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ANOTHER VIEW OF BIPARTITE RAMSEY NUMBERS

Zhenming Bi, Gary Chartrand

and

Ping Zhang

Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008, USA

e-mail: ping.zhang@wmich.edu

Abstract

For bipartite graphs F and H and a positive integer s, the s-bipartite Ramsey number $BR_s(F,H)$ of F and H is the smallest integer t with $t \geq s$ such that every red-blue coloring of $K_{s,t}$ results in a red F or a blue H. We evaluate this number for all positive integers s when $F = K_{2,2}$ and $H \in \{K_{2,3}, K_{3,3}\}$.

Keywords: Ramsey number, bipartite Ramsey number, s-bipartite Ramsey number.

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1. Introduction

In a *red-blue coloring* of a graph G, every edge of G is colored red or blue. For two graphs F and H, the *Ramsey number* $R(F,H)$ of F and H is the smallest positive integer n such that every red-blue coloring of the complete graph K_n of order n results in either a subgraph isomorphic to F all of whose edges are colored red (a *red* F) or a subgraph isomorphic to H all of whose edges are colored blue (a *blue* H). A graph (subgraph) all of whose edges are colored the same is called a *monochromatic graph* (*subgraph*). We refer to the book [4] for graph theory notation and terminology not described in this paper.

In [2] Beineke and Schwenk introduced a bipartite version of Ramsey numbers. For two bipartite graphs F and H, the *bipartite Ramsey number* $BR(F,H)$

of F and H is the smallest positive integer r such that every red-blue coloring of the r-regular complete bipartite graph $K_{r,r}$ results in either a red F or a blue H. Consequently, if $BR(F, H) = r$ for bipartite graphs F and H, then every red-blue coloring of $K_{r,r}$ results in a red F or a blue H, while there exists a red-blue coloring of $K_{r-1,r-1}$ for which there is neither a red F nor a blue H. Beineke and Schwenk [2] showed that $BR(F, H)$ exists for every two bipartite graphs F and H and showed that for the 4-cycle C_4, $BR(C_4, C_4) = 5$.

In [1], red-blue colorings of the intermediate graph $K_{r-1,r}$ were considered, which led to the concept of the 2-Ramsey number, the definition of which is more similar to the Ramsey number. For bipartite graphs F and H, the *2-Ramsey number* $R_2(F, H)$ of F and H is the smallest positive integer n such that every red-blue coloring of the complete bipartite graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ of order n results in a red F or a blue H. Thus, either $R_2(F, H) = 2BR(F, H)$ or $R_2(F, H) = 2BR(F, H) - 1$.

Since the bipartite Ramsey number $BR(C_4, C_4) = 5$, it follows that $R_2(C_4, C_4)$ is either 9 or 10. Because there is a red-blue coloring of $K_{4,5}$ that results in neither a red C_4 nor a blue C_4 (see Figure 1, where each solid edge is colored red and each dashed edge is colored blue), it follows that $R_2(C_4, C_4) \geq 10$ and so $R_2(C_4, C_4) = 10$.

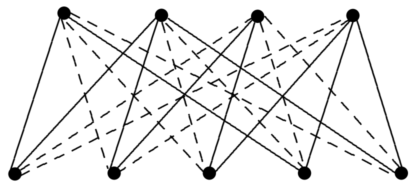

Figure 1. A red-blue coloring of $K_{4,5}$.

We now consider red-blue colorings of complete bipartite graphs when the numbers of vertices in the two partite sets need not differ by at most 1. Let F and H be two bipartite graphs. For a positive integer s, the *s-bipartite Ramsey number* $BR_s(F, H)$ of F and H is the smallest integer t with $t \geq s$ such that every red-blue coloring of $K_{s,t}$ results in a red F or a blue H. In [3], $BR_s(F, H)$ was studied for $F = H = C_4 = K_{2,2}$ and $F = H = K_{3,3}$. The following exact results were obtained.

Theorem 1.1 [3]. *For each integer $s \geq 2$,*

$$BR_s(K_{2,2}, K_{2,2}) = \begin{cases} \text{does not exist} & \text{if } s = 2, \\ 7 & \text{if } s = 3, 4, \\ s & \text{if } s \geq 5. \end{cases}$$

Theorem 1.2 [3]. *For each integer $s \geq 2$,*

$$BR_s(K_{3,3}, K_{3,3}) = \begin{cases} \text{does not exist} & \text{if } s = 2, 3, 4, \\ 41 & \text{if } s = 5, 6, \\ 29 & \text{if } s = 7, 8. \end{cases}$$

Here, we determine $BR_s(F, H)$ for all positive integers s when $F = K_{2,2}$ and $H \in \{K_{2,3}, K_{3,3}\}$, beginning with $H = K_{2,3}$.

2. The s-Bipartite Ramsey Number $BR_s(K_{2,2}, K_{2,3})$

We will see, for results presented in this section, that there is a connection with the concept of Steiner triple systems. For this reason, we briefly discuss this topic here. A *Steiner triple system* of order n is a set S with n elements and a collection T of 3-element subsets of S, called *triples*, such that every two distinct elements of S belong to a unique triple in T. A primary question here is that of determining those integers n for which a Steiner triple system of order n exists. An immediate observation is that there exists a Steiner triple system of order n if and only if K_n is K_3-decomposable. While it is not difficult to see that if there is a Steiner triple system of order n, then $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$, Kirkman [5] verified the converse in 1846, resulting in the following result.

Theorem 2.1. *A Steiner triple system of order $n \geq 3$ exists if and only if $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$.*

For example, there is a Steiner triple system of order 7. For the set $S = \{1, 2, \ldots, 7\}$, one Steiner triple system of order 7 has the following set of triples (where the triple $\{a, b, c\}$ is denoted by abc):

$$T = \{123, 145, 246, 356, 167, 257, 347\}. \tag{1}$$

Consequently, every pair of elements of S belongs to exactly one element of T. While no two triples of T have two elements of S in common, every two triples of T have exactly one element of S in common. To see that this is the case for every Steiner triple system of order 7, suppose that there is a Steiner triple system $S = \{a, b, c, d, e, f, g\}$ of order 7 with a set T of triples containing two disjoint triples, say $\{a, b, c\}$ and $\{d, e, f\}$. That is, there is a K_3-decomposition of K_7 with vertex set S, containing disjoint triangles with vertex sets $\{a, b, c\}$ and $\{d, e, f\}$. The vertex g belongs to three triples, where each triple contains one vertex (element) of $\{a, b, c\}$ and one vertex of $\{d, e, f\}$ (see these five triangles in Figure 2). The element a belongs to one other triple. However, no other pair of elements of S can form a triple with a that results in a Steiner triple system.

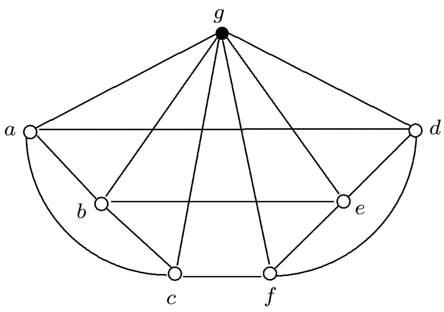

Figure 2. Five triangles in K_7.

Let us return to the Steiner triple system $S = \{1, 2, \ldots, 7\}$ of order 7 and the set T of triples shown in (1). Let G be the bipartite graph with partite sets $U = \{u_1, u_2, \ldots, u_7\}$ and $W = \{w_1, w_2, \ldots, w_7\}$. Denote the triples in (1) by

$$U_1 = 123,\ U_2 = 145,\ U_3 = 246,\ U_4 = 356,\ U_5 = 167,\ U_6 = 257,\ U_7 = 347.$$

Then $u_i w_j$ is an edge of G if $i \in U_j$, producing the 3-regular graph G shown in Figure 3. The graph G produces seven other triples, namely

$$W_1 = 125,\ W_2 = 136,\ W_3 = 147,\ W_4 = 237,\ W_5 = 246,\ W_6 = 345,\ W_7 = 567,$$

where $u_i w_j$ is an edge of G if $j \in W_i$. Thus, $\{W_1, W_2, \ldots, W_7\}$ is a second Steiner triple system for the set $S = \{1, 2, \ldots, 7\}$.

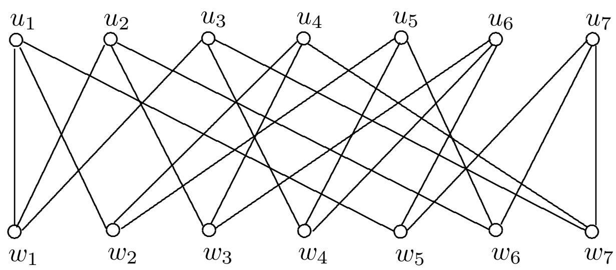

Figure 3. A 3-regular bipartite graph constructed from a Steiner triple system.

