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# A NOTE ON THE THUE CHROMATIC NUMBER OF LEXICOGRAPHIC PRODUCTS OF GRAPHS

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**Abstract**

A sequence is called *non-repetitive* if none of its subsequences forms a repetition (a sequence $r_1 r_2 \cdots r_{2n}$ such that $r_i = r_{n+i}$ for all $1 \leq i \leq n$). Let $G$ be a graph whose vertices are coloured. A colouring $\varphi$ of the graph $G$ is *non-repetitive* if the sequence of colours on every path in $G$ is non-repetitive. The *Thue chromatic number*, denoted by $\pi(G)$, is the minimum number of colours of a non-repetitive colouring of $G$.

In this short note we present two general upper bounds for the Thue chromatic number for the lexicographic product $G \circ H$ of graphs $G$ and $H$ with respect to some properties of the factors. One upper bound is then used to derive the exact values for $\pi(G \circ H)$ when $G$ is a complete multipartite graph and $H$ an arbitrary graph.

**Keywords:** non-repetitive colouring, Thue chromatic number, lexicographic product of graphs.

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## 1. Introduction and preliminaries

In 1906 the Norwegian mathematician Axel Thue started the systematic study of word structure. Thue [11] showed that there are arbitrarily long non-repetitive sequences over three symbols, where a sequence $a_1a_2\cdots$ is called *non-repetitive* if it does not contain a subsequence of consecutive elements, the first half of which is exactly the same as its second half. A sequence $r_1 \cdots r_{2n}$ such that $r_i = r_{n+i}$ for all $1 \leq i \leq n$ is called a *repetition*.

With the development of computer-science the research on string-type chains became more and more popular. Non-repetitive sequences found their applications besides mathematics or informatics in many very different areas from information security management to music.

Non-repetitive sequences were introduced also to graph theory by Alon, Grytczuk, Hałuszcak and Riordan [1]. Let $G$ be a simple graph and let $\varphi$ be a proper colouring of its vertices, $\varphi : V(G) \to \{1, \ldots, k\}$. We say that $\varphi$ is *non-repetitive* if for any simple path on vertices $v_1 \cdots v_{2n}$ in $G$ the associated sequence of colours $\varphi(v_1) \cdots \varphi(v_{2n})$ is not a repetition. The minimum number of colours in a non-repetitive colouring of a graph $G$ is the *Thue chromatic number* $\pi(G)$. For the case of list-colourings let the *Thue choice number* $\pi_{ch}(G)$ of a graph $G$ denote the smallest integer $k$ such that for every list assignment $L : V(G) \to 2^{\mathbb{N}}$ with minimum list length at least $k$, there is a colouring of the vertices of $G$ from the assigned lists such that the sequence of vertex colours of no path in $G$ forms a repetition. If a graph $G$ is non-repetitively list colourable for every list assignment $L$ with list size at least $k$, we call $G$ *non-repetitively $k$-choosable*. Hence, $\pi_{ch}(G)$ is the smallest integer $k$ such that $G$ is non-repetitively $k$-choosable.

A walk $v_1 \cdots v_{2n}$ is *boring* if $v_i = v_{n+i}$ for all $i \in \{1, \ldots, n\}$. A boring walk is repetitively coloured by every colouring. A colouring $\varphi$ is *walk non-repetitive* if the only walks that are repetitively coloured by $\varphi$ are boring. The *walk Thue chromatic number* $\pi_w(G)$ of a graph $G$ is the smallest integer $k$ such that $G$ is walk non-repetitively $k$ colourable. The walk Thue chromatic number was first investigated by Barát and Wood in [2]. They showed that every graph with treewidth $k$ and maximum degree $\Delta$ has a $O(k\Delta^3)$ walk non-repetitive colouring.

In [7] various questions concerning non-repetitive colourings of graphs have been formulated. We deal with the problem of finding the minimum number of colours that can be used to colour all vertices of a given graph such that the obtained colouring is non-repetitive. The problem of determining the Thue chromatic number of a graph was studied among others in [1, 3, 4, 10].

