# DESCRIBING NEIGHBORHOODS OF 5-VERTICES IN 3-POLYTOPES WITH MINIMUM DEGREE 5 AND WITHOUT VERTICES OF DEGREES FROM 7 TO 11¹ 

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#### Abstract

In 1940, Lebesgue proved that every 3 -polytope contains a 5 -vertex for which the set of degrees of its neighbors is majorized by one of the following sequences:


$$
(6,6,7,7,7),(6,6,6,7,9),(6,6,6,6,11)
$$

$(5,6,7,7,8),(5,6,6,7,12),(5,6,6,8,10),(5,6,6,6,17)$,
$(5,5,7,7,13),(5,5,7,8,10),(5,5,6,7,27)$,
$(5,5,6,6, \infty),(5,5,6,8,15),(5,5,6,9,11)$,
$(5,5,5,7,41),(5,5,5,8,23),(5,5,5,9,17)$, $(5,5,5,10,14),(5,5,5,11,13)$.

In this paper we prove that every 3-polytope without vertices of degree from 7 to 11 contains a 5 -vertex for which the set of degrees of its neighbors is majorized by one of the following sequences: $(5,5,6,6, \infty),(5,6,6,6,15)$, $(6,6,6,6,6)$, where all parameters are tight.
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## 1. Introduction

By a 3-polytope we mean a finite 3-dimensional convex polytope. As proved by Steinitz [31], the 3 -polytopes are in one to one correspondence with the 3 connected planar graphs.

The degree $d(v)$ of a vertex $v(r(f)$ of a face $f)$ in a 3 -polytope $P$ is the number of edges incident with it. By $\Delta$ and $\delta$ we denote the maximum and minimum vertex degrees of $P$, respectively. A $k$-vertex ( $k$-face) is a vertex (face) with degree $k$; a $k^{+}$-vertex has degree at least $k$, etc.

The weight of a face in $P$ is the degree sum of its boundary vertices, and $w(P)$, or simply $w$, denotes the minimum weight of $5^{-}$-faces in $P$.

In 1904, Wernicke [32] proved that every 3 -polytope with $\delta=5$ has a 5 -vertex adjacent with a $6^{-}$-vertex, which was strengthened by Franklin [15] in 1922, who proved that every 3 -polytope with $\delta=5$ has a 5 -vertex adjacent with two $6^{-}$vertices. Recently, Borodin and Ivanova [11] proved that every such 3-polytope has also a vertex of degree at most 6 adjacent to a 5 -vertex and another vertex of degree at most 6 , which is tight.

We say that $v$ is a vertex of type $\left(k_{1}, k_{2}, \ldots\right)$ or simply a $\left(k_{1}, k_{2}, \ldots\right)$-vertex if the set of degrees of the vertices adjacent to $v$ is majorized by the vector $\left(k_{1}, k_{2}, \ldots\right)$. If the order of neighbors in the type is not important, then we put a line over the corresponding degrees. The following description of the neighborhoods of 5 -vertices in a 3 -polytope with $\delta=5$ was given by Lebesgue [28, p. 36] in 1940, which includes the results of Wernicke [32] and Franklin [15].

Theorem 1 (Lebesgue [28]). Every triangulated 3-polytope with minimum degree 5 contains a 5 -vertex of one of the following types:

$$
\begin{gathered}
(\overline{6,6,7,7,7}),(\overline{6,6,6,7,9}),(\overline{6,6,6,6,11}) \\
(\overline{5,6,7,7,8}),(5,6, \overline{6,7}, 11),(5,6, \overline{6,8}, 8) \\
(5,6, \overline{6,9}, 7),(5,7,6,6,12),(5,8,6,6,10),(5,6,6,6,17) \\
(5,5,7,7,8),(5,13,5,7,7),(5,10,5,7,8), \\
(5,8,5,7,9),(5,7,5,7,10),(5,7,5,8,8), \\
(5,5,7,6,12),(5,5,8,6,10),(5,6,5,7,12), \\
(5,6,5,8,10),(5,17,5,6,7),(5,11,5,6,8), \\
(5,11,5,6,9),(5,7,5,6,13),(5,8,5,6,11),(5,9,5,6,10),(5,6,6,5, \infty) \\
(5,5,7,5,41),(5,5,8,5,23),(5,5,9,5,17),(5,5,10,5,14),(5,5,11,5,13) .
\end{gathered}
$$

Theorem 1, along with other ideas in Lebesgue [28], has many applications to plane graph coloring problems (first examples of such applications and a recent survey can be found in [7,30]). Some parameters of Lebesgue's Theorem were improved for narrow classes of plane graphs. For example, in 1963, Kotzig [27] proved that every plane triangulation with $\delta=5$ satisfies $w \leq 18$ and conjectured
that $w \leq 17$. In 1989, Kotzig's conjecture was confirmed by Borodin [3] in a more general form.

