# EXTREMAL IRREGULAR DIGRAPHS 

Joanna Górska, ZdZISŁAW Skupień<br>AGH Kraków<br>al. Mickiewicza 30, 30-059 Kraków, Poland<br>e-mail: gorska@agh.edu.pl<br>skupien@agh.edu.pl<br>Zyta Dziechcińska-Halamoda, Zofia Majcher

AND

Jerzy Michael<br>Institute of Mathematics and Informatics<br>Opole University<br>ul. Oleska 48, 45-052 Opole, Poland<br>e-mail: zdziech@uni.opole.pl<br>majcher@math.uni.opole.pl<br>michael@uni.opole.pl


#### Abstract

A digraph is called irregular if its distinct vertices have distinct degree pairs. An irregular digraph is called minimal (maximal) if the removal of any arc (addition of any new arc) results in a non-irregular digraph. It is easily seen that the minimum sizes among irregular $n$-vertex whether digraphs or oriented graphs are the same and are asymptotic to $(\sqrt{2} / 3) n^{3 / 2}$; maximum sizes, however, are asymptotic to $n^{2}$ and $n^{2} / 2$, respectively. Let $s$ stand for the sum of initial positive integers, $s=1,3,6, \ldots$ An oriented graph $H_{s}$ and a digraph $F_{s}$, both large (in terms of the size), minimal irregular, and on any such $s$ vertices, $s \geq 21$, are constructed in [Large minimal irregular digraphs, Opuscula Math. 23 (2003) 21-24], co-authored by Z. D-H. and three more of the present co-authors (Z.M., J.M., Z.S.). In the present paper we nearly complete these constructions. Namely, a large minimal irregular digraph $F_{n}$, respectively oriented graph $H_{n}$, are constructed for any of remaining orders $n, n>21$, and of size asymptotic to $n^{2}$, respectively to $n^{2} / 2$. Also a digraph $\Phi_{n}$ and an oriented graph $G_{n}$, both small maximal irregular of any order $n \geq 6$, are constructed. The asymptotic value of the size of $G_{n}$ is at least


$(\sqrt{2} / 3) n^{3 / 2}$ and is just the least if $n=s \rightarrow \infty$, but otherwise the value is at most four times larger and is just the largest if $n=s-1 \rightarrow \infty$. On the other hand, the size of $\Phi_{n}$ is of the asymptotic order $\Theta\left(n^{3 / 2}\right)$.
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## 1. Introduction

For terminology and notation we refer to Chartrand and Lesniak [1]. Let $D=$ $(V, A)$ be a digraph with vertex set $V=V(D)$ and arc set $A=A(D)$. A digraph without loops or 2-dicycles is called an oriented graph. Numbers of vertices and arcs of $D$ are denoted by $|D|$ and $\|D\|$ and are called the order and the size of $D$, respectively. The ordered pair $(a, b)$ comprising the two semi-degrees of a vertex, namely the outdegree $a$ followed by the indegree $b$, is called the degree pair of the vertex. The sum of both semi-degrees is called the degree of a vertex. Moreover, $\delta(D)$ and $\Delta(D)$ denote respectively the minimum and the maximum degree over vertices in $D$.

A digraph $D$ is called diregular ( $\rho$-diregular) if all outdegrees and all indegrees are mutually equal (and equal to $\rho$ ). At the other extreme, $D$ is said to be irregular if distinct vertices have distinct degree pairs, see Gargano, Kennedy and Quintas [4]. These digraphs were rediscovered and independently studied under the name fully irregular digraphs in $[6,7]$ and next in $[2,3,5]$ with due credit to predecessors. The $n$-vertex transitive tournament, denoted by $T_{n}$ (and coming from game theory), plays a prominent role in our study.

The following statements are well known and easily seen.
Theorem 1. The transitive tournament $T_{n}$ is the unique largest irregular oriented n-graph.

$$
\begin{equation*}
\left\|T_{n}\right\|=\frac{1}{2} n(n-1) \sim \frac{1}{2} n^{2} \tag{1}
\end{equation*}
$$

An $n$-set of $n$ degree pairs is called minimum and symmetric if the set is symmetric and with smallest possible sum of semi-degrees. In the paper [6], for each positive integer $n$, a minimum and symmetric $n$-set of degree pairs is presented. An oriented graph with those degree pairs is found. Consequently, a minimum $n$-vertex irregular digraph (i.e., with smallest size) is constructed. In the construction a few cases are considered, and the simplest of them is if $n$ is the sum of the first $t$ positive integers.