The *lexicographic product* or *graph composition* $G \circ H$ of graphs $G$ and $H$ is a graph such that the vertex set of $G \circ H$ is the Cartesian product $V(G) \times V(H)$ and any two vertices $(u, v)$ and $(x, y)$ are adjacent in $G \circ H$ if either $u$ is adjacent with $x$ in $G$ or $u = x$ and $v$ is adjacent with $y$ in $H$. For any vertex $v \in V(G)$ we call the set $\{(v, w) : w \in V(H)\}$ an $H$-*layer* (through $v$) and denote it by $v[H]$. A subgraph induced by $v[H]$ of $G \circ H$ is clearly isomorphic to $H$. If $G \circ H$ is a coloured graph, then we say that an $H$-layer $v[H]$ is *rainbow coloured* whenever all vertices of $v[H]$ have pairwise different colours. Note that the lexicographic product is in general non-commutative: $G \circ H \neq H \circ G$. The independence number of a lexicographic product may be easily calculated from that of its factors (see [6]): $\alpha(G \circ H) = \alpha(G)\alpha(H)$ and the clique number of a lexicographic product is multiplicative as well: $\omega(G \circ H) = \omega(G)\omega(H)$.

The Thue chromatic number of $G \circ H$ when $G$ is a path and $H$ is either an empty graph $E_k$ or a complete graph $K_k$ (also called the blow-up of G by H) was studied in [10]. Here we give some upper bounds for the Thue chromatic number of the lexicographic product of arbitrary graphs and demonstrate the tightness of the bounds by some examples. As a side result we show that for complete multipartite graphs the Thue chromatic number and the Thue choice number are the same[1].

An easy observation about non-repetitive sequences is the following: If a non-repetitive sequence is interrupted by non-repetitive sequences using a distinct set of symbols, then the resulting new sequence remains non-repetitive. Formally, we get the following lemma, proved in [9], where for a sequence of symbols $S = a_1 \cdots a_n$ with $a_i \in \mathbb{A}$, for all $1 \leq k \leq \ell \leq n$, the block $a_k a_{k+1} \cdots a_\ell$ is denoted by $S_{k,\ell}$.

**Lemma 1.1** (Havet *et al.*). *Let $A = a_1 \cdots a_m$ be a non-repetitive sequence with $a_i \in \mathbb{A}$ for every $i \in \{1, \ldots, m\}$. Let $B^i = b_1^i \cdots b_{m_i}^i$, $0 \leq i \leq r+1$, be non-repetitive sequences with $b_j^i \in \mathbb{B}$ for every $i \in \{0, \ldots, r+1\}$ and $j \in \{1, \ldots, m_i\}$. If $\mathbb{A} \cap \mathbb{B} = \emptyset$, then $S = B^0 A_{1,n_1} B^1 A_{n_1+1,n_2} \cdots B^r A_{n_r+1,m} B^{r+1}$ is a non-repetitive sequence.*

[1] Note that in general the Thue chromatic number and the Thue choice number of the same graph may have arbitrary large difference (see [5]), however the most interesting open problem from this area is whether the Thue chromatic number of a path equals its Thue choice number (see [8]).

A *rainbow* sequence, i.e., a sequence of pairwise different elements, is trivially non-repetitive. This implies the following corollary.

**Corollary 1.2.** *Let $A = a_1 \cdots a_m$ be a rainbow sequence with $a_i \in \mathbb{A}$ for all $i \in \{1, \ldots, m\}$. For $i \in \{0, \ldots, r+1\}$ let $b_i \notin \mathbb{A}$. Then $S = b_0 A_{1,n_1} b_1 A_{n_1+1,n_2} \cdots b_r A_{n_r+1,m} b_{r+1}$ is a non-repetitive sequence.*

## 2. Main Results

We start with a general upper bound for the Thue chromatic number of lexicographic products. An upper bound

$$\pi(G \circ H) \leq \pi_w(G)|V(H)|$$

was observed already by Keszegh *et al.* in [10]. We can improve this bound as follows.

**Theorem 2.1.** *For all simple graphs $G$ and $H$ we have that*

$$\pi(G \circ H) \leq \pi(H) + (\pi_w(G) - 1)|V(H)|.$$

***Proof.*** Let $\varphi' : V(G) \to \{1, \ldots, \pi_w(G)\}$ be a walk non-repetitive colouring of a graph $G$ and let $V_1, \ldots, V_{\pi_w(G)}$ be colour classes of $\varphi'$. Colour all $H$-layers corresponding to the vertices from $V_1$ with the set $C_1 = \{1_1, \ldots, \pi(H)_1\}$ of colours, so that the copy of a graph $H$ in each $H$-layer is coloured non-repetitively and for every two vertices $w', w \in V_1$ the colouring of $w'[H]$ is the same as the colouring of $w[H]$. For every other vertex $v \in V_i$, $2 \leq i \leq \pi_w(G)$, we rainbow colour $v[H]$ with colours from $C_i = \{1_i, \ldots, |V(H)|_i\}$. Obviously, such a colouring uses $\pi(H) + (\pi_w(G) - 1)|V(H)|$ colours. We claim that the obtained colouring, say $\varphi$, is a non-repetitive colouring of $G \circ H$.