Theorem 2 (Borodin [3]). Every 3-polytope with $\delta=5$ has a (5, 5, 7)-face or a $(5,6,6)$-face, where all parameters are tight.

By a minor $k$-star $S_{k}^{(m)}$ we mean a star with $k$ rays centered at a $5^{-}$-vertex. The Lebesgue's description [28, p.36] of the neighborhoods of 5 -vertices in 3polytopes with minimum degree $5, \mathbf{P}_{5}$, shows that there is a 5 -vertex with three $8^{-}$-neighbors. Another corollary of Lebesgue's description [28] is that $w\left(S_{3}^{(m)}\right) \leq 24$, which was improved in 1996 by Jendrol' and Madaras [23] to the sharp bound $w\left(S_{3}^{(m)}\right) \leq 23$. Furthermore, Jendrol' and Madaras [23] gave a precise description of minor 3 -stars in $\mathbf{P}_{5}$ : there is a $(6,6,6)$ - or $(5,6,7)$-star.

Also, Lebesgue [28] proved that $w\left(S_{4}^{(m)}\right) \leq 31$, which was strengthened by Borodin and Woodall [13] to the sharp bound $w\left(S_{4}^{(m)}\right) \leq 30$. Note that $w\left(S_{3}^{(m)}\right) \leq 23$ easily implies $w\left(S_{2}^{(m)}\right) \leq 17$ and immediately follows from $w\left(S_{4}^{(m)}\right) \leq 30$ (in both cases, it suffices to delete a vertex of maximum degree from a minor star of minimum weight). In [9], Borodin and Ivanova obtained a tight description of minor 4-stars in $\mathbf{P}_{5}$.

As for minor 5 -stars in $\mathbf{P}_{5}$, it follows from Lebesgue $[28$, p. 36$]$ that if there are no minor $(5,5,6,6)$-stars, then $w\left(S_{5}^{(m)}\right) \leq 68$ and $h\left(S_{5}^{(m)}\right) \leq 41$. Borodin, Ivanova, and Jensen [10] showed that the presence of minor (5,5,6,6)-stars can make $w\left(S_{5}^{(m)}\right)$ arbitrarily large and otherwise lowered Lebesgue's bounds to $w\left(S_{5}^{(m)}\right) \leq 55$ and $h\left(S_{5}^{(m)}\right) \leq 28$. On the other hand, a construction in [10] shows that $w\left(S_{5}^{(m)}\right) \geq 48$ and $h\left(S_{5}^{(m)}\right) \geq 20$. Recently, Borodin and Ivanova [12] proved that $w\left(S_{5}^{(m)}\right) \leq 51$ and $h\left(S_{5}^{(m)}\right) \leq 23$.

More results on the structure of edges and higher stars in various classes of 3 -polytopes can be found in $[1,2,4-6,8,9,14,16,19-22,24-26]$, with a detailed summary in [12].

In [28] Lebesgue did not give a proof of Theorem 1 and only gave its idea. In 2013, Ivanova and Nikiforov [17] gave a full proof of Theorem 1 and corrected the following imprecisions in its statement:
(1) in the type $(5,11,5,6,8)$ there should be 15 instead of 11 ;
(2) in the type $(5,17,5,6,7)$ there should be 27 instead of 17 ;
(3) in the type $(\overline{6,6,6,6,11})$ the line is not needed;
(4) instead of type $(\overline{5,6,7,7,8})$ there should be $(5,8, \overline{6,7,7})$ and $(5,7,6,8,7)$;
(5) the type $(5,6, \overline{6,9}, 7)$ is redundant;
(6) instead of $(5,5, \overline{7,7,8})$ it suffices to write $(5,5,7, \overline{7,8})$.

Later on, Ivanova and Nikiforov $[18,29]$ improved the corrected version of Theorem 1 by replacing 41 and 23 in the types ( $5,5,7,5,41$ ) and $(5,5,8,5,23)$ to 31 and 22 , respectively.

