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GENERALIZED HAMMING GRAPHS: SOME NEW RESULTS

Amari Bedrane

and

Berrachedi Abdelhafid

LIFORCE Laboratery
RO Department, USTHB, Algiers, Algeria
e-mail: bedrane.amari@gmail.com

Abstract

A projection of a vertex x of a graph G over a subset S of vertices is a vertex of S at minimal distance from x. The study of projections over quasi-intervals gives rise to a new characterization of quasi-median graphs.

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1. Introduction

All graphs considered in this paper are finite, undirected, without loops or multiple edges. We denote by $d(u,v)$ the length of a shortest (u,v)-path in the graph G. The *interval* $I(u,v)$ is the set of vertices of G lying on shortest (u,v)-paths: $I(u,v) = \{x : d(u,x) + d(x,v) = d(u,v)\}$. The *quasi-interval* $I^*(u,v)$ is the set of vertices x such that any shortest (u,x)-path and shortest (x,v)-path have only x as common vertex. That is, $I^*(u,v) = \{x : I(u,x) \cap I(x,v) = \{x\}\}$. This notion was introduced by Nebeský [10]. The *projection* (introduced by Berrachedi [4]) of a vertex x of a graph G over a subset S of vertices, is a subset of vertices of S which are at minimal distance from x. It is denoted by $P(x,S)$. A graph G is *Hilbertian* if $|P(x,I(u,v))| = 1$, for all $u,v,x \in G$. A graph G is *quasi-Hilbertian* if, for all u,v and x in G, $|P(x,I^*(u,v))| = 1$. Quasi-median graphs have been introduced by Mulder [9] as a natural generalization of median graphs, in fact, median graphs are just the bipartite quasi-median graphs. Many researchers are interested in studying this class of graphs. Among prominent examples of median graphs let us mention hypercubes, trees and grids. Berrachedi [4] proved that a graph G is median if and only if G is Hilbertian. From the fact that a quasi-interval is an enlarged interval and in median graphs a quasi-interval is also an interval, then another generalization of Hilbertian graphs is to consider graphs which are quasi-Hilbertian. In this paper, our aim is to show that the class of quasi-median graphs is the same as the class of quasi-Hilbertian graphs.

2. Preliminaries

In this section, we recall some classical definitions and notation following that of [7, 9]. Then we give a mini-review of some interesting results on median graphs, and results obtained analogously for quasi-median graphs. A connected subgraph H of a graph G is called *convex* if for any two vertices u and v from H all shortest (u, v)-paths are contained in H. The *convex closure* of a subgraph H of G is defined as the smallest convex subgraph of G which contains H. The *Cartesian product* $G\Box H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G\Box H)$ whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. *A clique* in G is a set of vertices $K \subseteq V(G)$ in which any two distinct vertices are adjacent. If K is a clique and $K = V(G)$, then G is the *complete graph* K_n, where n is the number of vertices of G. The graph $K_4 - e$ is the complete graph on four vertices minus an edge. $K_{n,m}$ is the *complete bipartite graph*, where n and m are the number of vertices of the first and the second part of the partition. For $u \in V(G)$, $N(u)$ is the set of vertices adjacent to the vertex u. A Cartesian product of complete graphs is called a *Hamming graph*, a Cartesian power of the K_2 is called a *hypercube*. A graph G satisfies the *triangle property* if for any vertices $u, x, y \in V(G)$, where $d(u, x) = d(u, y) = k$ such that $xy \in E(G)$, there exists a common neighbour v of x and y with $d(u, v) = k - 1$. A graph G satisfies the *quadrangle property* if for any $u, x, y, z \in V(G)$ such that $d(u, x) = d(u, y) = d(u, z) - 1$ and $d(x, y) = 2$, with z a common neighbour of x and y, there exists a common neighbour v of x and y such that $d(u, v) = d(u, x) - 1$. A graph which fulfils the quadrangle property and the triangle property is called a *weakly modular graph*.

2.1. Median graphs

A vertex x is a *median* of the triple of vertices u, v and w if

1. $d(u, x) + d(x, v) = d(u, v)$;
2. $d(v, x) + d(x, w) = d(v, w)$;

3. $d(w, x) + d(x, u) = d(w, u)$.

A graph G is a *median graph* if any three vertices u, v and w in G have a unique median. Mulder gave the following characterization of median graphs using the procedure of convex expansions, see [9] for the necessary details.

Theorem 1 (Mulder [9]). *A graph G is a median graph if and only if G can be obtained from K_1 by a sequence of convex expansions.*

Theorem 2 (Mulder [8]). *A graph G is a hypercube if and only if G is a regular median graph.*

A *retraction* f from a graph G to a subgraph H is a mapping f of the vertex set $V(G)$ of G onto the vertex set $V(H)$ of H such that for every edge uv in G the image $f(u)f(v)$ is an edge in H, and $f(w) = w$ for all vertices w of H. Using retraction, Bandelt [2] characterized hypercubes as median graphs.