Assume that there exists a repetitive path $P = v_1 \cdots v_r v_{r+1} \cdots v_{2r}$ in $G \circ H$, such that $\varphi(v_1) = \varphi(v_{r+1}), \ldots, \varphi(v_r) = \varphi(v_{2r})$. Let $P' = u_1 \cdots u_{2r}$ be a projection of $P$ to $G$. By the definition of $\varphi$ we have $\varphi'(u_1) = \varphi'(u_{r+1}), \ldots, \varphi'(u_r) = \varphi'(u_{2r})$. Suppose first that $u_1 = u_2 = \cdots = u_{2r}$. Clearly $u_1 \in V_1$, since for each vertex $z$ from $V(G) \setminus V_1$ we have a rainbow colouring of $z[H]$. This contradicts the definition of $\varphi$, where $u_1[H]$ is coloured non-repetitively. Hence we may assume that not all vertices of $P'$ are the same. In the sequences $u_1 \cdots u_r$ and $u_{r+1} \cdots u_{2r}$ delete every vertex $u_{i+1}$ whenever $u_{i+1} = u_i$ for $1 \leq i \leq r-1$ and $r+1 \leq i \leq 2r-1$, respectively. Notice that $u_i$ is deleted from the first sequence if and only if $u_{r+i}$ is deleted from the second sequence by the definition of $\varphi$ and $\varphi'$. The sequence obtained from amalgamation of remaining sequences yields a repetitive walk in $G$ which is a final contradiction with $\varphi'$ being a walk non-repetitive colouring of $G$. ∎

For the next upper bound recall that $\alpha(G)$ is the notation for the independence number of a graph $G$. The proof is similar to the proof of the previous theorem.

**Theorem 2.2.** *For all simple graphs $G$ and $H$ we have that*

$$\pi(G \circ H) \leq \pi(H) + (|V(G)| - \alpha(G))|V(H)|.$$

***Proof.*** Let $M$ be an independent set of vertices in $G$ of cardinality $\alpha(G)$. Colour all $H$-layers corresponding to the vertices from $M$ with the same set of $\pi(H)$ colours, so that the copy of a graph $H$ in each $H$-layer is coloured non-repetitively and for every two vertices $w', w \in M$ the colouring of $w'[H]$ is the same as the colouring of $w[H]$. All other vertices from $(V(G) \setminus M) \times V(H)$ are rainbow coloured with a new set of colours. Obviously such a colouring uses $\pi(H) + (|V(G)| - \alpha(G))|V(H)|$ colours. We claim that the obtained colouring, say $\varphi$, is a non-repetitive colouring of $G \circ H$.

Assume that there exists a repetitive path $P = v_1 \cdots v_r v_{r+1} \cdots v_{2r}$ in $G \circ H$, such that $\varphi(v_1) = \varphi(v_{r+1}), \ldots, \varphi(v_r) = \varphi(v_{2r})$. Note that no vertex from $(V(G) \setminus M) \times V(H)$ is on $P$, since it has a unique colour. Thus for each $j \in \{1, \ldots, 2r\}$ we have $v_j \in w[H]$ for some $w \in M$. Since every $H$ layer is coloured non-repetitively, not all vertices of $P$ can be in the same $H$ layer. Without loss of generality suppose that $v_1 \in w_1[H]$ and $v_2 \in w_2[H]$ where $w_1 \neq w_2$. As $e = v_1 v_2$ is an edge in $P$, there exists an edge $e' = w_1 w_2$ in $G$, a contradiction with $M$ being an independent set of vertices. Hence, our hypothesis was wrong and $\varphi$ is non-repetitive. ■

The bounds from Theorems 2.1 and 2.2 can behave quite differently. For instance $\pi_w(G)$ is bounded for paths or degree bounded trees by a constant, see [2], and does not depend on $|V(G)|$ as $|V(G)| - \alpha(G)$. On the other hand it is easy to see that $\pi_w(G) = |V(G)|$ for complete multipartite graphs, where $|V(G)| - \alpha(G)$ is better. It seems that the bound from Theorem 2.1 performs better for sparse graphs and the bound from Theorem 2.2 is better for dense graphs. Here we concentrate on the later class of graphs.

Before showing the sharpness of the bound for $\pi(G \circ H)$ from Theorem 2.2, we prove a result that is dealing with vertex list non-repetitive colourings. It is easy to verify that the statement of Theorem 2.3 holds, as it was already observed by several mathematicians. But for the sake of comprehensiveness we include its proof as well.

