

NOTE

ALMOST SELF-COMPLEMENTARY UNIFORM HYPERGRAPHS¹

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Abstract

A k -uniform hypergraph (k -hypergraph) is *almost self-complementary* if it is isomorphic with its complement in the complete k -uniform hypergraph minus one edge. We prove that an almost self-complementary k -hypergraph of order n exists if and only if $\binom{n}{k}$ is odd.

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1. INTRODUCTION

Let k be a positive integer and let V be a set of order $n > k$. By $\binom{V}{k}$ we denote the set of all k -subsets of V . A k -hypergraph with vertex set V and edge set $E \subset \binom{V}{k}$ is a pair $H = (V; E)$. The k -hypergraph $K_n^{(k)} = \left(V; \binom{V}{k}\right)$ is called *complete k -hypergraph*. The *complement of H in $K_n^{(k)}$* is then the k -hypergraph $\overline{H} = (V; \overline{E})$, where $\overline{E} = \binom{V}{k} - E$.

For a permutation σ of the set V we define the mapping σ^* on the set of edges by the formula $\sigma^*(e) = \{\sigma(x) : x \in e\}$, for every $e \in \binom{V}{k}$. If σ^* restricted to E induces a bijection onto $E' \subset \binom{V}{k}$ then we say that σ is an *isomorphism* of the k -hypergraph $H = (V; E)$ on $H' = (V; E')$. H and H' are then said to be *isomorphic*.

A k -hypergraph H is said to be *self-complementary* if it is isomorphic with $\overline{H}$. Clearly, if a k -hypergraph of order n is self-complementary then $\binom{n}{k}$ is even. It turns out that also a converse, in a sense, is true.

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Theorem 1.1 [8]. *Let n and k be positive integers, $k \leq n$. There is a k -uniform self-complementary hypergraph of order n if and only if $\binom{n}{k}$ is even.*

A k -uniform hypergraph $H = (V; E)$ is called *almost self-complementary* if it is isomorphic with $H' = \left(V; \binom{V}{k} - E - \{e_0\}\right)$ where e_0 is an element of the set $\binom{V}{k} - E$. Almost self-complementary k -hypergraphs of order n may be also called *self-complementary in $K_n^{(k)} - e_0$* . For $k = 2$ the almost self-complementary 2-hypergraphs are almost self-complementary graphs in $K_n - e_0$ defined by Clapham in [1]. The almost self-complementary 3-hypergraphs are considered in [3].

It is clear that an almost self-complementary k -hypergraph of order n may exist only when $\binom{n}{k}$ is odd. In Section 3 of the paper we prove the theorem generalizing the corresponding results of [1] for $k = 2$ and of [3] for $k = 3$.

Theorem 1.2. *Let n and k be positive integers, $k < n$. An almost self-complementary k -hypergraph of order n exists if and only if $\binom{n}{k}$ is odd.*

2. AUXILIARY RESULTS

In this section we give some results which are useful while proving Theorem 1.2.

The following is in fact a corollary of a celebrated theorem of Kummer [5] (see also [7] and [4]²).

Theorem 2.1 [5, 4]. *Let n and k be positive integers, $k < n$, $n = \sum_{i=0}^{\infty} a_i 2^i$, $k = \sum_{i=0}^{\infty} b_i 2^i$, where $a_i, b_i \in \{0, 1\}$ for all i . $\binom{n}{k}$ is odd if and only if $b_i \leq a_i$ for every i .*

For an integer $l = 2^p(2q + 1)$ we write $C_2(l) = p$ ($C_2(l)$ is the maximum integer p such that 2^p divides l). We shall use the following lemma which an immediate consequence of Lemmas 4 and 5 proved in [9].

Lemma 2.2. *Let k and m be positive integers, $k < m$, and let $\sigma = (1, 2, \dots, m)$ be a cyclic permutation. If $C_2(k) < C_2(m)$, then for every k -subset $e \subset \{1, 2, \dots, m\}$ and for every $q \in \mathbb{Z}$ we have $(\sigma^*)^{2q+1}(e) \neq e$.*

3. PROOF OF THEOREM 1.2

Let us suppose that $\binom{n}{k}$ is odd. We shall construct an almost self-complementary k -hypergraph of order n .

²Note that, very probably, neither the authors of [4], nor Glaisher [2] and Lucas [6] were aware of the result of Kummer while writing about the divisibility of Newton coefficients.

Write k and n in base-2 numeral system:

$$k = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_s},$$

$$n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_t},$$

where $k_1 < k_2 < \cdots < k_s$ and $n_1 < n_2 < \cdots < n_t$.