Theorem 3 (Ivanova, Nikiforov [17, 18, 29]). Every 3-polytope with minimum degree 5 contains a 5 -vertex of one of the following types:

$$
\begin{gathered}
(\overline{6,6,7,7,7}),(\overline{6,6,6,7,9}),(6,6,6,6,11), \overline{6}, 8) \\
(5,8,6,7,7),(5,7,6,8,7),(5,6, \overline{6,7}, 11),(5,6,6,8,8), \\
(5,7,6,6,12),(5,8,6,6,10),(5,6,6,6,17), \\
(5,5,7,7,8),(5,13,5,7,7),(5,10,5,7,8),(5,8,5,7,9) \\
(5,7,5,7,10),(5,7,5,8,8),(5,5,7,6,12),(5,5,8,6,10) \\
(5,6,5,7,12),(5,6,5,8,10),(5,27,5,6,7),(5,15,5,6,8), \\
(5,11,5,6,9),(5,7,5,6,13),(5,8,5,6,11),(5,9,5,6,10), \\
(5,6,6,5, \infty), \\
(5,5,7,5,31),(5,5,8,5,22),(5,5,9,5,17),(5,5,10,5,14),(5,5,11,5,13) .
\end{gathered}
$$

Theorem 1 subject to the corrections (1)-(6) implies the following fact.
Corollary 4. Every 3 -polytope with minimum degree 5 contains a 5 -vertex of one of the following types:

$$
\begin{gathered}
(\overline{6,6,7,7,7}),(\overline{6,6}, 6,7,9),(\overline{6,6,6,6,11}), \\
(\overline{5,6,7,7,8),(\overline{5,6,6,7,12}),(\overline{5,6,6,8,10}),(\overline{5,6,6,6,17}),} \\
(\overline{5,5,7,7,13}),(\overline{5,5,7,8,10}),(\overline{5,5,6,7,27}), \\
(\overline{5,5,6,6, \infty}),(\overline{5,5,6,8,15}),(\overline{5,5,6,9,11}), \\
(\overline{5,5,5,7,41}),(\overline{5,5,5,8,23}),(\overline{5,5,5,9,17}),(\overline{5,5,5,10,14}),(5,5,5,11,13) .
\end{gathered}
$$

We can see already from Theorem 1 that if vertices of degree from 7 to 11 are forbidden, then there is a 5 -vertex of one of the following types: $(\overline{5,5,6,6, \infty})$, $(\overline{5,6,6,6,17}),(6,6,6,6,6)$.

The purpose of this note is to obtain a precise description of 5 -stars in this subclass of $\mathbf{P}_{5}$.
Theorem 5. Every 3-polytope with minimum degree 5 and without vertices of degree from 7 to 11 contains a 5 -vertex of one of the following types: $(\overline{5,5,6,6, \infty})$, $(\overline{5,6,6,6,15}),(6,6,6,6,6)$, where all parameters are tight.

## 2. Proving Theorem 5

All parameters in Theorem 5 are best possible. Indeed, the following construction confirming the tightness of the type $(\overline{5,5,6,6, \infty})$ appears in [10]. Take three
concentric $n$-cycles $C^{i}=v_{1}^{i} \cdots v_{n}^{i}$, where $n$ is not limited and $1 \leq i \leq 3$, and join $C^{2}$ with $C^{1}$ by edges $v_{j}^{2} v_{j}^{1}$ and $v_{j}^{2} v_{j+1}^{1}$, where $1 \leq j \leq n$ (addition modulo $n$ ). Then do the same with $C^{2}$ and $C^{3}$. Finally, join all vertices of $C^{1}$ with a new $n$-vertex, and do the same for $C^{3}$.

The tightness of $(6,6,6,6,6)$ is confirmed by putting a 5 -vertex in each face of the dodecahedron.

To confirm the tightness of $(\overline{5,6,6,6,15})$, we take the dodecahedron and insert the fragment shown in Figure 1 into each face. As a result, we have a 3 -polytope with only ( $5,6,6,6,15$ )-vertices.


Figure 1. The insert in each face of the dodecahedron to produce a 3-polytope with 5 -vertices only of type $(5,6,6,6,15)$.

Now suppose a 3 -polytope $P^{\prime}$ is a counterexample to Theorem 5. Let $P$ be a counterexample on the same number of vertices with maximum possible number of edges.