We now determine $BR_s(K_{2,2}, K_{2,3})$ for each integer $s \geq 2$, beginning with an observation when $s = 2$.

Proposition 1. *The number* $BR_2(K_{2,2}, K_{2,3})$ *does not exist.*

Proof. For an arbitrary integer $t \geq 2$, the red-blue coloring of $K_{2,t}$, in which both red and blue subgraphs are $K_{1,t}$ produces neither a red $K_{2,2}$ nor a blue $K_{2,3}$. ■

Theorem 2.2. $BR_3(K_{2,2}, K_{2,3}) = 10$.

Proof. First, we show that there exists a red-blue coloring of $K_{3,9}$ that avoids both a red $K_{2,2}$ and a blue $K_{2,3}$. For $G = K_{3,9}$, let $U = \{u_1, u_2, u_3\}$ and $W = \{w_1, w_2, \ldots, w_9\}$ be the partite sets of G. Consider the following three 4-element subsets W_1, W_2, W_3 of W, where $\{w_a, w_b, w_c, w_d\}$ is denoted by $abcd$, and let $\overline{W}_i = W - W_i$ be the 5-element subset of W for $i = 1, 2, 3$.

W_i :	1378	5679	2489
$\overline{W}_i$:	24569	12348	13567

We now define a red-blue coloring of G by joining each vertex u_i $(1 \le i \le 3)$ to the four vertices in W_i by red edges and to the remaining five vertices in $\overline{W}_i$ by blue edges. This coloring is shown in Figure 4, where each solid line indicates a red edge and each dashed line indicates a blue edge. Since $|W_i \cap W_j| = 1$, $|\overline{W}_i \cap \overline{W}_j| = 2$ for $1 \le i \ne j \le 3$ and $|\overline{W}_1 \cap \overline{W}_2 \cap \overline{W}_3| = 0$, there is neither a red $K_{2,2}$ nor a blue $K_{2,3}$ in G. Hence, $BR_3(K_{2,2}, K_{2,3}) \ge 10$.

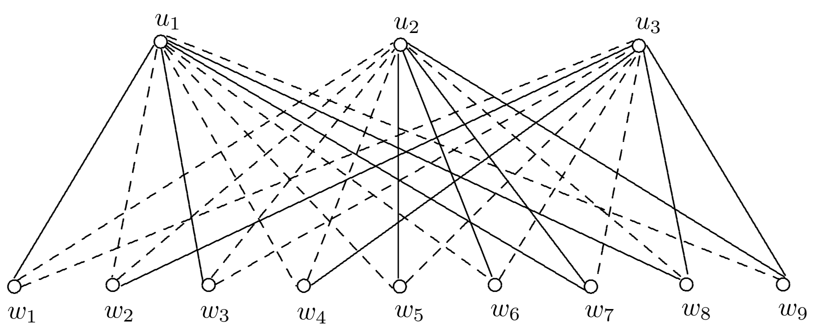

Figure 4. A red-blue coloring of $K_{3,9}$ avoiding both a red $K_{2,2}$ and a blue $K_{2,3}$.

Next, we verify that $BR_3(K_{2,2}, K_{2,3}) \le 10$ by showing that every red-blue coloring of $H = K_{3,10}$ results in a red $K_{2,2}$ or a blue $K_{2,3}$. Let there be given a red-blue coloring of H, where H_R denotes the red subgraph of H and H_B the blue subgraph. Let $U = \{u_1, u_2, u_3\}$ and $W = \{w_1, w_2, \ldots, w_{10}\}$ be the partite sets of H. First, suppose that W has at least four vertices of degree at most 1 in H_B; say $\deg_{H_B} w_i \le 1$ for $1 \le i \le 4$. Thus, $\deg_{H_R} w_i \ge 2$ for $1 \le i \le 4$. Since there are $\binom{3}{2} = 3$ distinct 2-element subsets of U, at least two vertices in $\{w_1, w_2, w_3, w_4\}$ are joined to the same pair of vertices of U by red edges, producing a red $K_{2,2}$. On the other hand, if W has at most three vertices of degree at most 1 in H_B, then W has at least seven vertices of degree 2 or more in H_B. Since there are only three distinct 2-element subsets of U, at least three vertices of W are joined to the same pair of vertices in U by blue edges, producing a blue $K_{2,3}$. In any case, there is either a red $K_{2,2}$ or a blue $K_{2,3}$ in H. Therefore, $BR_3(K_{2,2}, K_{2,3}) = 10$. ∎

Theorem 2.3. *If* $4 \leq s \leq 7$, *then* $BR_s(K_{2,2}, K_{2,3}) = 8$.

Proof. First, we show that there exists a red-blue coloring of $K_{7,7}$ that avoids both a red $K_{2,2}$ and a blue $K_{2,3}$, which then implies that $BR_s(K_{2,2}, K_{2,3}) \geq 8$ for $4 \leq s \leq 7$. Let $U = \{u_1, u_2, \ldots, u_7\}$ and $W = \{w_1, w_2, \ldots, w_7\}$ be the partite sets of $G = K_{7,7}$. Consider the seven 3-element subsets $U_1, U_2, \ldots, U_7$ of U shown below, where $\{u_a, u_b, u_c\}$ is denoted by abc and let $\overline{U}_i = U - U_i$ for $1 \leq i \leq 7$.

U_i :	123	145	246	356	167	257	347
$\overline{U}_i$:	4567	2367	1357	1247	2345	1346	1256

Notice that $U_1, U_2, \ldots, U_7$ are precisely the seven triples in the Steiner triple system described in (1). Thus, $|U_i \cap U_j| = 1$, $|\overline{U}_i \cap \overline{U}_j| = 2$ for $1 \leq i \neq j \leq 7$ and $|\overline{U}_i \cap \overline{U}_j \cap \overline{U}_k| \leq 1$ for $1 \leq i \neq j \neq k \neq i \leq 7$. We now define a red-blue coloring of G where w_i $(1 \leq i \leq 7)$ is joined to the three vertices in U_i by red edges and to the remaining four vertices in $\overline{U}_i$ by blue edges. Figure 3 shows the resulting red subgraph, which is exactly the graph of Figure 3 constructed from a Steiner triple system. By the definition of this red-blue coloring, the red-neighborhood of w_i is $N_R(w_i) = U_i$ and the blue-neighborhood of w_i is $N_B(w_i) = \overline{U}_i$ for $1 \leq i \leq 7$. Furthermore, for each integer j with $1 \leq j \leq 7$, let $W_j = N_R(u_j)$ and $\overline{W}_j = N_B(u_j) = W - W_j$. Then, denoting w_r, w_s, w_t by rst, we have the following:

W_j :	125	136	147	237	246	345	567
$\overline{W}_j$:	3467	2457	2356	1456	1357	1267	1234

As we saw, $W_1, W_2, \ldots, W_7$ are also the triples in a Steiner triple system of order 7. Let G_R and G_B be the resulting red and blue subgraphs of G. Then G_R is 3-regular and G_B is 4-regular. It remains to show that there is neither a red $K_{2,2}$ nor a blue $K_{2,3}$ in G. First, each 2-element subset of W belongs to at most one of $U_1, U_2, \ldots, U_7$ and each 2-element subset of W belongs to at most one of $W_1, W_2, \ldots, W_7$. Hence, there is no red $K_{2,2}$. Next, each 3-element subset of U belongs to at most one of the sets $\overline{U}_i$ for $1 \leq i \leq 7$ and each 3-element subset of W belongs to at most one of the sets $\overline{W}_j$ for $1 \leq j \leq 7$. Hence, there is no blue $K_{2,3}$. Therefore, there is neither a red $K_{2,2}$ nor a blue $K_{2,3}$ in G and so $BR_7(K_{2,2}, K_{2,3}) \geq 8$.

Next, we show that every red-blue coloring of $H = K_{4,8}$ results in a red $K_{2,2}$ or a blue $K_{2,3}$, which then implies that $BR_s(K_{2,2}, K_{2,3}) \leq 8$ for $4 \leq s \leq 7$. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B such that there is no red $K_{2,2}$. We show that there is a blue $K_{2,3}$. Let $U = \{u_1, u_2, u_3, u_4\}$ and $W = \{w_1, w_2, \ldots, w_8\}$ be the partite sets of H.