By Theorem 2.1, for every $i \in \{1, 2, \dots, s\}$ there exists $j_i \in \{1, 2, \dots, t\}$ such that $n_{j_i} = k_i$ (the sequence $(k_i)_{i=1}^s$ is a subsequence of $(n_j)_{j=1}^t$).

Let V be a set of cardinality n and let $V = V_1 \cup V_2 \cup \cdots \cup V_t$ be such a partition of V that $|V_j| = 2^{n_j}$, $V_j = \{a_1^j, a_2^j, \dots, a_{2^{n_j}}^j\}$ say, for $j = 1, 2, \dots, t$.

Set $e_0 = V_{j_1} \cup V_{j_2} \cup \cdots \cup V_{j_s}$. We see at once that $|e_0| = k$. Write

$$\sigma_j = (a_1^j, a_2^j, \dots, a_{2^{n_j}}^j)$$

for every $j = 1, 2, \dots, t$ and

$$\sigma = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_t.$$

Let e be any k -subset of V . If $e = e_0$ then $\sigma^*(e) = e$. If $e \neq e_0$ then there is an index $l_0 \in \{1, 2, \dots, t\}$ such that $e \cap V_{l_0} \neq \emptyset$ and $e \neq V_{l_0}$. It is easily seen that

$$C_2(|e \cap V_{l_0}|) < C_2(|V_{l_0}|).$$

By Lemma 2.2, for every integer q

$$(\sigma^*)^{2q+1}(|e \cap V_{l_0}|) \neq |e \cap V_{l_0}|$$

and by consequence

$$(\sigma^*)^{2q+1}(e) \neq e.$$

We color now all the k -subsets of V , except e_0 , with colors blue and red by applying the following algorithm.

If e is a k -subset of V which is not yet colored and $e \neq e_0$, then color all the edges of the form $(\sigma^*)^{2m+1}(e)$ red, and all the edges of the form $(\sigma^*)^{2m}(e)$ blue.

It is clear that we may color in this way all the k -subsets of V except e_0 and that the hypergraph H_b with the vertex set V and the set E_b of edges consisting of all the blue k -subsets of V is isomorphic with the k -hypergraph H_r with vertex set V and edge set E_r consisting of all the red k -subsets V . Moreover, σ is an isomorphism of $H_b = (V; E_b)$ with $H_r = (V; E_r)$, $E_b \cap E_r = \emptyset$, $e_0 \notin E_b \cup E_r$ and $E_b \cup E_r \cup \{e_0\} = \binom{V}{k}$. This finishes the proof.

It is easy to adopt the proof of Theorem 1.2 given above to obtain a proof of Theorem 1.1. This proof is much simpler than the one presented in [8].

REFERENCES

- [1] C.R.J. Clapham, *Graphs self-complementary in $K_n - e$* , Discrete Math. **81** (1990) 229–235.
doi:10.1016/0012-365X(90)90062-M
- [2] J.W.L. Glaisher, *On the residue of a binomial coefficient with respect to a prime modulus*, Quart. J. Math. **30** (1899) 150–156.
- [3] L.N. Kamble, C.M. Deshpande and B.Y. Bam, *Almost self-complementary 3-uniform hypergraphs*, Discuss. Math. Graph Theory **37** (2017) 131–140.
doi:10.7151/dmgt.1919
- [4] S.H. Kimball, T.R. Hatcher, J.A. Riley and L. Moser, *Solution to problem E1288 : Odd binomial coefficients*, Amer. Math. Monthly **65** (1958) 368–369.
doi:10.2307/2308812
- [5] E.E. Kummer, *Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen*, J. Reine Angew. Math. **44** (1852) 93–146.
doi:10.1515/crll.1852.44.93
- [6] E. Lucas, *Sur les congruences des nombres eulériens et des coefficients différentiels*, Bull. Soc. Math. France **6** (1878) 49–54.
doi:10.24033/bsmf.127
- [7] G. Ringenboim, *Fermat’s Last Theorem for Amateurs* (Springer Verlag, 1999).
- [8] A. Szymański and A.P. Wojda, *A note on k -uniform self-complementary hypergraphs of given order*, Discuss. Math. Graph Theory **29** (2009) 199–202.
doi:10.7151/dmgt.1440
- [9] A.P. Wojda, *Self-complementary hypergraphs*, Discuss. Math. Graph Theory **26** (2006) 217–224.
doi:10.7151/dmgt.1314

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