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DESCRIBING NEIGHBORHOODS OF 5-VERTICES IN 3-POLYTOPES WITH MINIMUM DEGREE 5 AND WITHOUT VERTICES OF DEGREES FROM 7 TO 11[1]

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Abstract

In 1940, Lebesgue proved that every 3-polytope contains a 5-vertex for which the set of degrees of its neighbors is majorized by one of the following sequences:

$$(6,6,7,7,7),\ (6,6,6,7,9),\ (6,6,6,6,11),$$
$$(5,6,7,7,8),\ (5,6,6,7,12),\ (5,6,6,8,10),\ (5,6,6,6,17),$$
$$(5,5,7,7,13),\ (5,5,7,8,10),\ (5,5,6,7,27),$$
$$(5,5,6,6,\infty),\ (5,5,6,8,15),\ (5,5,6,9,11),$$
$$(5,5,5,7,41),\ (5,5,5,8,23),\ (5,5,5,9,17),$$
$$(5,5,5,10,14),\ (5,5,5,11,13).$$

In this paper we prove that every 3-polytope without vertices of degree from 7 to 11 contains a 5-vertex for which the set of degrees of its neighbors is majorized by one of the following sequences: $(5,5,6,6,\infty)$, $(5,6,6,6,15)$, $(6,6,6,6,6)$, where all parameters are tight.

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1. Introduction

By a 3-*polytope* we mean a finite 3-dimensional convex polytope. As proved by Steinitz [31], the 3-polytopes are in one to one correspondence with the 3-connected planar graphs.

The *degree* $d(v)$ of a vertex v ($r(f)$ of a face f) in a 3-polytope P is the number of edges incident with it. By Δ and δ we denote the maximum and minimum vertex degrees of P, respectively. A k-*vertex* (k-*face*) is a vertex (face) with degree k; a k^+-*vertex* has degree at least k, etc.

The *weight* of a face in P is the degree sum of its boundary vertices, and $w(P)$, or simply w, denotes the minimum weight of 5^--faces in P.

In 1904, Wernicke [32] proved that every 3-polytope with $\delta = 5$ has a 5-vertex adjacent with a 6^--vertex, which was strengthened by Franklin [15] in 1922, who proved that every 3-polytope with $\delta = 5$ has a 5-vertex adjacent with two 6^--vertices. Recently, Borodin and Ivanova [11] proved that every such 3-polytope has also a vertex of degree at most 6 adjacent to a 5-vertex and another vertex of degree at most 6, which is tight.

We say that v is a *vertex of type* $(k_1, k_2, \ldots)$ or simply a $(k_1, k_2, \ldots)$-*vertex* if the set of degrees of the vertices adjacent to v is majorized by the vector $(k_1, k_2, \ldots)$. If the order of neighbors in the type is not important, then we put a line over the corresponding degrees. The following description of the neighborhoods of 5-vertices in a 3-polytope with $\delta = 5$ was given by Lebesgue [28, p. 36] in 1940, which includes the results of Wernicke [32] and Franklin [15].

Theorem 1 (Lebesgue [28]). *Every triangulated 3-polytope with minimum degree 5 contains a 5-vertex of one of the following types:*

$$(\overline{6,6,7,7,7}),\ (\overline{6,6,6,7,9}),\ (\overline{6,6,6,6,11}),$$
$$(\overline{5,6,7,7,8}),\ (5,6,\overline{6,7},11),\ (5,6,\overline{6,8},8),$$
$$(5,6,\overline{6,9},7),\ (5,7,6,6,12),\ (5,8,6,6,10),\ (5,6,6,6,17),$$
$$(5,5,\overline{7,7,8}),\ (5,13,5,7,7),\ (5,10,5,7,8),$$
$$(5,8,5,7,9),\ (5,7,5,7,10),\ (5,7,5,8,8),$$
$$(5,5,7,6,12),\ (5,5,8,6,10),\ (5,6,5,7,12),$$
$$(5,6,5,8,10),\ (5,17,5,6,7),\ (5,11,5,6,8),$$
$$(5,11,5,6,9),\ (5,7,5,6,13),\ (5,8,5,6,11),\ (5,9,5,6,10),\ (5,6,6,5,\infty),$$
$$(5,5,7,5,41),\ (5,5,8,5,23),\ (5,5,9,5,17),\ (5,5,10,5,14),\ (5,5,11,5,13).$$

Theorem 1, along with other ideas in Lebesgue [28], has many applications to plane graph coloring problems (first examples of such applications and a recent survey can be found in [7, 30]). Some parameters of Lebesgue's Theorem were improved for narrow classes of plane graphs. For example, in 1963, Kotzig [27] proved that every plane triangulation with $\delta = 5$ satisfies $w \leq 18$ and conjectured that $w \leq 17$. In 1989, Kotzig's conjecture was confirmed by Borodin [3] in a more general form.