If there are two vertices of W having degree at most 1 in H_B, say $\deg_{H_B} w_1 \leq 1$ and $\deg_{H_B} w_2 \leq 1$, then $\deg_{H_R} w_1 \geq 3$ and $\deg_{H_R} w_2 \geq 3$. Then w_1 and w_2 are joined to at least two vertices of U by red edges and so there is a red $K_{2,2}$, a contradiction. Thus, at most one vertex of W has degree at most 1 in H_B. We consider two cases.

Case 1. *Exactly one vertex of* W *has degree at most* 1 *in* H_B, *say* $\deg_{H_B} w_1 \leq 1$. First, suppose that $\deg_{H_B} w_1 = 0$. Thus, $\deg_{H_R} w_1 = 4$; that is, w_1 is joined to each vertex in U by a red edge. If there is a vertex w_i, where $2 \leq i \leq 8$, that is joined to two or more vertices of U by red edges, then there is a red $K_{2,2}$, a contradiction. Thus, we may assume that $\deg_{H_R} w_i \leq 1$ for $2 \leq i \leq 8$ and so $\deg_{H_B} w_i \geq 3$. Since there are $\binom{4}{3} = 4$ distinct 3-element subsets of U, there are at least two vertices of W that are joined to the same three vertices in U by blue edges, producing a blue $K_{2,3}$.

Next, suppose that $\deg_{H_B} w_1 = 1$. Thus, $\deg_{H_R} w_1 = 3$, say $N_{H_R}(w_1) = \{u_1, u_2, u_3\}$. If any of $\{u_1, u_2\}$, $\{u_1, u_3\}$ or $\{u_2, u_3\}$ belongs to the red-neighborhood $N_{H_R}(w_i)$ of some vertex w_i where $2 \leq i \leq 8$, then there is a red $K_{2,2}$, a contradiction. Hence, none of $\{u_1, u_2\}$, $\{u_1, u_3\}$ or $\{u_2, u_3\}$ belongs to the red-neighborhood $N_{H_R}(w_i)$ of any vertex w_i where $2 \leq i \leq 8$. Therefore, for each integer i with $2 \leq i \leq 8$, at least one of the sets $\{u_1, u_2\}$, $\{u_1, u_3\}$ or $\{u_2, u_3\}$ belongs to $N_{H_B}(w_i)$. Hence, one of these three sets belongs to $N_{H_B}(w_i)$ for three vertices w_i for $2 \leq i \leq 8$ and so H_B contains a blue $K_{2,3}$.

Case 2. *No vertex of* W *has degree at most* 1 *in* H_B. Thus, $\deg_{H_B} w_i \geq 2$ for $1 \leq i \leq 8$. First, we verify the following:

(2) If $w', w'' \in W$ such that $\deg_{H_B} w' = \deg_{H_B} w'' = 2$, then $N_{H_B}(w') \neq N_{H_B}(w'')$.

If w' and w'' are joined to the same two vertices of U by blue edges, then w' and w'' are joined to the remaining two vertices of U by red edges. This then produces a red $K_{2,2}$, which is impossible. Therefore, (2) holds.

Since there are six distinct 2-element subsets of U, it follows that W has at most six vertices of degree 2 in H_B. Hence, at least two vertices in W have degree 3 or more in H_B. Furthermore, we may assume that no vertex of W have degree 4 in H_B (for otherwise, there is a blue $K_{2,3}$). Hence, every vertex of W has degree 2 or 3 in H_B and least two of them have degree 3. Since there are only four distinct 3-element subsets of U, we may assume that W has at most four vertices of degree at least 3 in H_B. If W has exactly four vertices of degree 3 in H_B, say $\deg_{H_B} w_i = 3$ for $2 \leq i \leq 5$, then we may assume that $N_{H_B}(w_2) = \{u_1, u_2, u_3\}$, $N_{H_B}(w_3) = \{u_1, u_2, u_4\}$, $N_{H_B}(w_4) = \{u_1, u_3, u_4\}$ and $N_{H_B}(w_5) = \{u_2, u_3, u_4\}$ (for otherwise, there is a blue $K_{2,3}$). Since $\deg_{H_B} w_6 = 2$, we may assume that w_6u_1 and w_6u_2 are blue. Then there is a blue $K_{2,3}$ with partite sets $\{u_1, u_2\}$ and $\{w_2, w_3, w_6\}$. Hence, either W has exactly two vertices of degree 3 in H_B or W has exactly three vertices of degree 3 in H_B. We consider these two subcases.

Subcase 2.1. *W has exactly two vertices of degree* 3 *in* H_B. We may assume that $\deg_{H_B} w_1 = \deg_{H_B} w_2 = 3$. Then $\deg_{H_B} w_i = 2$ for $3 \le i \le 8$ and $|N_{H_B}(w_1) \cap N_{H_B}(w_2)| = 2$; for otherwise, if $|N_{H_B}(w_1) \cap N_{H_B}(w_2)| = 3$, then there is a blue $K_{2,3}$. Suppose that $N_{H_B}(w_1) \cap N_{H_B}(w_2) = \{u_1, u_2\}$. If there is some vertex w_i, where $3 \le i \le 8$, such that $N_{H_B}(w_i) = \{u_1, u_2\}$, say w_3, then there is a blue $K_{2,3}$ with partite sets $\{u_1, u_2\}$ and $\{w_1, w_2, w_3\}$. So we may assume that $N_{H_B}(w_i) \neq \{u_1, u_2\}$ for $3 \le i \le 8$. However, since there are only $\binom{4}{2} - 1 = 5$ distinct 2-element subsets of U that are available for the neighborhoods of six vertices w_i for $3 \le i \le 8$, at least two of these vertices of degree 2 have the same neighborhood in H_B, which is a contradiction by (2).

Subcase 2.2. *W has exactly three vertices of degree* 3 *in* H_B. We may assume that $\deg_{H_B} w_i = 3$ for $i = 1, 2, 3$. Furthermore, we may assume that $N_{H_B}(w_1) = \{u_1, u_2, u_3\}$, $N_{H_B}(w_2) = \{u_1, u_2, u_4\}$ and $N_{H_B}(w_3) = \{u_1, u_3, u_4\}$. Then for each integer i with $4 \le i \le 8$, the blue-neighborhood $N_{H_B}(w_i)$ of w_i cannot be any of $\{u_1, u_2\}$, $\{u_1, u_3\}$ and $\{u_1, u_4\}$ (for otherwise, there is a blue $K_{2,3}$). Since there are $\binom{4}{2} - 3 = 3$ distinct 2-element subsets of U that are available as the blue-neighborhood of the five vertices w_i for $4 \le i \le 8$, it follows that at least two of these vertices of degree 2 have the same neighborhood in H_B, which is a contradiction by (2). ■

By Theorem 2.3, $BR(K_{2,2}, K_{2,3}) = BR_7(K_{2,2}, K_{2,3}) = 8$ and so we have the following result.

Theorem 2.4. *For each integer $s \ge 2$,*

$$BR_s(K_{2,2}, K_{2,3}) = \begin{cases} \text{does not exist} & \text{if } s = 2, \\ 10 & \text{if } s = 3, \\ 8 & \text{if } 4 \le s \le 7, \\ s & \text{if } s \ge 8. \end{cases}$$

3. The s-Bipartite Ramsey Number $BR_s(K_{2,2}, K_{3,3})$

We now determine $BR_s(K_{2,2}, K_{3,3})$ for each integer $s \ge 2$, beginning with an observation for $s = 2, 3$.

Proposition 2. *The number $BR_s(K_{2,2}, K_{3,3})$ does not exist for $s = 2, 3$.*

Proof. For an arbitrarily integer $t \ge s$, the red-blue coloring of $K_{3,t}$ in which the red subgraph is $K_{1,t}$ and the blue subgraph is $K_{2,t}$ produces neither a red $K_{2,2}$ nor a blue $K_{3,3}$. ■

Theorem 3.1. $BR_4(K_{2,2}, K_{3,3}) = 15$.

Proof. First, we show that there exists a red-blue coloring of $K_{4,14}$ that avoids both a red $K_{2,2}$ and a blue $K_{3,3}$. For $G = K_{4,14}$, let $U = \{u_1, u_2, u_3, u_4\}$ and $W = \{w_1, w_2, \ldots, w_{14}\}$ be the partite sets of G. Consider the following subsets $U_1, U_2, \ldots, U_{14}$ of U, where $\{u_a, u_b, \ldots\}$ is denoted by $ab\cdots$, and let $\overline{U}_i = U - U_i$ for $1 \le i \le 14$.