Notation. Throughout the paper the symbol $s_{t}$ (or $s$ as abbrevation) stands for the sum of the first $t$ positive integers for some integer $t$,

$$
\begin{equation*}
s_{t}(=s):=1+2+\cdots+t=\frac{1}{2} t(t+1) \tag{2}
\end{equation*}
$$

Proposition 2. For each s, a minimum s-set of degree pairs is unique. Namely, the following union

$$
\begin{equation*}
U_{s}:=T_{1} \cup T_{2} \cup \cdots \cup T_{t} \tag{3}
\end{equation*}
$$

of pairwise vertex disjoint tournaments $T_{1}, T_{2}, \ldots, T_{t}$ is an oriented graph which is a digraphic realization of the minimum s-set. That realization is not unique if $t \geq 3$.

Related examples. 1. The set of degree pairs of the union $T_{2} \cup T_{3}$ can be realized by orienting the edges of the path $P_{5}$ (in three ways), see Figure 1.


Figure 1. Three more realizations of the set of degree pairs of $T_{2} \cup T_{3}$.
2. Let $v_{1}, v_{2}$ and $w_{1}, w_{2}, w_{3}, w_{4}$ be consecutive vertices of tournaments $T_{2}$ and $T_{4}$, respectively. Let $D_{6}$ be the digraph induced by the following set of arcs $A\left(D_{6}\right)=\left\{\left(v_{1}, w_{4}\right),\left(w_{1}, v_{2}\right),\left(w_{1}, w_{3}\right),\left(w_{1}, w_{4}\right),\left(w_{2}, w_{3}\right),\left(w_{2}, w_{4}\right),\left(w_{3}, w_{2}\right)\right\}$. Then $D_{6}$ includes a 2-dicycle $\vec{C}_{2}, \vec{C}_{2}=\left(w_{2}, w_{3}, w_{2}\right)$, and is a realization of the degree pairs in the union $T_{2} \cup T_{4}$, see Figure 2.


Figure 2. The digraph $D_{6}$ which is a realization of the set of degree pairs of $T_{2} \cup T_{4}$.
Theorem 3. Since $U_{s}$ is among the smallest irregular digraphs, the complement $\overline{U_{s}}$ (in the complete symmetric digraph $K_{s}^{*}$ ) is one of the largest digraphs among irregular digraphs of order $s$.

One can see the following consequence of (2):

$$
\begin{equation*}
t=\frac{1}{2}(\sqrt{8 s+1}-1) \sim \sqrt{2 s} \tag{4}
\end{equation*}
$$

which implies the equality $t^{2}=2 s-t$. Therefore, the asymptotic values of the sizes of $U_{s}$ and $\overline{U_{s}}$ (which will be very useful) are as follows:

$$
\begin{align*}
\left\|U_{s}\right\|=\frac{1}{2} \sum_{i=1}^{t} i(i-1)=\frac{1}{6} t\left(t^{2}-1\right) & =\frac{1}{6}\left(t^{3}-t\right) \sim \frac{\sqrt{2}}{3} s^{3 / 2}  \tag{5}\\
& =\frac{1}{6} t(2 s-t-1)=\frac{1}{3} t s+O\left(t^{2}\right), \tag{6}
\end{align*}
$$

$$
\begin{equation*}
\left\|\overline{U_{s}}\right\|=\left\|K_{s}^{*}\right\|-\left\|U_{s}\right\|=s(s-1)-\left\|U_{s}\right\| \sim s^{2} . \tag{7}
\end{equation*}
$$

An irregular digraph is called minimal if the removal of any arc spoils irregularity. Obviously, $U_{s}$ is an example of minimal irregular oriented graph (and digraph). In the paper [2] a large minimal irregular oriented graph $H_{s}$ and analogous digraph $F_{s}$, where $s \geq 21$ is the sum of six or more initial positive integers, are constructed. The sizes of $H_{s}$ and $F_{s}$ are asymptotically the largest possible since they are asymptotic to $s^{2} / 2$ and $s^{2}$, respectively. We are going to construct large minimal irregular structures (an oriented graph $H_{n}$ and a digraph $F_{n}$ ) of any remaining order $n>21$ and with the same corresponding asymptotic sizes.