Theorem 2 (Borodin [3]). *Every* 3*-polytope with* $\delta = 5$ *has a* $(5,5,7)$*-face or a* $(5,6,6)$*-face, where all parameters are tight.*

By a *minor* k*-star* $S_k^{(m)}$ we mean a star with k rays centered at a 5^--vertex. The Lebesgue's description [28, p.36] of the neighborhoods of 5-vertices in 3-polytopes with minimum degree 5, $\mathbf{P}_5$, shows that there is a 5-vertex with three 8^--neighbors. Another corollary of Lebesgue's description [28] is that $w\left(S_3^{(m)}\right) \leq 24$, which was improved in 1996 by Jendrol' and Madaras [23] to the sharp bound $w\left(S_3^{(m)}\right) \leq 23$. Furthermore, Jendrol' and Madaras [23] gave a precise description of minor 3-stars in $\mathbf{P}_5$: there is a $(6,6,6)$- or $(5,6,7)$-star.

Also, Lebesgue [28] proved that $w\left(S_4^{(m)}\right) \leq 31$, which was strengthened by Borodin and Woodall [13] to the sharp bound $w\left(S_4^{(m)}\right) \leq 30$. Note that $w\left(S_3^{(m)}\right) \leq 23$ easily implies $w\left(S_2^{(m)}\right) \leq 17$ and immediately follows from $w\left(S_4^{(m)}\right) \leq 30$ (in both cases, it suffices to delete a vertex of maximum degree from a minor star of minimum weight). In [9], Borodin and Ivanova obtained a tight description of minor 4-stars in $\mathbf{P}_5$.

As for minor 5-stars in $\mathbf{P}_5$, it follows from Lebesgue [28, p. 36] that if there are no minor $(5,5,6,6)$-stars, then $w\left(S_5^{(m)}\right) \leq 68$ and $h\left(S_5^{(m)}\right) \leq 41$. Borodin, Ivanova, and Jensen [10] showed that the presence of minor $(5,5,6,6)$-stars can make $w\left(S_5^{(m)}\right)$ arbitrarily large and otherwise lowered Lebesgue's bounds to $w\left(S_5^{(m)}\right) \leq 55$ and $h\left(S_5^{(m)}\right) \leq 28$. On the other hand, a construction in [10] shows that $w(S_5^{(m)}) \geq 48$ and $h\left(S_5^{(m)}\right) \geq 20$. Recently, Borodin and Ivanova [12] proved that $w\left(S_5^{(m)}\right) \leq 51$ and $h\left(S_5^{(m)}\right) \leq 23$.

More results on the structure of edges and higher stars in various classes of 3-polytopes can be found in [1, 2, 4–6, 8, 9, 14, 16, 19–22, 24–26], with a detailed summary in [12].

In [28] Lebesgue did not give a proof of Theorem 1 and only gave its idea. In 2013, Ivanova and Nikiforov [17] gave a full proof of Theorem 1 and corrected the following imprecisions in its statement:

(1) in the type $(5,11,5,6,8)$ there should be 15 instead of 11;

(2) in the type $(5,17,5,6,7)$ there should be 27 instead of 17;

(3) in the type $(\overline{6,6,6,6,11})$ the line is not needed;

(4) instead of type $(\overline{5,6,7,7,8})$ there should be $(5,8,\overline{6,7,7})$ and $(5,7,6,8,7)$;

(5) the type $(5, 6, \overline{6, 9}, 7)$ is redundant;
(6) instead of $(5, 5, \overline{7, 7, 8})$ it suffices to write $(5, 5, 7, \overline{7, 8})$.