Theorem 3 (Bandelt [2]). *The retracts of hypercubes are precisely the median graphs.*

Berrachedi in [4] introduced the class of Hilbertian graphs, using projections over intervals, he showed the following.

Theorem 4 (Berrachedi [4]). *Let G be a graph. Then G is Hilbertian if and only if G is a median graph.*

Other characterizations of median graphs using projections over intervals and convex sets are given by Berrachedi and Mollard in [5].

2.2. Quasi-median graphs

A triple of vertices (x, y, z) is a quasi-median of (u, v, w) if we have:

1. $d(u, x) + d(x, y) + d(y, v) = d(u, v)$;
   $d(v, y) + d(y, z) + d(z, w) = d(v, w)$;
   $d(w, z) + d(z, x) + d(x, u) = d(w, u)$.
2. $d(x, y) = d(y, z) = d(z, x) = k$.
3. k is minimal under the two above conditions.

Mulder [9] defines a quasi-median graph G as follows.

(i) Each ordered triple of vertices of G has a unique quasi-median;

(ii) G does not admit $K_4 - e$ as induced subgraph;

(iii) Each induced C_6 in G has $K_3 \Box K_3$ or Q_3 as convex closure.

He characterized the quasi-median graphs with the quasi-median expansion procedure.

Theorem 5 (Mulder [9]). *A graph G is quasi-median if and only if G can be obtained from K_1 by a sequence of quasi-median expansions.*

Theorem 6 (Mulder [9]). *A graph G is a Hamming graph if and only if G is a regular quasi-median graph.*

Theorem 7 (Wilkeit [11]). *The retracts of Hamming graphs are precisely the quasi-median graphs.*

Chung *et al.* [6], characterized quasi-median graphs as weakly modular graphs without $K_4 - e$ or $K_{2,3}$ as induced subgraph.

Theorem 8 (Chung *et al.* [6]). *A graph G is quasi-median if and only if G is weakly modular and does not contain $K_4 - e$ or $K_{2,3}$ as an induced subgraph.*

More characterizations of quasi-median graphs can be found in [1, 3, 6, 9, 11].

3. Quasi-Hilbertian Graphs

In this section we shall prove that quasi-Hilbertian graphs are precisely quasi-median graphs. Chung *et al.* [6], established a relation between the quasi-median graphs and weakly modular graphs. We use their relation and some proprieties of quasi-Hilbertian graphs to prove that quasi-Hilbertian graphs are precisely quasi-median graphs.

Theorem 9 (the main result). *A graph G is a quasi-median graph if and only if G is a quasi-Hilbertian graph.*

This Theorem will be proved using a series of Lemmas that follow.

Lemma 10. *A quasi-median graph is quasi-Hilbertian.*

Proof. Let u, v, w be three vertices of a quasi-median graph G. We assume that $P(u, I^*(v, w))$ contains at least two vertices x and x'. We take the triple (x, v, w). As known in [9], there exists a unique vertex y in $I(x, v) \cap I(v, w)$ with $I(x, v) \cap I(v, w) = I(v, y)$. Also, with the triple (x, y, w) we get $I(x, w) \cap I(y, w) = I(w, z)$. In the same way, starting by the triple (x', v, w), we find $I(x', v) \cap I(v, w) = I(v, y')$ and $I(x', w) \cap I(y', w) = I(w, z')$. Thus, (x, y, z) and (x', y', z') are two quasi-medians of (u, v, w) in G, which is a contradiction. ∎

Lemma 11. *A quasi-Hilbertian graph is $K_{2,3}$-free.*

Proof. Let u, v, w, x and y be five vertices that induce a $K_{2,3}$ in the quasi-Hilbertian graph G. Let v, w and u be the vertices of degree 2. Consider the quasi-interval $I^*(v, w)$. Since $I(v, u) \cap I(u, w) \supseteq \{u, x, y\}$, $u \notin I^*(v, w)$. The vertices v, w, x and y are in $I^*(v, w)$. As $d(u, x) = d(u, y) = 1$, $P(u, I^*(v, w)) \supseteq \{x, y\}$. This contradicts the fact that G is a quasi-Hilbertian graph. ∎

Lemma 12. *A quasi-Hilbertian graph is $K_4 - e$-free.*

Proof. Let u, v, w and z be four vertices that induce a $K_4 - e$ in the quasi-Hilbertian graph G. Let u and w be the vertices of degree 2. Consider the quasi-interval $I^*(v, w)$. The vertices v, w and z are in $I^*(v, w)$, but $u \notin I^*(v, w)$. As $d(u, v) = d(u, z) = 1$, $P(u, I^*(v, w)) \supseteq \{v, z\}$. This contradicts the fact that G is a quasi-Hibertian graph. ∎

Lemma 13. *In a quasi-Hilbertian graph G, for all $vw \in E(G)$ and for all $x \in I^*(v, w) \setminus \{v, w\}$, we have $d(v, x) = d(w, x) = 1$.*

Proof. By contrary. Let vw be an edge in a quasi-Hilbertian graph G and $x \in I^*(v, w) \setminus \{v, w\}$. Let us consider the two possible cases.