An irregular digraph (irregular oriented graph) is called maximal if the addition of any new arc spoils irregularity (or spoils being an oriented graph). Small maximal irregular structures (an oriented graph $G_{n}$ and a digraph $\Phi_{n}$ ) of arbitrary order $n \geq 6$ and with sizes of asymptotic order $\Theta\left(n^{3 / 2}\right)$ will be constructed. In the special case when $n=s=s_{t}$, we construct a maximal oriented graph, $G_{s}$, with size asymptotic to $(\sqrt{2} / 3) s^{3 / 2}$.

In the constructions which follow we assume that $t_{0}, t, s, m$ and $n$ are positive integers such that

$$
t \geq t_{0}, s=s_{t}, 0 \leq m \leq t, n=s+m
$$

Consequently, due to (4), $m=O(t)=O(\sqrt{s})$. Moreover,

$$
\begin{equation*}
s \sim n, t \sim \sqrt{2 n} \text { and } m=O(\sqrt{n}) \tag{8}
\end{equation*}
$$

## 2. Large Minimal Irregular Digraphs $H_{n}$ and $F_{n}$

In this section we assume that

$$
\begin{equation*}
t \geq t_{0}:=6 \quad \text { whence } \quad n \geq s=s_{t} \geq 21 \tag{9}
\end{equation*}
$$

We first recall the construction (see [2]) of the large minimal irregular oriented graph $H_{s}$.

## Construction 1.

- Let $D_{s}^{\prime}$ be a $\rho$-diregular digraph on $s$ vertices such that $\left.\rho=\lfloor(s-1) / 2)\right\rfloor$, $V\left(D_{s}^{\prime}\right)=\mathbb{Z}_{s}$, and $A\left(D_{s}^{\prime}\right)=\left\{(i, i+j): i, j \in \mathbb{Z}_{s}, 1 \leq j \leq \rho\right\} ;$
- split the vertex sequence $(0,1, \ldots, s-1)$ into strings (initial sections) of decreasing lengths: $2 t-1$ and next $t-2, t-3, \ldots, 1$;
- split the first string $(0,1, \ldots, 2 t-2)$ into two disjoint subsequences which make up sequences: $V_{t}:=(0,2, \ldots, 2 t-2)$ and $V_{t-1}:=(1,3, \ldots, 2 t-3)$. Denote the remaining strings by $V_{t-2}, V_{t-3}, \ldots, V_{1}$;
- let $U_{s}^{\prime}$ be the union of $t$ subgraphs of $D_{s}^{\prime}$ induced by all $t$ sequences $V_{j}$;
- $H_{s}:=D_{s}^{\prime}-A\left(U_{s}^{\prime}\right)$.

Theorem 4. Under the assumptions (2) and (9), the digraph $U_{s}^{\prime}$ is isomorphic to $U_{s}$, which is defined in (3) within Proposition 2.

Proof. Due to the above definition of the digraph $D_{s}^{\prime}$, it is enough to prove, for the longest sequence $V_{t}$ which is $V_{t}=(0,2, \ldots, 2 t-2)$, the following two properties:
(i) an arc exists which joins the initial vertex 0 to the terminal vertex $2 t-2$,
(ii) no arc of the digraph joins the terminal vertex $2 t-2$ to another vertex of the sequence.
It is so because then $V_{t}$ and each shorter sequence induce transitive tournaments.
The properties (i) and (ii) are clearly implied by the respective inequalities:
(i') $2 t-2 \leq \rho$, and (ii') $2 t-2+\rho \leq s-1$. It remains to prove (i') for $t \geq 6$ because then, for $\rho=\lfloor(s-1) / 2\rfloor$, we have $2 \rho \leq s-1$. To this end, we introduce the following notation.
$L_{t}=2 t-2, \rho=\rho_{t}$ where $s=s_{t}$ is involved, and next we use induction on $t \geq 6$. Note that equality holds in (i') if the initial $t=6$. Assume that the inequality (i'), that is, $\rho_{t} \geq L_{t}$, holds for some $t \geq 6$. Then, for the next value $t+1$, we have $L_{t+1}=L_{t}+2, s_{t+1}=s_{t}+t+1$ whence $\rho_{t+1}=\left\lfloor\left(s_{t+1}-1\right) / 2\right\rfloor=$ $\left\lfloor\left(\left(s_{t}-1\right)+t+1\right) / 2\right\rfloor \geq \rho_{t}+t / 2 \geq L_{t+1}$, which ends the induction.

