_{-N}) \leq \max_{u \in V(G_{-N})} m(u)$, where $m(u)$ now stands for usual average vertex degree in the neighbourhood of u (cf. [10, p.34]).

Observe that (5) holds for disconnected signed graphs and that this lower bound may be trivial. A decrease in the number of negative edges (obtained by repeating the procedure described in the proof of Lemma 2) results in an increase of this bound. Moreover, the following lemma holds.

Lemma 11. *Equality in (5) holds if $\dot{G}$ is a regular signed graph whose positive and negative edges induce two spanning regular signed subgraphs such that the $n \times 1$ all-1 eigenvector $\mathbf{j}$ corresponds to the largest eigenvalue of $\dot{G}$.*

Proof. The graphs G_P and G_{-N} are regular and $\mathbf{j}$ is an eigenvector corresponding to their indices. Moreover, it is easy to check that $\mathbf{j}$ also corresponds to the eigenvalue $\lambda_1(G_P) - \lambda_1(G_{-N}) = \sqrt{\overline{d_+^2}} - \max_{u \in V(\dot{G})} m_-(u)$ of $\dot{G}$, and the result follows. ■

4. REPRESENTATIVES OF SMALL SWITCHING EQUIVALENT SIGNED GRAPHS

Recall from Section 2 that all switching isomorphic signed graphs share the same spectrum and that each of them can be considered as a representative of the corresponding switching equivalence class. Here we present computational results related to determination of class representatives of comparatively small order. In particular, we provide the theoretical basis for our computational approach and give some numerical data, while the corresponding signed graphs can be found on <http://www.math.rs/~zstancic/siggr.htm>.

We use publicly available library of programs nauty [7] to generate all connected graphs having a given number of vertices. We also use the results of the forthcoming theorems.

For a moment we include isomorphic signed graphs by introducing labelling of their vertices. A labelled (signed) graph is a (signed) graph with each vertex labelled differently and so they are distinguished one from another. Usually, the vertices are labelled by the numbers $1, 2, \dots, n$. Clearly, labelled signed graphs G and H are switching equivalent whenever they are switching equivalent if considered as unlabelled. It can easily be seen that the number of (not necessarily connected) labelled simple graphs with n vertices is $2^{\binom{n}{2}}$. In the following theorem we consider labelled signed graphs that share the same underlying graph.

Recall that a connected (signed) graph is called *k-cyclic* if its order and size satisfy $m = n + k - 1$.

Theorem 12. *If $\mathcal{S}(G)$ denotes the set of all labelled signed graphs that are not pairwise switching equivalent and share given connected *k-cyclic* underlying graph G , then $|\mathcal{S}(G)| = 2^k$.*

Proof. Our proof is based on induction arguments. For $k = 0$, the statement holds trivially. Assume that the statement holds for any connected $(k - 1)$ -cyclic underlying graph.

Let G be a *k-cyclic* connected graph and e its edge contained in at least one cycle. Then, by the induction hypothesis, the set $\mathcal{S}(G - e)$ counts exactly 2^{k-1} labelled signed graphs. By adding the edge e , first with positive and then with negative signature, to each of them we get the required set $\mathcal{S}(G)$. Indeed, by its construction, $\mathcal{S}(G)$ does not contain any pair of switching equivalent signed graphs. In addition, if $\dot{G}$ is a labelled signed graph with G as an underlying graph, then $\dot{G} - e \in \mathcal{S}(G - e)$ and consequently $\dot{G} \in \mathcal{S}(G)$. ■

Determination of these signed graphs is considered in our next result.

Theorem 13. *If T is a spanning tree of a *k-cyclic* graph G and $e_1, e_2, \dots, e_k$ are the edges outside T , then the 2^k labelled signed graphs of $\mathcal{S}(G)$ are obtained by taking all possible combinations for the signature of the edges $e_1, e_2, \dots, e_k$.*

Proof. Clearly, in the described way we obtain 2^k signed graphs. To complete the proof we need to show that no two of them are switching equivalent. Since switching equivalent signed graphs share the same set of balanced cycles, the latter follows from the fact that each of our labelled signed graphs has a unique set of balanced elementary cycles (with respect to T). ■

Now it is easy to do our computer search. For every simple graph generated by nauty we determine a spanning tree, then using the last theorem we determine all signed graphs that are derived from it, and simultaneously eliminate switching isomorphic ones. We denote by $T(n)$ the total number of connected signed graphs that are not switching isomorphic and have n vertices. Here is an overview of the results including signed graphs having at most 8 vertices.

n	3	4	5	6	7	8
$T(n)$	3	12	79	1123	42 065	4 880 753
% of underlying graphs	66.67	50.00	26.58	9.97	2.03	0.23

As expected, the proportion of underlying graphs in the total number decreases dramatically.

On the basis of these results we may consider the cospectrality of signed graphs. No signed graph with at most 4 vertices has a cospectral mate. We give the numerical data on graphs of $T(n)$, where $n \in \{5, 6, 7\}$, that have at least one cospectral mate. The number of such signed graphs is denoted by $C(n)$. Note that disconnected signed graphs are included as possible cospectral mates (they are easily derived by combining connected ones). The results are as follows.

n	5	6	7
$C(n)$	2	131	8219
% in $T(n)$	2.53	11.67	19.54

One may observe that $C(5)$ is equal to 2, contrary to the case of simple graphs where there is a unique connected graph that have a (disconnected) cospectral mate. In addition, the proportion of connected cospectral signed graphs with 6 or 7 vertices is larger than that for simple graphs (for the corresponding data, we refer to [5]).

REFERENCES

- [1] F. Belardo, E.M. Li Marzi and S.K. Simić, *Combinatorial approach for computing the characteristic polynomial of a matrix*, Linear Algebra Appl. **433** (2010) 1513–1523.
doi:10.1016/j.laa.2010.05.010
- [2] F. Belardo and P. Petecki, *Spectral characterizations of signed lollipop graphs*, Linear Algebra Appl. **480** (2015) 144–167.
doi:10.1016/j.laa.2015.04.022
- [3] D. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs — Theory and Application*, 3rd Edition (Johann Ambrosius Barth Verlag, Heidelberg–Leipzig, 1995).
- [4] D. Cvetković, P. Rowlinson and S. Simić, *An Introduction to the Theory of Graph Spectra* (Cambridge University Press, Cambridge, 2010).
- [5] W.H. Haemers and E. Spence, *Enumeration of cospectral graphs*, European J. Combin. **25** (2004) 199–211.
doi:10.1016/S0195-6698(03)00100-8
- [6] T. Koledin and Z. Stanić, *Connected signed graphs of fixed order, size, and number of negative edges with maximal index*, Linear Multilinear Algebra **65** (2017) 2187–2198.
doi:10.1080/03081087.2016.1265480
- [7] B.D. McKay and A. Piperno, *Practical graph isomorphism*, II, J. Symbolic Comput. **60** (2014) 94–112.
doi:10.1016/j.jsc.2013.09.003
- [8] G. Pólya, *Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen*, Acta Math. **68** (1937) 145–254.
doi:10.1007/BF02546665

- [9] S.K. Simić and Z. Stanić, *Polynomial reconstruction of signed graphs*, Linear Algebra Appl. **501** (2016) 390–408.
doi:10.1016/j.laa.2016.03.036
- [10] Z. Stanić, *Inequalities for Graph Eigenvalues* (Cambridge University Press, Cambridge, 2015).

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