Remark 6. In $P$, each $4^{+}$-face $f=v_{1} \cdots v_{d(f)}$ with $d\left(v_{1}\right)=5$ or $d\left(v_{1}\right) \geq 15$ satisfies $d\left(v_{i}\right) \geq 6$ whenever $3 \leq i \leq d(f)-1$. Otherwise, we could put a diagonal $v_{1} v_{i}$, which contradicts the maximality of $P$.
Corollary 7. In P, each $4^{+}$-face has at most two vertices with degree 5 and/or at least 15. Moreover, if there are precisely two such vertices, then they are adjacent to each other.

### 2.1. Discharging

The sets of vertices, edges, and faces of $P$ are denoted by $V, E$, and $F$, respectively. Euler's formula $|V|-|E|+|F|=2$ for $P$ implies

$$
\begin{equation*}
\sum_{v \in V}(d(v)-6)+\sum_{f \in F}(2 r(f)-6)=-12 \tag{1}
\end{equation*}
$$

We assign an initial charge $\mu(v)=d(v)-6$ to every vertex $v$ and $\mu(f)=$ $2 d(f)-6$ to every face $f$, so that only $5^{-}$-vertices have negative charge. Using the properties of $P$ as a counterexample, we define a local redistribution of charges, preserving their sum, such that the new charge $\mu^{\prime}(x)$ is non-negative whenever $x \in V \cup F$. This will contradict the fact that the sum of the new charges is, by (1), equal to -12 . The technique of discharging is often used in solving structural and coloring problems on plane graphs.

Let $v_{1}, \ldots, v_{d(v)}$ denote the neighbors of a vertex $v$ in a cyclic order round $v$, and let $f_{1}, \ldots, f_{d(v)}$ be the faces incident with $v$ in the same order.

We use the following rules of discharging (see Figure 2).


Figure 2. Rules of discharging.
R1. Every $4^{+}$-face gives 1 to every incident 5 -vertex.
R2. Every $12^{+}$-vertex $v$ gives a simplicial 5 -vertex $v_{2}$ the following charge through a face $f=v_{2} v v_{3}$ :
(a) $\frac{1}{4}$ if $d\left(v_{3}\right)=5$,
(b) $\frac{1}{2}$ if $d\left(v_{3}\right) \geq 6$, with the following exception.
(e) If $d(v) \geq 16, d\left(v_{1}\right)=5, d\left(v_{3}\right)=d(x)=d(y)=6$, where $v_{2}$ is incident to face $v_{2} x y$, then $v$ gives $\frac{2}{3}$ to $v_{2}$ through face $v_{2} v v_{3}$ and $\frac{1}{3}$ through face $v_{1} v v_{2}$.
R3. Suppose a simplicial 5 -vertex $v$ is adjacent to a 16 -vertex $v_{1}$, simplicial 5 vertices $v_{2}$ and $v_{5}$, and $v_{2}$ is surrounded by $v_{1}, v, v_{3}, x, y$, where $d\left(v_{3}\right)=d(x)=$ $d(y)=6$, (consequently $\left.d\left(v_{4}\right) \geq 12\right)$, while $v_{5}$ is surrounded by $v_{1}, v, v_{4}, w, z$, where $d(z) \geq 6$. Then $v$ gives $\frac{1}{4}$ to $v_{1}$.

### 2.2. Proving $\boldsymbol{\mu}^{\prime}(x) \geq 0$ whenever $x \in V \cup \boldsymbol{F}$

First consider a face $f$ in $P$. If $d(f)=3$, then $f$ does not participate in discharging, and so $\mu^{\prime}(v)=\mu(f)=2 \times 3-6=0$. Note that every $4^{+}$-face is incident with at most two 5 -vertices due to Corollary 7 , which implies that $\mu^{\prime}(v)=$ $2 d(f)-6-2 \times 1 \geq 0$ by R1.

Now let $v$ be a vertex in $P$.
Case 1. $d(v)=5$. If $v$ is incident with a $4^{+}$-face, then $\mu^{\prime}(v) \geq 5-6+1=0$ due to R1. In what follows we can assume that $v$ is simplicial.

Subcase 1.1. $v$ is incident only with $6^{+}$-vertices. Then there is at least one $v_{i}$ with $d\left(v_{i}\right) \geq 12$ due to the absence of $(6,6,6,6,6)$-vertices in $P$. Hence, $\mu^{\prime}(v) \geq$ $-1+2 \times \frac{1}{2}=0$ by R2(b).