Theorem 5 (Theorem 1 in [2]). Under the assumptions (2) and (9), the digraph $H_{s}$ obtained by Construction 1 is a minimal irregular oriented graph of size $\left\|H_{s}\right\| \sim s^{2} / 2, s=21,28, \ldots$.

Construction 2 (extension of Construction 1 under the assumption $n=s+m$, $s=s_{t}$ and $\left.0<m \leq t\right)$.

- Let $S_{m}$ be an oriented graph on $m$ new vertices $v_{1}, v_{2}, \ldots, v_{m}$ and with the following arc set:

$$
A\left(S_{m}\right)=\left\{\left(v_{i}, v_{j}\right): 1 \leq i \leq\left\lfloor\frac{m}{2}\right\rfloor, m-i+1 \leq j \leq m\right\}
$$

- $H_{n}^{\prime}:=H_{s} \cup S_{m}$ where $H_{s}$ is the digraph obtained by Construction 1 (and $0<m \leq t)$.

Lemma 6. The digraph $S_{m}$, see Construction 2, is a minimal irregular oriented graph of size $O(n)$.

Proof. The structure is clearly correct if $m=1,2,3$. Let $m \geq 4$. The outdegrees of initial vertices $1,2, \ldots,\lfloor m / 2\rfloor$ grow by one from 1 to $\lfloor m / 2\rfloor$, the indegrees of next vertices grow analogously, either from 0 if $m$ is odd or from 1 otherwise, up to $\lfloor m / 2\rfloor$, and all remaining semi-degrees of vertices in $S_{m}$ equal 0 . Hence $S_{m}$ is irregular. Moreover, each arc of $S_{m}$ is incident with a vertex which has a semidegree larger than 1. This proves minimality. Furthermore, $\left\|S_{m}\right\|=\sum_{i=1}^{\lfloor m / 2\rfloor} i \leq$ $1+2+\cdots+\lfloor t / 2\rfloor \leq t(t+2) / 8=O(n)$ due to (8).

Let

$$
H_{n}= \begin{cases}H_{s} & \text { if } n=s=s_{t}(m=0) \\ H_{n}^{\prime} & \text { if } n=s+m, s=s_{t} \text { and } 0<m \leq t .\end{cases}
$$

Theorem 7. The digraph $H_{n}$ is a large minimal irregular oriented graph of order $n$ and size $\left\|H_{n}\right\| \sim n^{2} / 2$.

Proof. Due to Theorem 5, it remains to consider the case $m>0$ and $H_{n}=H_{n}^{\prime}$. By Lemma 6, $S_{m}$ is a minimal irregular oriented graph. Moreover, digraphs $H_{s}$ and $S_{m}$ are vertex-disjoint, and the maximum degree in $S_{m}$ is not greater than $t / 2$ while the minimum degree in $H_{s}$ (see Construction 1) is $\delta\left(H_{s}\right)=\delta\left(D_{s}^{\prime}\right)-\Delta\left(U_{s}\right)=$ $2 \rho-(t-1) \geq(s-2)-t+1=s-(t+1)=(t+1)(t-2) / 2$ due to (2) and then $\delta\left(H_{s}\right)>t / 2$ due to (9). Therefore, the digraph $H_{n}^{\prime}$ is irregular, too. It is clear that the removal of any arc from $H_{s}$ or from $S_{m}$ spoils irregularity of $H_{n}^{\prime}$. On the other hand, by Theorem 5 and Lemma 6 , since $s \sim n,\left\|H_{n}^{\prime}\right\|=\left\|H_{s}\right\|+\left\|S_{m}\right\| \sim n^{2} / 2$.