Case 1. $d(v, x) \neq d(w, x)$. We assume without loss of generality that $d(v, x) < d(w, x)$, then $d(v, x) + 1 \leq d(w, x)$, which implies that $I(v, x) \subset I(w, x)$. Thus $I(v, x) \cap I(w, x) = I(v, x)$, this is a contradiction with $x \in I^*(v, w) \setminus \{v, w\}$.

Case 2. $d(v, x) = d(w, x) > 1$. We suppose that $d(v, x)$ is minimal. Let x_1 be a vertex in $I(v, x) \cap N(v)$. As $I(x_1, v) \cap I(v, x) = \{v, x_1\}$, $v \notin I^*(x_1, x)$. $I(x, w) \cap I(w, x_1) \neq \{w\}$, otherwise $P(v, I^*(x, x_1)) \supseteq \{w, x_1\}$. Necessarily, there exists $x_2 \in I(x, w) \cap I(w, x_1) \setminus \{w\}$ and $d(x_1, x_2) = 1$. If $v \in N(x_2)$ and $w \notin N(x_1)$, then $K_4 - e$ will be an induced subgraph. The same result holds if $v \notin N(x_2)$ and $w \in N(x_1)$. If $v \in N(x_2)$ and $w \in N(x_1)$, then $P(v, I^*(x, x_1)) \supseteq \{x_1, x_2\}$ and $P(w, I^*(x, x_1)) \supseteq \{x_1, x_2\}$. Thus $d(v, x_2) = d(w, x_1) = 2$. From the minimality of $d(v, x)$, we have $d(x, x_1) = d(x, x_2) = 1$, so that $P(x_1, I^*(v, w)) \supseteq \{v, x\}$. Contradiction with the fact that G is a quasi-Hilbertian graph. Consequently, we have $d(v, x) = d(w, x) = 1$, for all $x \in I^*(v, w) \setminus \{v, w\}$ with $vw \in E(G)$. ∎

Lemma 14. *For every two adjacent vertices v and w of a quasi-Hilbertian graph G, the quasi-interval $I^*(v, w)$ induces a complete subgraph.*

Proof. Let $I^*(v, w)$ be the quasi-interval such that $d(v, w) = 1$, and $x, y \in I^*(v, w)$ such that $x \neq y$. From Lemma 13, we have

$$\begin{cases} d(v, x) = d(w, x) = d(v, w) = 1, \\ d(v, y) = d(w, y) = d(v, w) = 1. \end{cases}$$

If $x = v$ or $x = w$, then $d(x, y) = 1$. The same result hold if $y = v$ or $y = w$. Else, if $d(x, y) \neq 1$, then the vertices v, w, x and y induce a forbidden $K_4 - e$. ∎

Lemma 15. *A quasi-Hilbertian graph satisfies the triangle property.*

Proof. Consider three vertices u, v and w of a quasi-Hilbertian graph such that $d(u,v) = d(u,w) = k$ and $d(w,v) = 1$. If $k = 1$, we have the triangle property. Suppose that $k \geq 2$. Since $I^*(w,v)$ induce a complete subgraph, u is not in $I^*(w,v)$. So, there exists x in $I(w,u) \cap I(u,v) \setminus \{u\}$ such that $x \in I^*(w,v)$. Hence, $d(x,v) = d(w,x) = 1$ and $d(u,x) = k-1$. ∎

Lemma 16. *A quasi-Hilbertian graph satisfies the quadrangle property.*

Proof. Let u, v, w and z be four vertices in a quasi-Hilbertian graph such that $d(u,v) = d(u,z) = d(u,w) - 1 = k$, $d(z,v) = 2$, and $w \in I(v,z)$.

Consider the quasi-interval $I^*(u,z)$. If $k = 1$, we have the quadrangle property. Suppose that $k \geq 2$. $I(u,v) \cap I(v,z) \neq \{v\}$, otherwise $P(w, I^*(u,z)) \supseteq \{z,v\}$. Necessarily, there exists $x \in I(z,v) \cap I(v,u) \setminus \{v\}$, then $d(z,x) = d(v,x) = 1$ and $d(u,x) = k-1$. ∎

***Proof of Theorem* 9.** From Lemma 10, a quasi-median graph is quasi-Hilbertian. As a quasi-Hilbertian graph is weakly modular (Lemmas 15 and 16), and does not contain $K_{2,3}$ or $K_4 - e$ as an induced subgraph, it is a quasi-median graph (Theorem 8). ∎

Theorems 9 and 6 give a new characterization of Hamming graphs.

Theorem 17. *A graph G is a Hamming graph if and only if*

$$\begin{cases} G \textit{ is regular}, \\ \textit{for all } \; u, v, w \in G \textit{ we have } |P(w, I^*(u,v)| = 1. \end{cases}$$

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