Subcase 1.2. $v$ is incident with precisely one 5 -vertex. Since there is no ( $5,6,6$, $6,15)$-vertex in $P$, we can assume that $v$ has either at least two $12^{+}$-neighbors, or precisely one $16^{+}$-neighbor. So we have either $\mu^{\prime}(v) \geq-1+2 \times \frac{1}{2}+2 \times \frac{1}{4}>0$ by R2(a), (b), or $\mu^{\prime}(v)=-1+\frac{2}{3}+\frac{1}{3}=0$ by R2(e), respectively.

Subcase 1.3. $v$ is incident with at least two 5 -vertices. Note that now R2(e) is not applicable to $v$. Also note that $v$ cannot be incident with more than three 5 -vertices due to the absence of $(5,5,6,6, \infty)$-vertices in $P$, which implies that $v$ has at least two $12^{+}$-neighbors. If $v$ is incident with precisely three 5 -vertices, then we have $\mu^{\prime}(v) \geq-1+4 \times \frac{1}{4}=0$ by R2(a),(b).

Suppose $v$ is incident with precisely two 5 -vertices. If $v$ does not participate in R3, then $\mu^{\prime}(v) \geq-1+3 \times \frac{1}{4}+\frac{1}{2}>0$ by R2(a),(b). Note that if $v$ participates in R3, then it gives $\frac{1}{4}$ only to one 16 -neighbor, hence $\mu^{\prime}(v) \geq-1+3 \times \frac{1}{4}+\frac{1}{2}-\frac{1}{4}=0$.

Case 2. $d(v)=6$. Since $v$ does not participate in discharging, we have $\mu^{\prime}(v)=\mu^{\prime}(v)=6-6=0$.

Case 3. $12 \leq d(v) \leq 15$. Now R2(e) is not applicable to $v$, so $v$ sends at most $\frac{1}{2}$ through each face by R2(a),(b), which implies that $\mu^{\prime}(v) \geq d(v)-6-d(v) \times \frac{1}{2}=$ $\frac{d(v)-12}{2} \geq 0$.

Case 4. $16 \leq d(v) \leq 17$. Note that $v$ gives at most $\frac{2}{3}$ through each 3-face and only to a simplicial 5 -vertex. If $v$ gives nothing through at least one incident face, then $\mu^{\prime}(v) \geq 16-6-15 \times \frac{2}{3}=0$ by R1, R2. Further, we can assume that $v$ is simplicial and each face takes away some positive charge from $v$, which implies that each face at $v$ is incident with a 5 -vertex, and all 5 -vertices adjacent to $v$ are simplicial. Thus, $\mu^{\prime}(v) \geq d(v)-6-d(v) \times \frac{2}{3}=\frac{d(v)-18}{3}$, and we have the deficiency $\frac{1}{3}$ for a 17 -vertex and $\frac{2}{3}$ for a 16 -vertex with respect to donating $\frac{2}{3}$ per face.

Suppose $S_{k}=v_{1}, \ldots, v_{k}$ is a sequence of neighbors of $v$ with $d\left(v_{1}\right) \geq 6$, $d\left(v_{k}\right) \geq 6$, while $d\left(v_{i}\right)=5$ whenever $2 \leq i \leq k-1$ and $k \geq 3$, and $f_{1}, \ldots, f_{k-1}$ are the corresponding faces. (It is not excluded that $S_{k}=S_{d(v)}$, which happens when $v$ has precisely one $6^{+}$-neighbor.) We say that the sequence of faces $f_{1}, \ldots, f_{k-1}$ saves $\varepsilon$ with respect to the level of $\frac{2}{3}$ if these faces take away the total of $(k-$ 1) $\times \frac{2}{3}-\varepsilon$ from $v$.

Remark 8. Only $v_{2}$ and $v_{k-1}$ in $S_{k}$ can receive the charge $\frac{2}{3}$ from $v$ by R2(e), while each of the other 5 -vertices $v_{i}$ receives precisely $\frac{1}{4}$ from $v$ through each incident face. So, if $k \geq 5$, then $v_{2}$ receives at most 1 , and $v_{3}$ receives $\frac{1}{2}$ from $v$ through incident faces.