U_i :	1	1	2	2	3	3	4	4	12	13	14	23	24	34
$\overline{U}_i$:	234	234	134	134	124	124	123	123	34	24	23	14	13	12

We now define a red-blue coloring of G by joining each vertex w_i $(1 \le i \le 14)$ to the vertices in U_i by red edges and to the remaining vertices in $\overline{U}_i$ by blue edges. The resulting red subgraph of this red-blue coloring is shown in Figure 5. Since $|U_i \cap U_j| \le 1$ and $|\overline{U}_i \cap \overline{U}_j \cap \overline{U}_k| \le 2$ for $1 \le i \ne j \ne k \le 14$ and $i \ne k$, there is neither a red $K_{2,2}$ nor a blue $K_{3,3}$ in G and so $BR_4(K_{2,2}, K_{3,3}) \ge 15$.

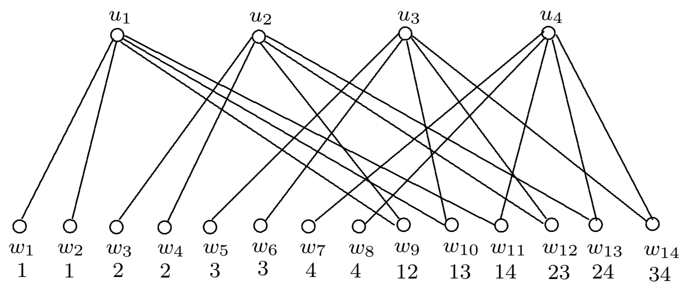

Figure 5. The red subgraph in a red-blue coloring of $K_{4,14}$.

Next, we show that $BR_4(K_{2,2}, K_{3,3}) \le 15$. That is, we show that every red-blue coloring of $H = K_{4,15}$ results in a red $K_{2,2}$ or a blue $K_{3,3}$. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. Suppose that there is no red $K_{2,2}$. We show that there is a blue $K_{3,3}$. Let $U = \{u_1, u_2, u_3, u_4\}$ and $W = \{w_1, w_2, \ldots, w_{15}\}$ be the partite sets of H. First, we claim that W contains at least 9 vertices of degree at least 3 in H_B. This is certainly true if the maximum degree of vertices of W in H_R is at most 1. Thus, we may assume that the maximum degree of vertices of W in H_R is 4, 3 or 2. We consider these three cases.

Case 1. *The maximum degree of vertices of* W *in* H_R *is* 4. Since there is no red $K_{2,2}$, it follows that W contains exactly one vertex of degree 4 in H_R, say $\deg_{H_R} w_1 = 4$. This implies that $\deg_{H_R} w_i \le 1$ for each integer $2 \le i \le 15$. Consequently, $\deg_{H_B} w_i \ge 3$ for $2 \le i \le 15$ and so W contains 14 vertices of degree at least 3 in H_B.

Case 2. *The maximum degree of vertices of* W *in* H_R *is* 3. Since there is no red $K_{2,2}$, it follows that W contains exactly one vertex of degree 3, say $\deg_{H_R} w_1 = 3$ and $N_{H_R}(w_1) = \{u_1, u_2, u_3\}$. If $\deg_{H_R} w_i = 2$ for some integer i with $2 \le i \le 15$, then $N_{H_R}(w_i)$ cannot be a subset of $N_{H_R}(w_1)$, namely $N_{H_R}(w_i)$ cannot be any of the three sets $\{u_1, u_2\}, \{u_1, u_3\}, \{u_2, u_3\}$; for otherwise, there is a red $K_{2,2}$. Since there are only $\binom{4}{2} - 3 = 3$ available 2-element subsets of U for $N_{H_R}(w_i)$ for $2 \le i \le 15$, it follows that W has at most 3 vertices of degree 2 in H_R. Consequently, W has at least 11 vertices of degree at most 1 in H_R and so W has at least 11 vertices of degree at least 3 in H_B.

Case 3. *The maximum degree of vertices of* W *in* H_R *is* 2. Since there is no red $K_{2,2}$ and there are only six distinct 2-element subsets of U, it follows that W contains at most 6 vertices of degree 2 in H_R. Consequently, W contains at least 9 vertices of degree at most 1 in H_R and so W contains at least 9 vertices of degree at least 3 in H_B.

In each case, W contains at least 9 vertices of degree at least 3 in H_B, as claimed. Since there are only 4 distinct 3-element subsets of U, at least three vertices of W are joined to the same three vertices in U by blue edges, producing a blue $K_{3,3}$. Therefore, $BR_4(K_{2,2}, K_{3,3}) \le 15$ and so $BR_4(K_{2,2}, K_{3,3}) = 15$. ■

Theorem 3.2. $BR_5(K_{2,2}, K_{3,3}) = BR_6(K_{2,2}, K_{3,3}) = 12$.

Proof. It suffices to show (1) there exists a red-blue coloring of $K_{6,11}$ that avoids both a red $K_{2,2}$ and a blue $K_{3,3}$ and (2) every red-blue coloring of $K_{5,12}$ results in a red $K_{2,2}$ or a blue $K_{3,3}$. We begin with (1). For $G = K_{6,11}$, let $U = \{u_1, u_2, \ldots, u_6\}$ and $W = \{w_1, w_2, \ldots, w_{11}\}$ be the partite sets of G. Consider the following subsets $U_1, U_2, \ldots, U_{11}$ of U, where $\{u_a, u_b, \ldots\}$ is denoted by $ab\cdots$, and let $\overline{U}_i = U - U_i$ for $1 \le i \le 11$.

U_i :	123	14	15	16	24	25	26	34	35	36	456
$\overline{U}_i$:	456	2356	2346	2345	1356	1346	1345	1256	1246	1245	123

We now define a red-blue coloring of G by joining each vertex w_i ($1 \le i \le 11$) to the vertices in U_i by red edges and to the remaining vertices in $\overline{U}_i$ by blue edges. The resulting red subgraph of this red-blue coloring is shown in Figure 6. Since $|U_i \cap U_j| \le 1$ and $|\overline{U}_i \cap \overline{U}_j \cap \overline{U}_k| \le 2$ for $1 \le i \ne j \ne k \le 11$ and $i \ne k$, there is neither a red $K_{2,2}$ nor a blue $K_{3,3}$ in G and so $BR_6(K_{2,2}, K_{3,3}) \ge 12$. This also implies that $BR_5(K_{2,2}, K_{3,3}) \ge 12$.

Next, we show (2), that is, we show that every red-blue coloring of $H = K_{5,12}$ results in a red $K_{2,2}$ or a blue $K_{3,3}$. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. Suppose that there is no red $K_{2,2}$. We show that there is a blue $K_{3,3}$. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ and $W = \{w_1, w_2, \ldots, w_{12}\}$ be the partite sets of H. First, we verify two claims:

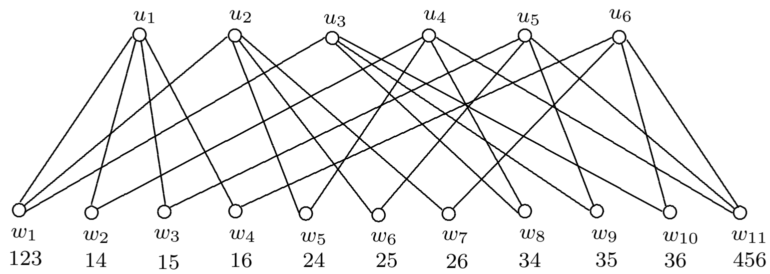

Figure 6. The red subgraph in a red-blue coloring of $K_{6,11}$.

Claim 1. *If W contains at least six vertices of degree at least 4 in H_B, then H contains a blue $K_{3,3}$.*

To show Claim 1, we may assume that $\deg_{H_B} w_i \geq 4$ for $1 \leq i \leq 6$. Since there are $\binom{5}{4} = 5$ distinct 4-element subsets of U, at least two vertices in $\{w_1, w_2, \ldots, w_6\}$ are joined to the same four vertices in U by blue edges, say $\{u_1, u_2, u_3, u_4\} \subseteq N_{H_B}(w_1) \cap N_{H_B}(w_2)$. Since $\deg_{H_B} w_3 \geq 4$, it follows that w_3 is joined to at least three vertices of $\{u_1, u_2, u_3, u_4\}$ by blue edges, producing a blue $K_{3,3}$. Thus, Claim 1 holds.

Claim 2. *No two vertices of W having degree 3 in H_B can be joined to the same three vertices of U by blue edges.*

If Claim 2 is false, then there are two vertices of W having degree 3 in H_B that are joined to the same three vertices of U by blue edges. However then, these two vertices are joined to the two remaining vertices of U by red edges, producing a red $K_{2,2}$, a contradiction. Thus, Claim 2 holds.

By Claim 1, if the maximum degree of vertices of W in H_R is at most 1, then H contains a blue $K_{3,3}$. Hence, we may assume that the maximum degree of vertices of W in H_R is 5, 4, 3 or 2. We consider these four cases.