A large minimal irregular digraph $F_{n}$ of order $n, n=s+m$ with $0 \leq m \leq t$, is obtained in a similar way. Namely, $F_{n}^{\prime}=F_{s} \cup S_{m}$ with $m>0$ is the disjoint union, where $S_{m}$ is the digraph defined above, and digraphs $F_{s}, F_{n}^{\prime}$ and next $F_{n}$ are constructed in the following way.

Construction 3 (under the assumption $n=s+m, s=s_{t}$ and $0<m \leq t$ ).

- Let $D_{s}^{\prime \prime}$ be the digraph with vertex set $V\left(D_{s}^{\prime \prime}\right)=\mathbb{Z}_{s}$ obtained from the complete digraph $K_{s}^{*}$ by deleting arcs $(i, i-1)$ and $(i, i-2 j)$, for $i, j \in \mathbb{Z}_{s}$ and $j=2,3, \ldots, t-1$;
- let $U_{s}$ be the union of transitive tournaments on sequences $V_{1}, V_{2}, \ldots, V_{t}$, cf. Construction 1;
- let $F_{s}=D_{s}^{\prime \prime}-A\left(U_{s}\right)$;
- let $F_{n}^{\prime}=F_{s} \cup S_{m}$, with $m>0$ and $S_{m}$ as in Construction 2.

Hence $\left\|D_{s}^{\prime \prime}\right\|=\left\|K_{s}^{*}\right\|-s(t-1)$ and

$$
\begin{equation*}
\left\|F_{s}\right\|=\left\|D_{s}^{\prime \prime}\right\|-\left\|U_{s}\right\|=\left\|K_{s}^{*}\right\|-s(t-1)-\left\|U_{s}\right\| . \tag{10}
\end{equation*}
$$

Let

$$
F_{n}= \begin{cases}F_{s} & \text { if } n=s=s_{t}(m=0),  \tag{11}\\ F_{n}^{\prime}=F_{s} \cup S_{m} & \text { if } n=s+m, s=s_{t} \text { and } 0<m \leq t\end{cases}
$$

Theorem 8 (Theorem 2 in [2]). Under the assumptions (2) and (9), the digraph $F_{s}$ obtained by Construction 3 is a minimal irregular digraph of size $\left\|F_{s}\right\|=$ $s(s-t)-\left\|U_{s}\right\| \sim s^{2}, s=21,28, \ldots$.

Theorem 9. The digraph $F_{n}$, see (11), is a large minimal irregular digraph of order $n$ and size $\left\|F_{n}\right\| \sim n^{2}$.

The proof of Theorem 9 is analogous to that of Theorem 7 because $\delta\left(F_{s}\right)>t / 2 \geq$ $\Delta\left(S_{m}\right)$.
3. Small Maximal Irregular Digraphs $G_{s}, G_{n}$, and $\Phi_{s}, \Phi_{n}$

In this section we assume that

$$
t \geq t_{0}:=3 \quad \text { whence } \quad n \geq s=s_{t} \geq 6
$$

We now present a construction of a small maximal irregular oriented graph $G_{s}$ of order $s$ and with the spanning cycle $\overrightarrow{C_{s}}=(0,1,2, \ldots, s-1,0)$.

## Construction 4.

- Assume that the sequence $V_{t}, V_{t-1}, \ldots, V_{2}, V_{1}$ represents a partition of the vertex set $V\left(\overrightarrow{C_{s}}\right)$ into $i$-sets $V_{i}, i=t, t-1, \ldots, 2,1$. Subsets $V_{i}$ are chosen in such a way that in all $\lfloor t / 2\rfloor$ pairs $\left(V_{i}, V_{i-1}\right)$ with $i=t, t-2, \ldots$ down to $i=3$ or $i=2$, depending on the parity of $t$, the sets $V_{i}, V_{i-1}$ intertwine along the cycle $\vec{C}_{s}$. For instance, $V_{t}=\{0,2, \ldots, 2 t-2\}$ and $V_{t-1}=\{1,3, \ldots, 2 t-3\}$. Consequently, each $V_{i}$ is an independent subset of $V\left(\overrightarrow{C_{s}}\right)$;
- for $k=1,2, \ldots, t$, let $T_{k}$ be the transitive tournament such that $V\left(T_{k}\right)=V_{k}$ and for $i, j \in V_{k},(i, j) \in A\left(T_{k}\right) \Leftrightarrow i<j$;
- let $U_{s}=\bigcup_{i=1}^{t} T_{i}$;
- let $G_{s}=\vec{C}_{s}+A\left(U_{s}\right)$.