Remark 9. If $v$ is completely surrounded by 5 -vertices, then $\mu^{\prime}(v) \geq d(v)-6-$ $\frac{d(v)}{2}=\frac{d(v)-12}{2}>0$, and hence we can assume from now on that the neighborhood of $v$ is partitioned into $S_{k} \mathrm{~s}$.
(P1) If $k=3$, then $\varepsilon=\frac{1}{3}$. Indeed, here $v_{2}$ receives $\frac{1}{2}$ through each of the faces $v_{1} v v_{2}$ and $v_{2} v v_{3}$ by R2(b), whence $\varepsilon=2 \times \frac{2}{3}-2 \times \frac{1}{2}=\frac{1}{3}$.
(P2) If $k=4$, then $\varepsilon=0$. Now each of $v_{2}$ and $v_{3}$ receives at most 1 from $v$ by Remark 8 , so $\varepsilon=3 \times \frac{2}{3}-2=0$.
(P3) If $k=5$, then $\varepsilon=\frac{2}{3}$. Suppose $w_{1}, \ldots, w_{4}$ are the neighbors of $v_{1}, \ldots, v_{5}$ such that there are the faces $v_{i} w_{i} v_{i+1}$, where $1 \leq i \leq 4$.

If $v_{2}$ receives 1 by $\mathrm{R} 2(\mathrm{e})$, then $d\left(w_{1}\right)=d\left(w_{2}\right)=6$. Hence, $d\left(w_{3}\right) \geq 12$ due to the absence of a $(5,5,6,6, \infty)$-vertex in $P$, which implies that $v_{4}$ is adjacent to two $12^{+}$-vertices, whence it receives $\frac{1}{2}$ from $v$ through $f_{4}$ and $\frac{1}{4}$ through $f_{3}$. Moreover, $v_{3}$ gives $\frac{1}{4}$ to $v$ by R3. Hence, $\varepsilon=4 \times \frac{2}{3}-1-\frac{1}{2}-\frac{3}{4}+\frac{1}{4}=\frac{2}{3}$.

If R2(e) is not applicable to $v$, then $\varepsilon=4 \times \frac{2}{3}-4 \times \frac{1}{2}=\frac{2}{3}$.
$(\mathbf{P} 4)$ If $k=6$, then $\varepsilon=\frac{1}{3}$. Here, each of $v_{2}$ and $v_{5}$ receives at most 1 , while each of $v_{3}$ and $v_{4}$ receives $\frac{1}{2}$ from $v$ by Remark 8 , so $\varepsilon=5 \times \frac{2}{3}-2 \times 1-2 \times \frac{1}{2}=\frac{1}{3}$.
(P5) If $k=7$, then $\varepsilon=\frac{1}{2}$. Now we have $\varepsilon=6 \times \frac{2}{3}-2 \times 1-3 \times \frac{1}{2}=\frac{1}{2}$ by Remark 8.
(P6) If $k \geq 8$, then $\varepsilon \geq \frac{2}{3}$. Now we have $\varepsilon=(k-1) \times \frac{2}{3}-2 \times 1-(k-4) \times \frac{1}{2}=$ $\frac{k-4}{6} \geq \frac{2}{3}$.

If $d(v)=17$, then it suffices to assume that the neighborhood of $v$ consists of pairs of 5 -vertices separated from each other by $6^{+}$-vertices by (P1)-(P6) (since otherwise we pay off the deficiency), which is impossible due to the fact that 17 is not divisible by 3 .

Suppose that $d(v)=16$ and $\mu^{\prime}(v)<0$. As follows from (P1)-(P6), the neighborhood of $v$ can have at most one of the paths $S_{t+2}$ of $t$ vertices of degree 5 , where $t \in\{1,4,5\}$, while all other vertices are partitioned into pairs of 5 vertices separated from each other by 6 -vertices. Indeed, if there are either two paths with $t \in\{1,4,5\}$, or at least one path with $t=3$ or $t \geq 6$, then we can pay off the deficiency $\frac{2}{3}$, a contradiction. But none of these cases is possible due to the divisibility by 3 . Namely, if $t=1$ we have $16-2=14$ faces to be divided into triplets of faces with a sequence $S_{4}$ of neighbors of $v$ as in (P2), or $16-5=11$ and $16-6=10$ faces for $t=4$ and $t=5$, respectively; a contradiction.

Case 6. $d(v) \geq 18$. Now $\mu^{\prime}(v) \geq d(v)-6-d(v) \times \frac{2}{3}=\frac{d(v)-18}{3} \geq 0$ by R2.
Thus we have proved $\mu^{\prime}(x) \geq 0$ for every $x \in V \cup F$, which contradicts (1) and completes the proof of Theorem 5 .

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