**Theorem 2.3.** *If $G$ is a graph on $n$ vertices, then the following statements hold.*

(i) $\pi(G) \leq \pi_{ch}(G) \leq n - \alpha(G) + 1$.

(ii) *If $G$ is a complete multipartite graph, then $\pi(G) = \pi_{ch}(G) = n - \alpha(G) + 1$.*

***Proof.*** (i) Let $G$ be a graph on $n$ vertices. As every non-repetitive $k$-colouring of $G$ can be considered as a non-repetitive list-colouring of $G$ from identical lists of size $k$, the first inequality ($\pi(G) \leq \pi_{ch}(G)$) trivially holds.

To show the second inequality let $M$ be a maximum independent set of vertices from $V(G)$ with $|M| = \alpha(G)$, and let $L : V(G) \rightarrow 2^{\mathbb{N}}$ be any list assignment such that each list length is at least $n - \alpha(G) + 1$. Colour the vertices belonging to $V(G) \setminus M$ with pairwise different colours from their lists, and remove all colours used by any of these vertices from the lists of the vertices of $M$. As each list is of length at least $n - \alpha(G) + 1$, at least one colour from the list of each vertex $x \in M$ remains, and this will be used to colour the vertex $x$. Now consider the sequence of vertex colours of any path in $G$. The subsequence of colours of this sequence which belong to the vertices of $V(G) \setminus M$ constitute a rainbow sequence, which is interrupted by single colours belonging to the vertices of $M$. Hence, by Corollary 1.2 such a colouring is non-repetitive, which proves that $\pi(G) \leq \pi_{ch}(G) \leq n - \alpha(G) + 1$.

(ii) Let $G$ be a complete multipartite graph of order $n$ with partite sets $V_1, \ldots, V_m$. To prove the statement it is sufficient to show that $\pi(G) \geq n - \alpha(G) + 1$. This will be proven by a contradiction. Assume there is a non-repetitive $(n - \alpha(G))$-colouring $\varphi$ of $G$. Because there are $n$ vertices coloured by $n - \alpha(G)$ different colours, by pigeon-hole principle the set $M = \{x \in V(G) : \exists\, x' \in V(G) \setminus \{x\} : \varphi(x) = \varphi(x')\}$ of vertices without unique colour consists of at least $\alpha(G) + 1$ vertices. Because all partite sets consist of at most $\alpha(G)$ vertices there exist two vertices $x$ and $y$ of $M$ belonging to different partite sets. Without loss of generality we assume that $x \in V_1$ and $y \in V_2$. Since adjacent vertices must receive different colours the colour $\varphi(x)$ can only appear in $V_1$ and the colour $\varphi(y)$ can only appear in $V_2$. Hence, there must be a vertex $x' \in V_1$, $x' \neq x$, with $\varphi(x') = \varphi(x)$ and a vertex $y' \in V_2$, $y' \neq y$, with $\varphi(y') = \varphi(y)$. But then the colour sequence of the path $P = (xyx'y')$ is repetitive, a contradiction. ∎

As an immediate corollary we obtain several infinite subclasses of complete multipartite graphs, where the graph parameters $\pi$ and $\pi_{ch}$ coincide. These results on $\pi(G)$ and $\pi_{ch}(G)$ are well known.

**Corollary 2.4.** *The following statements hold.*

(i) $\pi(K_n) = \pi_{ch}(K_n) = n$ *for the complete graph* $K_n$ *on* $n$ *vertices.*

(ii) $\pi(S_n) = \pi_{ch}(S_n) = 2$ *for a star* $S_n$ *on* $n+1$ *vertices.*

(iii) $\pi(K_{m,n}) = \pi_{ch}(K_{m,n}) = \min\{m, n\} + 1$ *for a complete bipartite graph* $K_{m,n}$.

For a lower bound we strongly suspect that the following is true.

**Conjecture 2.5.** *For all simple graphs* $G$ *and* $H$ *we have that*

$$\pi(H) + (\pi(G) - 1)|V(H)| \leq \pi(G \circ H).$$

This conjecture is true for $\pi(P_n \circ E_k)$ and for $\pi(P_n \circ K_k)$ by Theorem 1.2 and Theorem 1.4, respectively, of [10]. Moreover, it is sharp for $\pi(P_n \circ E_k)$ for $n \geq 4$ and $k > 2$, but not for $\pi(P_n \circ K_k)$ for $n \geq 28$ by the same theorems. We show that Conjecture 2.5 holds also for the lexicographic product of the complete multipartite graph $K_{n_1,\ldots,n_k}$ with any graph $H$. This represents another good reason to believe that Conjecture 2.5 is true: namely paths are very sparse graphs while complete multipartite graphs represent dense graphs with respect to the number of edges.