Theorem 10. The digraph $G_{s}$ obtained by Construction 4 is a small maximal irregular oriented graph of order s and size $\left\|G_{s}\right\| \sim(\sqrt{2} / 3) s^{3 / 2}$.

Proof. To show maximality is to prove that, for any arc $e$ which joins nonadjacent vertices of $G_{s}$, the digraph $G_{s}+e$ is not irregular. To this end, let an arc $e$ join a vertex, say $x \in V_{i}$, with a vertex, say $x^{\prime}$, of some $V_{j}$ such that $j>i$ and vertices $x, x^{\prime}$ are nonadjacent in $G_{s}$. Because each subset $V_{k}$ with $1 \leq k \leq t$ induces a transitive tournament in $G_{s},\left|V_{k}\right|$ is the number of degree pairs in $G_{s}$ of vertices of that subset. Consequently, if $k=j<t$ and $z=x^{\prime} \in V_{k}$ or $k=i<j-1$ and $z=x \in V_{k}$, then the degree pair $p_{G_{s}+e}(z)$ coincides with degree pair $p_{G_{s}}(y)$ of a vertex $y \in V_{k+1}$. The case which still remains is $i+1=j=t$. Then the degree pair $p_{G_{s}+e}(x)$ coincides with that of a neighbor of $x$ on the cycle $\vec{C}_{s}$. This shows maximality of $G_{s}$. Note that $\left\|G_{s}\right\|=|A(\vec{C})|+\left|A\left(U_{s}\right)\right|=s+\left\|U_{s}\right\| \sim(\sqrt{2} / 3) s^{3 / 2}$ due to (5).

We now construct a small maximal irregular oriented graph $G_{n}^{\prime}$ of order $n=s+m$ with $0<m \leq t$.

Construction 5 (under the assumption $n=s+m, s=s_{t}$ and $0<m \leq t$ ).

- Let $G_{s}$ be the oriented graph as in Construction 4;
- let $T_{m}$ be the transitive tournament on $m$ new vertices $v_{1}, v_{2}, \ldots, v_{m}$;
- let $G_{n}^{\prime}=G_{s} \cup T_{m}+A_{m}$ where $m>0$ and $A_{m}$ is the set of all arcs which go from vertices of $T_{m}$ to those of $G_{s}, A_{m}=\left\{(u, v): u \in V\left(T_{m}\right), v \in V\left(G_{s}\right)\right\}$.
Let

$$
G_{n}= \begin{cases}G_{s} & \text { if } n=s=s_{t}(m=0) \\ G_{n}^{\prime} & \text { if } n=s+m, s=s_{t} \text { and } 0<m \leq t\end{cases}
$$

The following theorem complements Theorem 10.
Theorem 11. The digraph $G_{n}$ is a small maximal irregular oriented graph of order $n$ and size $\left\|G_{n}\right\|=\Theta\left(n^{3 / 2}\right)$. Moreover, if $m=t$ and $n=s_{t}+t$, then $\left\|G_{n}\right\| \sim(4 \sqrt{2} / 3) n^{3 / 2}$ wherein the asymptotic coefficient is the largest possible.

Proof. Due to Theorem 10, it remains to consider the case $m>0$ and $G_{n}=G_{n}^{\prime}$. It is easy to see that $G_{n}^{\prime}$ is an irregular oriented graph. Note that only an arc with both endvertices in the subgraph $G_{s}$ can be added to $G_{n}^{\prime}$ so that the property of being an oriented graph could be preserved. However, adding any such arc spoils irregularity, cf. the above proof of Theorem 10 . This shows maximality.