Case 1. *The maximum degree of vertices of W in H_R is* 5. Since there is no red $K_{2,2}$, it follows that W contains exactly one vertex of degree 5 in H_R, say $\deg_{H_R} w_1 = 5$. Thus, $\deg_{H_R} w_i \leq 1$ for each integer $2 \leq i \leq 12$ and so $\deg_{H_B} w_i \geq 4$ for $2 \leq i \leq 12$. Therefore, W contains at least 11 vertices of degree at least 4 in H_B and so H contains a blue $K_{3,3}$ by Claim 1.

Case 2. *The maximum degree of vertices of W in H_R is* 4. Since there is no red $K_{2,2}$, it follows that W contains exactly one vertex of degree 4 in H_R, say $\deg_{H_R} w_1 = 4$ and $N_{H_R}(w_1) = \{u_1, u_2, u_3, u_4\}$, and W contains no vertex of degree 3 in H_R. If $\deg_{H_R} w_i = 2$ for some integer i with $2 \leq i \leq 12$, then $N_{H_R}(w_i)$ is one of $\{u_1, u_5\}, \{u_2, u_5\}, \{u_3, u_5\}$ and $\{u_4, u_5\}$; for otherwise, there is a red $K_{2,2}$. Hence, W has at most four vertices of degree 2 in H_R and so W has at least seven vertices of degree at most 1 in H_R. This implies that W has at least seven vertices of degree at least 4 in H_B and so H contains a blue $K_{3,3}$ by Claim 1.

Case 3. *The maximum degree of vertices of* W *in* H_R *is* 3. Since there is no red $K_{2,2}$, it follows that W contains at most two vertices of degree 3 in H_R. If W contains two vertices of degree 3, say $\deg_{H_R} w_1 = \deg_{H_R} w_2 = 3$, then $|N_{H_R}(w_1) \cap N_{H_R}(w_2)| = 1$. We may assume that $N_{H_R}(w_1) = \{u_1, u_2, u_3\}$ and $N_{H_R}(w_2) = \{u_3, u_4, u_5\}$. If $\deg_{H_R} w_i = 2$ for some integer i with $3 \le i \le 12$, then $N_{H_R}(w_i)$ is one of $\{u_1, u_4\}, \{u_1, u_5\}, \{u_2, u_4\}$ and $\{u_2, u_5\}$; for otherwise, there is a red $K_{2,2}$. Hence, W has at most four vertices of degree 2 in H_R and so W has at least six vertices of degree at most 1 in H_R. This implies that W contains at least six vertices of degree at least 4 in H_B. Thus, H contains a blue $K_{3,3}$ by Claim 1. So, we may assume that W contains exactly one vertex of degree 3 in H_R, say $\deg_{H_R} w_1 = 3$ and $N_{H_R}(w_1) = \{u_1, u_2, u_3\}$. Then $\deg_{H_R} w_i \le 2$ for $2 \le i \le 12$. If $\deg_{H_R} w_j = 2$ for some $2 \le j \le 12$, then, since there is no red $K_{2,2}$, none of $\{u_1, u_2\}$, $\{u_1, u_3\}$ and $\{u_2, u_3\}$ belongs to $N_{H_R}(w_j)$. So there are at most $\binom{5}{2} - 3 = 7$ vertices of degree 2 in H_R. Thus, W contains at least four vertices of degree at most 1 in H_R. This implies that W contains at least four vertices of degree at least 4 in H_B. We may assume that there is no vertex of degree 5 in H_B and any two vertices of degree 4 in H_B have different neighborhoods (for otherwise, there is a blue $K_{3,3}$). By Claim 1, we may further assume that W contains at most five vertices of degree 4 in H_B. Thus, the number of vertices of W having degree 4 in H_B is 4 or 5. We consider these two subcases.

Subcase 3.1. *W contains exactly four vertices of degree* 4 *in* H_B. We may assume that $\deg_{H_B} w_i = 4$ for $2 \le i \le 5$ and $N_{H_B}(w_2) = \{u_1, u_2, u_3, u_4\}$, $N_{H_B}(w_3) = \{u_1, u_2, u_3, u_5\}$, $N_{H_B}(w_4) = \{u_1, u_2, u_4, u_5\}$ and $N_{H_B}(w_5) = \{u_1, u_3, u_4, u_5\}$. Then $\deg_{H_B} w_i = 3$ for $6 \le i \le 12$. Then we may assume that none of $\{u_1, u_2, u_3\}$, $\{u_1, u_2, u_4\}$, $\{u_1, u_3, u_4\}$, $\{u_1, u_2, u_5\}$, $\{u_1, u_3, u_5\}$ and $\{u_1, u_4, u_5\}$ belongs to $N_{H_B}(w_i)$ for $6 \le i \le 12$; for otherwise, there is a blue $K_{3,3}$. So there are $\binom{5}{3} - 6 = 4$ distinct 3-element subsets that are available for $N_{H_B}(w_i)$ where $6 \le i \le 12$. However then, at least two vertices in $\{w_6, w_7, \ldots, w_{12}\}$ are joined to the same three vertices of U in H_B, which contradicts Claim 2.

Subcase 3.2. *W contains exactly five vertices of degree* 4 *in* H_B. We may assume that $\deg_{H_B} w_i = 4$ for $2 \le i \le 6$ and $N_{H_B}(w_2) = \{u_1, u_2, u_3, u_4\}$, $N_{H_B}(w_3) = \{u_1, u_2, u_3, u_5\}$, $N_{H_B}(w_4) = \{u_1, u_2, u_4, u_5\}$, $N_{H_B}(w_5) = \{u_1, u_3, u_4, u_5\}$ and $N_{H_B}(w_6) = \{u_2, u_3, u_4, u_5\}$. Then $\deg_{H_B} w_i = 3$ for $7 \le i \le 12$. Each triple of U belongs to exactly two of $N_{H_B}(w_i)$ for $2 \le i \le 6$. Since $\deg_{H_B} w_7 = 3$, we may assume that $N_{H_B}(w_7) = \{u_1, u_2, u_3\}$. Then there is a blue $K_{3,3}$ with partite sets $\{u_1, u_2, u_3\}$ and $\{w_2, w_3, w_7\}$.

Case 4. *The maximum degree of vertices of* W *in* H_R *is* 2. Then $\deg_{H_B} w_i \geq 3$ for each integer i with $1 \leq i \leq 12$. Since there are $\binom{5}{2} = 10$ distinct 2-element subsets of U, there are at most ten vertices of W of degree 2 in H_R, or equivalently, there are at most ten vertices of W of degree 3 in H_B. This implies that there are at least two vertices of W of degree at least 4 in H_B, say $\deg_{H_B} w_1 \geq 4$ and $\deg_{H_B} w_2 \geq 4$. Hence, $|N_{H_B}(w_1) \cap N_{H_B}(w_2)| \geq 3$. There are two subcases.

Subcase 4.1. $|N_{H_B}(w_1) \cap N_{H_B}(w_2)| \geq 4$. We may assume, without loss of generality, that $\{u_1, u_2, u_3, u_4\} \subseteq N_{H_B}(w_1) \cap N_{H_B}(w_2)$. If there is a vertex w_j where $3 \leq j \leq 12$ such that w_j is joined to three vertices in $\{u_1, u_2, u_3, u_4\}$ by blue edges, then H contains a blue $K_{3,3}$. Hence, $\deg_{H_B} w_j = 3$ for $3 \leq j \leq 12$. Furthermore, no 3-element subset of $\{u_1, u_2, u_3, u_4\}$ can be $N_{H_B}(w_j)$ for each j with $3 \leq j \leq 12$. Hence, there are at most $\binom{5}{3} - \binom{4}{3} = 6$ distinct 3-element subsets of U that are available for $N_{H_B}(w_j)$ for all j with $3 \leq j \leq 12$. Since there are exactly ten vertices of degree 3 in H_B, at least two of these vertices are joined to the same three vertices in U by blue edges, which is impossible by Claim 2.

Subcase 4.2. $|N_{H_B}(w_1) \cap N_{H_B}(w_2)| = 3$. We may assume that

$$N_{H_B}(w_1) = \{u_1, u_2, u_3, u_4\} \text{ and } N_{H_B}(w_2) = \{u_2, u_3, u_4, u_5\}. \tag{3}$$

We claim that W contains at least three vertices of degree 4 in H_B. If this were not the case, then $\deg_{H_B} w_j = 3$ for $3 \leq j \leq 12$. If there is a vertex w_j where $3 \leq j \leq 12$ that is joined to each vertex of $\{u_2, u_3, u_4\}$ by a blue edge, then there is a blue $K_{3,3}$. Hence, there are only $\binom{5}{3} - 1 = 9$ distinct 3-element subsets of U that are available for $N_{H_B}(w_j)$ for $3 \leq j \leq 12$. This implies that at least two vertices of W having degree 3 in H_B are joined to the same three vertices of U by blue edges, which is impossible by Claim 2. Therefore, there are at least three vertices of W having degree 4 in H_B. By Claim 1, we may assume that W contains at most five vertices of degree 4 in H_B. Thus, the number of vertices of W having degree 4 in H_B is 5, 4 or 3.