**Theorem 2.6.** *For a complete multipartite graph $G$ and an arbitrary graph $H$ we have that*

$$\pi(H) + (\pi(G) - 1)|V(H)| \leq \pi(G \circ H).$$

***Proof.*** Let $G$ be a complete multipartite graph $K_{n_1,\ldots,n_k}$, where $V_1, \ldots, V_k$ form a partition of $V(G)$ with $|V_1| = n_1, \ldots, |V_k| = n_k$ and $n_1 + \cdots + n_k = n$. Towards a contradiction suppose that there exists a non-repetitive colouring $\varphi$ of $G \circ H$ using less than $\pi(H) + (\pi(G) - 1)|V(H)|$ colours. Since $\pi(H) \leq |V(H)|$, we have less than $\pi(G)|V(H)|$ colours for $\varphi$. Hence there exist two vertices in $G \circ H$, say $(g, h)$ and $(g', h')$ with the same colour. We may assume that $\varphi(g, h) = 1 = \varphi(g', h')$. Clearly $g$ and $g'$ belong to the same set $V_i$, say with $i = 1$, since they are nonadjacent. If there exists $g'' \in V(G) \setminus V_1$ with two different vertices in $g''[H]$, say $(g'', h_1)$ and $(g'', h_2)$, of the same colour, then we have a repetition on $(g, h)(g'', h_1)(g', h')(g'', h_2)$, which is a contradiction. Therefore for every $g'' \in V(G) \setminus V_1$ the layer $g''[H]$ must be rainbow coloured. Let now $g_1, g_2 \in V(G) \setminus V_1$, $g_1 \neq g_2$. If there are vertices of the same colour in $g_1[H]$ and in $g_2[H]$, say $(g_1, h_3)$ and $(g_2, h_4)$, then we have again a repetition $(g, h)(g_1, h_3)(g', h')(g_2, h_4)$, which is not possible. Thus for all pairs of different vertices $g_1, g_2 \in V(G) \setminus V_1$ all colours in $g_1[H]$ and $g_2[H]$ must be pairwise different. This means we have used $|(V(G) \setminus V_1) \times V(H)|$ colours on $(G - V_1) \circ H$. By Theorem 2.3, $\pi(G) = n - \max\{n_1, \ldots, n_k\} + 1$. Considering the number of colours in $\varphi$ and the number of colours we have used for $(G - V_1) \circ H$, the number of available colours for the layer $g[H]$ is less than

$$\begin{aligned}
&\pi(H) + (\pi(G) - 1) \cdot |V(H)| - |(V(G) \setminus V_1)| \cdot |V(H)| \\
&= \pi(H) + (n - \max\{n_1, \ldots, n_k\} + 1 - 1) \cdot |V(H)| - (n - n_1) \cdot |V(H)| \\
&= \pi(H) + (n - \max\{n_1, \ldots, n_k\} - n + n_1) \cdot |V(H)| \\
&= \pi(H) + (n_1 - \max\{n_1, \ldots, n_k\}) \cdot |V(H)| \leq \pi(H).
\end{aligned}$$

This yields a final contradiction, since there are less than $\pi(H)$ colours left for $g[H]$, which results in a repetition in the $H$-layer $g[H]$. ∎

The family of complete multipartite graphs is one that establishes the tightness of the bounds for $G \circ H$ given by Theorem 2.2 and Theorem 2.6.

**Theorem 2.7.** *Let $G$ be a complete multipartite graph. Then*

$$\pi(H) + (\pi(G) - 1)|V(H)| = \pi(G \circ H) = \pi(H) + (|V(G)| - \alpha(G))|V(H)|.$$

***Proof.*** Theorem 2.3 shows that $\pi(G) = n - \alpha(G) + 1$, where $\alpha(G)$ is the independence number of $G$ being a complete multipartite graph on $n$ vertices. The thesis of Theorem 2.7 then directly follows from Theorem 2.2 and Theorem 2.6. ■

Theorem 2.7 and Corollary 2.4 then give the following corollary.

**Corollary 2.8.** *For any graph $H$ we have that*

(i) $\pi(K_n \circ H) = \pi(H) + (n - 1)|V(H)|$, *where $K_n$ is a complete graph on $n$ vertices;*

(ii) $\pi(S_n \circ H) = \pi(H) + |V(H)|$, *where $S_n$ is a star on $n + 1$ vertices;*

(iii) $\pi(K_{m,n} \circ H) = \pi(H) + \min\{m, n\} \cdot |V(H)|$, *where $K_{m,n}$ is a complete bipartite graph on $m + n$ vertices.*

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