Let $n=s+t$ where, due to (2), $s=t(t+1) / 2=O\left(t^{2}\right)$. Then $\left\|G_{s}\right\|=$ $s+\left\|U_{s}\right\|=t^{3} / 6+O\left(t^{2}\right)$ due to (2) and (6), $\left\|T_{t}\right\|=O\left(t^{2}\right)$ due to (1), and $\left|A_{m}\right| \leq\left|A_{t}\right|=t s=t^{3} / 2+O\left(t^{2}\right)$ due to (2) whence $\left\|G_{n}^{\prime}\right\| \leq\left\|G_{s+t}^{\prime}\right\|=\left\|G_{s}\right\|+$ $\left\|T_{t}\right\|+\left|A_{t}\right|=t^{3} / 6+t^{3} / 2+O\left(t^{2}\right) \sim 2 t^{3} / 3 \sim(4 \sqrt{2} / 3) n^{3 / 2}$ due to (8).

Proposition 12. The complement of a (large) minimal irregular digraph is a (small) maximal irregular digraph, and conversely.

Using (11) we define

$$
\Phi_{n}= \begin{cases}\overline{F_{s}} & \text { if } n=s=s_{t}(m=0),  \tag{12}\\ \overline{F_{n}^{\prime}} & \text { if } n=s+m, s=s_{t} \text { and } 0<m \leq t\end{cases}
$$

Theorem 13. We refer to (12), (11) and Construction 3 under the assumptions (2) and (9). The digraph $\Phi_{n}$ is a small maximal irregular digraph of order $n$ and with size $\left\|\Phi_{n}\right\|=\Theta\left(n^{3 / 2}\right)$ where $\left\|\Phi_{s}\right\| \sim(4 \sqrt{2} / 3) s^{3 / 2}$ if $n=s(m=0)$. The largest asymptotic coefficient is if $n=s+t(m=t)$ and then $\left\|\Phi_{n}\right\| \sim$ $(10 \sqrt{2} / 3) n^{3 / 2}$.
Proof. The digraph $\Phi_{n}$ is maximal irregular due to Proposition 12 and Theorems 8 and 9. Note that, due to (12) and (10), $\left\|\Phi_{s}\right\|=\left\|K_{s}^{*}\right\|-\left\|F_{s}\right\|=s(t-1)+\left\|U_{s}\right\|$.

Hence, due to (4) and (5),

$$
\begin{equation*}
\left\|\Phi_{s}\right\| \sim s \sqrt{2 s}+\frac{\sqrt{2}}{3} s^{3 / 2}=4 \frac{\sqrt{2}}{3} s^{3 / 2} \tag{13}
\end{equation*}
$$

Next if $n=s+m$ and $0<m \leq t$ then, due to (11) and (10), $\left\|\Phi_{s+m}\right\|=$ $\left\|\overline{F_{s} \cup S_{m}}\right\|=\left\|K_{s+m}^{*}\right\|-\left\|F_{s}\right\|-\left\|S_{m}\right\|=s(s-1)-\left\|F_{s}\right\|+2 s m+m(m-1)-\left\|S_{m}\right\|=$ $\left\|\Phi_{s}\right\|+2 s m+\left\|S S_{m}\right\|=\left\|\Phi_{s}\right\|+2 s m+O\left(m^{2}\right)$. Using this equality for $m=t$ and $n=s+t$, and using asymptotic formulas (8) and (13), we get $O\left(m^{2}\right)=O(s)$, $2 s m=2 s t \sim(2 s)^{3 / 2}$, and $\left\|\Phi_{s+t}\right\| \sim(4 \sqrt{2} / 3) s^{3 / 2}+2 \sqrt{2} s^{3 / 2}=(10 \sqrt{2} / 3) s^{3 / 2}$, which ends the proof because $s \sim n$.

## 4. Concluding Remarks

Constructed in this paper are large minimal irregular oriented graphs $H_{n}$ (respectively digraphs $F_{n}$ ) of order $n \geq 21$ on one hand, and small maximal irregular oriented graphs $G_{n} /$ digraphs $\Phi_{n}$, both of order $n \geq 6$, on the other hand. They have sizes which are asymptotically best possible as compared to the corresponding smallest/largest irregular structures, see Theorems 7, 9, 11, 13 versus Theorems 1, 3 and asymptotics in formulas (1), (5) and (7).

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