- If W contains exactly 5 vertices of degree 4 in H_B, then by (3) we may assume that $N_{H_B}(w_3) = \{u_1, u_2, u_3, u_5\}$, $N_{H_B}(w_4) = \{u_1, u_2, u_4, u_5\}$ and $N_{H_B}(w_5) = \{u_1, u_3, u_4, u_5\}$. Since $\deg_{H_B} w_6 = 3$, we may assume that w_6u_1, w_6u_2 and w_6u_3 are blue. Then there is a blue $K_{3,3}$ with partite sets $\{u_1, u_2, u_3\}$ and $\{w_1, w_3, w_6\}$.

- If W contains exactly 4 vertices of degree 4 in H_B, then by (3) we may assume that $N_{H_B}(w_3) = \{u_1, u_2, u_3, u_5\}$ and $N_{H_B}(w_4) = \{u_1, u_2, u_4, u_5\}$. Since $\{u_1, u_3, u_4\}$, $\{u_1, u_3, u_5\}$, $\{u_1, u_4, u_5\}$ and $\{u_3, u_4, u_5\}$ are the only 3-element subsets of U that belong to at most one of $N_{H_B}(w_i)$ for $1 \leq i \leq 5$, we may assume that $N_{H_B}(w_i)$ is one of $\{u_1, u_3, u_4\}$, $\{u_1, u_3, u_5\}$, $\{u_1, u_4, u_5\}$ and $\{u_3, u_4, u_5\}$ for each integer j with $5 \leq i \leq 12$; for otherwise, there is a blue $K_{3,3}$. Since W contains exactly eight vertices of degree 3 in H_B, at least two of these vertices are joined to the same three vertices of U by blue edges, which is impossible by Claim 2.

• If W contains exactly 3 vertices of degree 4 in H_B, then by (3) we may assume that $N_{H_B}(w_3) = \{u_1, u_2, u_3, u_5\}$. Since each of $\{u_1, u_2, u_3\}$, $\{u_2, u_3, u_4\}$ and $\{u_2, u_3, u_5\}$ belongs to exactly two of $N_{H_B}(w_i)$ for $1 \le i \le 3$, we may assume that $N_{H_B}(w_j)$ is not any of $\{u_1, u_2, u_3\}$, $\{u_2, u_3, u_4\}$, $\{u_2, u_3, u_5\}$ for each integer j with $4 \le j \le 12$; for otherwise, there is a blue $K_{3,3}$. Hence, there are only $\binom{5}{3} - 3 = 7$ distinct 3-element subsets of U for $N_{H_B}(w_j)$ for $4 \le j \le 12$ and so at least two vertices of W having degree 3 in H_B are joined to the same three vertices of U by blue edges, which is impossible by Claim 2.

Thus, every red-blue coloring of $K_{5,12}$ results in a red $K_{2,2}$ or a blue $K_{3,3}$. Hence, $BR_5(K_{2,2}, K_{3,3}) \le 12$ and so $BR_5(K_{2,2}, K_{3,3}) = 12$. This also implies that every red-blue coloring of $K_{6,12}$ results in a red $K_{2,2}$ or a blue $K_{3,3}$. Therefore, $BR_6(K_{2,2}, K_{3,3}) \ge 12$ and so $BR_6(K_{2,2}, K_{3,3}) = 12$. ■

Theorem 3.3. $BR_7(K_{2,2}, K_{3,3}) = BR_8(K_{2,2}, K_{3,3}) = 9$.

Proof. It suffices to show (1) there exists a red-blue coloring of $K_{8,8}$ that avoids both a red $K_{2,2}$ and a blue $K_{3,3}$ and (2) every red-blue coloring of $K_{7,9}$ results in a red $K_{2,2}$ or a blue $K_{3,3}$. We begin with (1). Although one could use the known fact that $BR(K_{2,2}, K_{3,3}) = 9$ to show that $BR_8(K_{2,2}, K_{3,3}) \ge 9$, we provide an independent proof here for completion. For $G = K_{8,8}$, let $U = \{u_1, u_2, \ldots, u_8\}$ and $W = \{w_1, w_2, \ldots, w_8\}$ be the partite sets of G. Consider the following subsets $U_1, U_2, \ldots, U_8$ of U, where $\{u_a, u_b, \ldots\}$ is denoted by $ab\cdots$, and let $\overline{U}_i = U - U_i$ for $1 \le i \le 8$.

U_i :	123	145	1678	246	257	356	347	38
$\overline{U}_i$:	45678	23678	2345	13578	13468	12478	12568	124567

We now define a red-blue coloring of G by joining each vertex w_i $(1 \le i \le 8)$ to the vertices in U_i by red edges and to the remaining vertices in $\overline{U}_i$ by blue edges. The resulting red subgraph of this red-blue coloring is shown in Figure 7. Since $|U_i \cap U_j| \le 1$ and $|\overline{U}_i \cap \overline{U}_j \cap \overline{U}_k| \le 2$ for $1 \le i \ne j \ne k \le 8$ and $i \ne k$, there is neither a red $K_{2,2}$ nor a blue $K_{3,3}$ in G and so $BR_8(K_{2,2}, K_{3,3}) \ge 9$. This also implies that there exists a red-blue coloring of $K_{7,8}$ that avoids both a red $K_{2,2}$ and a blue $K_{3,3}$ and so $BR_7(K_{2,2}, K_{3,3}) \ge 9$.

Next, we show (2); that is, we show that every red-blue coloring of $H = K_{7,9}$ results in a red $K_{2,2}$ or a blue $K_{3,3}$. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. Suppose that there is no red $K_{2,2}$. We show that there is a blue $K_{3,3}$. Let $U = \{u_1, u_2, \ldots, u_7\}$ and $W = \{w_1, w_2, \ldots, w_9\}$ be the partite sets of H. First, we verify the following two claims.

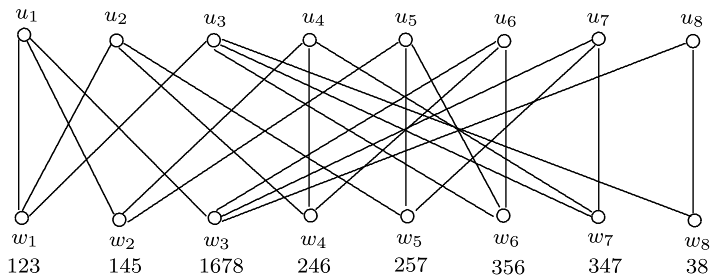

Figure 7. The red subgraph in a red-blue coloring of $K_{8,8}$.

Claim 1. *If W contains three vertices such that the union of their neighborhoods in H_R consists of at most four vertices, then H contains a blue $K_{3,3}$.*

To show Claim 1, we may assume that $|N_{H_R}(w_1) \cup N_{H_R}(w_2) \cup N_{H_R}(w_3)| \leq 4$. Since $|U| = 7$, there are at least three vertices of $|U|$, say u_1, u_2 and u_3, that are not joined to w_1, w_2 and w_3 by red edges. It follows that u_1, u_2 and u_3 are joined to w_1, w_2 and w_3 by blue edges, producing a blue $K_{3,3}$ with partite sets $\{u_1, u_2, u_3\}$ and $\{w_1, w_2, w_3\}$. Thus, Claim 1 holds.

Claim 2. *If W contains at least five vertices of degree at most 2 in H_R, then H contains a blue $K_{3,3}$.*

To show Claim 2, we may assume that $\deg_{H_R} w_i \leq 2$ for $1 \leq i \leq 5$. By Claim 1, we may assume that W contains no vertex of degree 0 in H_R and contains at most one vertex of degree 1 in H_R. First, suppose that W contains exactly one vertex of degree 1 in H_R, say $\deg_{H_R} w_1 = 1$ and $N_{H_R}(w_1) = \{u_1\}$. Thus, $\deg_{H_R} w_i = 2$ for $2 \leq i \leq 5$. By Claim 1 then, $u_1 \notin N_{H_R}(w_i)$ and $N_{H_R}(w_j) \cap N_{H_R}(w_k) = \emptyset$ for $2 \leq i, j, k \leq 5$ and $j \neq k$. So, there are only three possibilities for $N_{H_R}(w_i)$ where $2 \leq i \leq 5$. Hence, at least two vertices of $\{w_2, w_3, w_4, w_5\}$ are joined to the same two vertices by red edges, producing a red $K_{2,2}$, which is a contradiction. Next, suppose that W contains no vertex of degree 1 in H_R. Hence, $\deg_{H_R} w_i = 2$ for $1 \leq i \leq 5$. Then at least two vertices in $\{w_1, w_2, w_3, w_4, w_5\}$ are joined to the same vertex by red edges, say $N_{H_R}(w_1) = \{u_1, u_2\}$ and $N_{H_R}(w_2) = \{u_2, u_3\}$. By Claim 1, $N_{H_R}(w_i) \subseteq \{u_4, u_5, u_6, u_7\}$ for $3 \leq i \leq 5$, producing a blue $K_{3,3}$. Therefore, Claim 2 holds.

By Claim 2, if the maximum degree of vertices of W in H_R is at most 2, then H contains a blue $K_{3,3}$. If the maximum degree of vertices of W in H_R is at least 6, say $\deg_{H_R} w_1 \geq 6$, then, since there is no red $K_{2,2}$, it follows that $\deg_{H_R} w_i \leq 2$ for $2 \leq i \leq 9$. Again, by Claim 2, there is a blue $K_{3,3}$. Hence, we may assume that the maximum degree of vertices of W in H_R is 5, 4 or 3. We consider these three cases.

Case 1. *The maximum degree of vertices of W in H_R is* 5. Since there is no red $K_{2,2}$, it follows that W contains at most one vertex of degree 3, no vertex of degree 4, and exactly one vertex of degree 5 in H_R. Then W contains at least seven vertices of degree at most 2 in H_R. Thus, H contains a blue $K_{3,3}$ by Claim 2.

Case 2. *The maximum degree of vertices of W in H_R is* 4. Since there is no red $K_{2,2}$, it follows that W contains at most two vertices of degree 4 in H_R. First suppose that W contains exactly two vertices of degree 4 in H_R, say $\deg_{H_R} w_1 = \deg_{H_R} w_2 = 4$. Since there is no red $K_{2,2}$, we may assume that $N_{H_R}(w_1) = \{u_1, u_2, u_3, u_4\}$ and $N_{H_R}(w_2) = \{u_4, u_5, u_6, u_7\}$ and so $\deg_{H_R} w_i \leq 2$ for $3 \leq i \leq 9$. Thus, W contains seven vertices of degree at most 2 in H_R. By Claim 2, H contains a blue $K_{3,3}$. Next, suppose that W contains exactly one vertex of degree 4 in H_R, say $\deg_{H_R} w_1 = 4$ and $N_{H_R}(w_1) = \{u_1, u_2, u_3, u_4\}$. Again, since there is no red $K_{2,2}$, any vertex of degree 3 in H_R must be joined to at least two vertices of $\{u_5, u_6, u_7\}$ by red edges. If there is a vertex of degree 3 in H_R that is joined to u_5, u_6 and u_7 by red edges, then, since there is no red $K_{2,2}$, it follows that W contains at most one vertex of degree 3. If W contains no vertex of degree 3 in H_R that are joined to u_5, u_6 and u_7 by red edges, then any vertex of degree 3 in H_R must be joined to exactly two vertices of $\{u_5, u_6, u_7\}$ by red edges. In this case, W has at most three vertices of degree 3 in H_R. This implies that W has at least five vertices of degree at most 2 in H_R. Thus, H contains a blue $K_{3,3}$ by Claim 2.

Case 3. *The maximum degree of vertices of W in H_R is* 3. If W contains two vertices of degree 1 in H_R, say $\deg_{H_R} w_1 = \deg_{H_R} w_2 = 1$, then, by Claim 1, W contains no vertex of degree 2 in H_R and $N_{H_R}(w_1) \neq N_{H_R}(w_2)$. This implies that $\deg_{H_R} w_i = 3$ for $3 \leq i \leq 9$. We may assume that $N_{H_R}(w_1) = \{u_1\}$ and $N_{H_R}(w_2) = \{u_2\}$. Then, by Claim 1, $\{u_1, u_2\} \not\subseteq N_{H_R}(w_i)$ for each i with $3 \leq i \leq 9$. Since there is no Steiner triple system of order 5, it follows that W has at least two vertices of degree 3 in H_R that are joined to the same two vertices of U by red edges, producing a red $K_{2,2}$, which is a contradiction. So W contains at most one vertex of degree 1 in H_R.

Since there is a Steiner triple system of order 7, it follows that W contains at most seven vertices of degree 3 in H_R. If W has exactly seven vertices of degree 3 in H_R, say $\deg_{H_R} w_i = 3$ for $1 \leq i \leq 7$, then any two vertices of U are joined to exactly one of these seven vertices by red edges. Since there is no red $K_{2,2}$, it follows that W contains no vertex of degree 2 in H_R. This implies that $\deg_{H_R} w_8 = \deg_{H_R} w_9 = 1$, which is a contradiction. Thus, W contains at most six vertices of degree 3 in H_R. By Claim 2, W contains at least five vertices of degree 3 in H_R. Hence, we consider two subcases, according to whether W has five vertices of degree 3 or six vertices of degree 3 in H_R.

Subcase 3.1. *W contains exactly five vertices of degree* 3 *in* H_R. We assume that $\deg_{H_R} w_i = 3$ for $1 \le i \le 5$ and $\deg_{H_R} w_i \le 2$ for $6 \le i \le 9$. First, suppose that W contains one vertex of degree 1 in H_R, say $\deg_{H_R} w_6 = 1$ and $N_{H_R}(w_6) = \{u_1\}$. Then $\deg_{H_R} w_i = 2$ for $7 \le i \le 9$. By Claim 1, it follows that $|N_{H_R}(w_j) \cap N_{H_R}(w_k)| = 0$ and $u_1 \notin N_{H_R}(w_i)$ for $7 \le i, j, k \le 9$ and $j \ne k$. So, we may assume that $N_{H_R}(w_7) = \{u_2, u_3\}$, $N_{H_R}(w_8) = \{u_4, u_5\}$, and $N_{H_R}(w_9) = \{u_6, u_7\}$. Then by Claim 1, $u_1 \notin N_{H_R}(w_i)$ for $1 \le i \le 5$. Since there is no Steiner triple system of order 6, it follows that W has at least two vertices of degree 3 in H_R that are joined to the same two vertices of U by red edges, producing a red $K_{2,2}$, which is a contradiction.

Next, suppose that W contains no vertex of degree 1 in H_R, that is, $\deg_{H_R} w_i = 2$ for $6 \le i \le 9$. Then there are at least two vertices of $\{w_6, w_7, w_8, w_9\}$ that are joined to the same vertex of U by red edges, say $N_{H_R}(w_6) = \{u_1, u_2\}$ and $N_{H_R}(w_7) = \{u_2, u_3\}$. By Claim 1, $N_{H_R}(w_8) \cup N_{H_R}(w_9) \subseteq \{u_4, u_5, u_6, u_7\}$. If $|N_{H_R}(w_8) \cap N_{H_R}(w_9)| = 1$, then we may assume that $N_{H_R}(w_8) = \{u_4, u_5\}$ and $N_{H_R}(w_9) = \{u_5, u_6\}$. By Claim 1, we may further assume that none of the 2-element sets $\{u_1, u_2\}$, $\{u_1, u_3\}$, $\{u_2, u_3\}$, $\{u_4, u_5\}$, $\{u_4, u_6\}$, $\{u_5, u_6\}$ is a subset of $N_{H_R}(w_i)$ for each i with $1 \le i \le 5$. Since there are $\binom{7}{2} = 21$ distinct 2-element subsets of U, there are $21 - 6 = 15$ distinct 2-element sets that are available for the 2-elements subsets of $N_{H_R}(w_i)$ where $1 \le i \le 5$. Since there is no red $K_{2,2}$, each of these 15 distinct 2-element sets must belong to exactly one of $N_{H_R}(w_i)$ for $1 \le i \le 5$, say $\{u_1, u_4\} \subseteq N_{H_R}(w_1)$ and $\{u_1, u_5\} \subseteq N_{H_R}(w_2)$. Then we must have $N_{H_R}(w_1) = \{u_1, u_4, u_7\}$ and $N_{H_R}(w_2) = \{u_1, u_5, u_7\}$, producing a red $K_{2,2}$ with the partite sets $\{u_1, u_7\}$ and $\{w_1, w_2\}$, which is a contradiction. So $N_{H_R}(w_8) \cap N_{H_R}(w_9) = \emptyset$, say $N_{H_R}(w_8) = \{u_4, u_5\}$ and $N_{H_R}(w_9) = \{u_6, u_7\}$. By Claim 1, none of the 2-element sets $\{u_1, u_2\}$, $\{u_1, u_3\}$, $\{u_2, u_3\}$, $\{u_4, u_5\}$ and $\{u_6, u_7\}$ is a subset of $N_{H_R}(w_i)$ for each i with $1 \le i \le 5$. Furthermore, each w_i is joined to exactly one vertex of $\{u_1, u_2, u_3\}$ and exactly two vertices of $\{u_4, u_5, u_6, u_7\}$. Since (1) there are only $\binom{4}{2} = 6$ distinct 2-element subsets of $\{u_4, u_5, u_6, u_7\}$ and (2) $\{u_4, u_5\}$ and $\{u_6, u_7\}$ are not subsets of $N_{H_R}(w_i)$ for any i with $1 \le i \le 5$, there are four possibilities for $N_{H_R}(w_i)$ where $1 \le i \le 5$. Thus, there are at least two vertices of W that are joined to the same two vertices of $\{u_4, u_5, u_6, u_7\}$ by red edges, producing a red $K_{2,2}$, which is a contradiction.

Subcase 3.2. *W contains exactly six vertices of degree* 3 *in* H_R*, say* $\deg_{H_R} w_i = 3$ *for* $1 \le i \le 6$ *and* $\deg_{H_R} w_i \le 2$ *for* $7 \le i \le 9$. First, suppose that W contains one vertex of degree 1 in H_R, say $\deg_{H_R} w_7 = 1$ and $N_{H_R}(w_7) = \{u_1\}$. By Claim 1, $N_{H_R}(w_8) \cap N_{H_R}(w_9) = \emptyset$ and $u_1 \notin N_{H_R}(w_8) \cup N_{H_R}(w_9)$. So we may assume that $N_{H_R}(w_8) = \{u_2, u_3\}$ and $N_{H_R}(w_9) = \{u_4, u_5\}$. Since there are $\binom{7}{2} = 21$ distinct 2-element subsets of U, there are $21 - 2 = 19$ distinct 2-element sets available for the 18 distinct 2-element subsets of $N_{H_R}(w_i)$ where $1 \le i \le 6$. By Claim 1, $\{u_1, u_3\}$ and $\{u_1, u_2\}$ cannot belong to any of $N_{H_R}(w_i)$ for $1 \le i \le 6$.

So, there are at most 17 distinct 2-element subsets for the 18 distinct 2-element subsets of $N_{H_R}(w_i)$ where $1 \le i \le 6$. Thus, W contains two vertices of degree 3 in H_R that are joined to the same two vertices of U by red edges, producing a red $K_{2,2}$, which is a contradiction.

Next, suppose that W contains no vertex of degree 1 in H_R and so $\deg_{H_R} w_i = 2$ for $7 \le i \le 9$. Then there are exactly $21 - 3 = 18$ distinct 2-element sets for the 18 distinct 2-element subsets of $N_{H_R}(w_i)$ where $1 \le i \le 6$. If there are two vertices in $\{w_7, w_8, w_9\}$ that are joined to the same vertex of U by red edges, say $N_{H_R}(w_7) = \{u_1, u_2\}$ and $N_{H_R}(w_8) = \{u_2, u_3\}$, then by Claim 1, we may assume that $N_{H_R}(w_9) = \{u_4, u_5\}$. Since $\{u_1, u_3\}$ must be a subset of $N_{H_R}(w_i)$ for some i with $1 \le i \le 6$, say $\{u_1, u_3\} \subseteq N_{H_R}(w_1)$, it follows that $|N_{H_R}(w_1) \cup N_{H_R}(w_7) \cup N_{H_R}(w_8)| \le 4$. Thus, there is a blue $K_{3,3}$ by Claim 1. Hence, we may assume that $N_{H_R}(w_i) \cap N_{H_R}(w_j) = \emptyset$ for $7 \le i, j \le 9$ and $i \ne j$, say $N_{H_R}(w_7) = \{u_1, u_2\}$, $N_{H_R}(w_8) = \{u_3, u_4\}$, and $N_{H_R}(w_9) = \{u_5, u_6\}$. Since $\{u_1, u_7\}$ and $\{u_2, u_7\}$ must belong to the neighborhoods of two distinct vertices w_i in H_R for $1 \le i \le 6$, we may assume that (1) $N_{H_R}(w_1) = \{u_1, u_7, u_3\}$ and (2) $N_{H_R}(w_2) = \{u_2, u_7, u_4\}$ or $N_{H_R}(w_2) = \{u_2, u_7, u_5\}$. First, suppose that $N_{H_R}(w_1) = \{u_1, u_7, u_3\}$ and $N_{H_R}(w_2) = \{u_2, u_7, u_4\}$. Since $\{u_5, u_7\}$ is contained in the neighborhood of some w_i in H_R for $3 \le i \le 6$, we may assume that $\{u_5, u_7\} \subseteq N_{H_R}(w_3)$. So w_3 is joined to one of the vertices in $\{u_1, u_2, u_3, u_4, u_6\}$ by a red edge. If w_3 is joined to one of the vertices in $\{u_1, u_2, u_3, u_4\}$ by a red edge, say w_3u_1 is red, then H contains a red $K_{2,2}$ with partite sets $\{u_1, u_7\}$ and $\{w_1, w_3\}$, a contradiction; while if w_3 is joined to u_6 by a red edge, then H contains a red $K_{2,2}$ with partite sets $\{u_5, u_6\}$ and $\{w_3, w_9\}$, which is a contradiction. Next suppose that $N_{H_R}(w_1) = \{u_1, u_7, u_3\}$ and $N_{H_R}(w_2) = \{u_2, u_7, u_5\}$. Since each 2-element subset of U belongs to the neighborhood of exactly one of the vertices of W, we may assume that $\{u_4, u_7\} \subseteq N_{H_R}(w_3)$. Since there is no red $K_{2,2}$, it follows that $N_{H_R}(w_3) = \{u_4, u_7, u_6\}$. Now, consider $\{u_1, u_4\}$ and we may assume that $\{u_1, u_4\} \subseteq N_{H_R}(w_4)$. Then $N_{H_R}(w_4) = \{u_1, u_4, u_5\}$. Next, consider $\{u_2, u_4\}$ and we may assume that $\{u_2, u_4\} \subseteq N_{H_R}(w_5)$. Since $|N_{H_R}(w_5)| = 3$, it follows that $N_{H_R}(w_5) = \{u_2, u_4, u_i\}$, where $i \in \{1, 3, 5, 6, 7\}$. However then, this produces a red $K_{2,2}$. For example, $N_{H_R}(w_5) = \{u_2, u_4, u_1\}$, then there is a red $K_{2,2}$ whose partite sets are $\{u_1, u_2\}$ and $\{w_5, w_7\}$. Therefore, a contradiction is produced.

Thus, every red-blue coloring of $K_{7,9}$ results in a red $K_{2,2}$ or a blue $K_{3,3}$. Hence, $BR_7(K_{2,2}, K_{3,3}) \le 9$ and so $BR_7(K_{2,2}, K_{3,3}) = 9$. This also implies that every red-blue coloring of $K_{8,9}$ results in a red $K_{2,2}$ or a blue $K_{3,3}$. Therefore, $BR_8(K_{2,2}, K_{3,3}) \ge 9$ and so $BR_8(K_{2,2}, K_{3,3}) = 9$. ■

By Theorem 3.3, it follows that $BR(K_{2,2}, K_{3,3}) = BR_9(K_{2,2}, K_{3,3}) = 9$. Hence, we have the following result.

Theorem 3.4. *For each integer $s \geq 2$,*

$$BR_s(K_{2,2}, K_{3,3}) = \begin{cases} \textit{does not exist} & \textit{if } s = 2, 3, \\ 15 & \textit{if } s = 4, \\ 12 & \textit{if } s = 5, 6, \\ 9 & \textit{if } s = 7, 8, \\ s & \textit{if } s \geq 9. \end{cases}$$

There is a familiar problem corresponding to the Ramsey number $R(K_3, K_3)$, which is stated as follows: What is the smallest number of people who must be present at a gathering, where every two people are either acquaintances or strangers, such that there are three among them who are either mutual acquaintances or mutual strangers? Since $R(K_3, K_3) = 6$, the answer to this question is 6. On the other hand, for a gathering of people, six of whom are women, what is the smallest number of men who must also be present at the gathering so that there are four among them, two women and two men, where each woman is an acquaintance of each man, or there are six among them, three women and three men, where each woman is a stranger of each man. Since $BR_6(K_{2,2}, K_{3,3}) = 12$, the required number of men to be present is 12.

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