Later on, Ivanova and Nikiforov [18, 29] improved the corrected version of Theorem 1 by replacing 41 and 23 in the types $(5, 5, 7, 5, 41)$ and $(5, 5, 8, 5, 23)$ to 31 and 22, respectively.

Theorem 3 (Ivanova, Nikiforov [17, 18, 29])**.** *Every* 3*-polytope with minimum degree* 5 *contains a* 5*-vertex of one of the following types:*

$$(\overline{6, 6, 7, 7, 7}),\ (\overline{6, 6, 6, 7, 9}),\ (6, 6, 6, 6, 11),$$
$$(5, 8, \overline{6, 7, 7}),\ (5, 7, 6, 8, 7),\ (5, 6, \overline{6, 7}, 11),\ (5, 6, \overline{6, 8}, 8),$$
$$(5, 7, 6, 6, 12),\ (5, 8, 6, 6, 10),\ (5, 6, 6, 6, 17),$$
$$(5, 5, 7, \overline{7, 8}),\ (5, 13, 5, 7, 7),\ (5, 10, 5, 7, 8),\ (5, 8, 5, 7, 9),$$
$$(5, 7, 5, 7, 10), (5, 7, 5, 8, 8),\ (5, 5, 7, 6, 12),\ (5, 5, 8, 6, 10),$$
$$(5, 6, 5, 7, 12),\ (5, 6, 5, 8, 10),\ (5, 27, 5, 6, 7),\ (5, 15, 5, 6, 8),$$
$$(5, 11, 5, 6, 9),\ (5, 7, 5, 6, 13),\ (5, 8, 5, 6, 11),\ (5, 9, 5, 6, 10),$$
$$(5, 6, 6, 5, \infty),$$
$$(5, 5, 7, 5, 31),\ (5, 5, 8, 5, 22),\ (5, 5, 9, 5, 17),\ (5, 5, 10, 5, 14),\ (5, 5, 11, 5, 13).$$

Theorem 1 subject to the corrections (1)–(6) implies the following fact.

Corollary 4. *Every* 3*-polytope with minimum degree* 5 *contains a* 5*-vertex of one of the following types:*

$$(\overline{6, 6, 7, 7, 7}),\ (\overline{6, 6, 6, 7, 9}),\ (\overline{6, 6, 6, 6, 11}),$$
$$(\overline{5, 6, 7, 7, 8}),\ (\overline{5, 6, 6, 7, 12}),\ (\overline{5, 6, 6, 8, 10}),\ (\overline{5, 6, 6, 6, 17}),$$
$$(\overline{5, 5, 7, 7, 13}),\ (\overline{5, 5, 7, 8, 10}),\ (\overline{5, 5, 6, 7, 27}),$$
$$(\overline{5, 5, 6, 6, \infty}),\ (\overline{5, 5, 6, 8, 15}),\ (\overline{5, 5, 6, 9, 11}),$$
$$(\overline{5, 5, 5, 7, 41}),\ (\overline{5, 5, 5, 8, 23}),\ (\overline{5, 5, 5, 9, 17}),\ (\overline{5, 5, 5, 10, 14}),\ (\overline{5, 5, 5, 11, 13}).$$

We can see already from Theorem 1 that if vertices of degree from 7 to 11 are forbidden, then there is a 5-vertex of one of the following types: $(\overline{5, 5, 6, 6, \infty})$, $(\overline{5, 6, 6, 6, 17})$, $(6, 6, 6, 6, 6)$.

The purpose of this note is to obtain a precise description of 5-stars in this subclass of $\mathbf{P}_5$.

Theorem 5. *Every* 3*-polytope with minimum degree* 5 *and without vertices of degree from* 7 *to* 11 *contains a* 5*-vertex of one of the following types:* $(\overline{5, 5, 6, 6, \infty})$, $(\overline{5, 6, 6, 6, 15})$, $(6, 6, 6, 6, 6)$*, where all parameters are tight.*

2. Proving Theorem 5

All parameters in Theorem 5 are best possible. Indeed, the following construction confirming the tightness of the type $(\overline{5, 5, 6, 6, \infty})$ appears in [10]. Take three concentric n-cycles $C^i = v_1^i \cdots v_n^i$, where n is not limited and $1 \le i \le 3$, and join C^2 with C^1 by edges $v_j^2 v_j^1$ and $v_j^2 v_{j+1}^1$, where $1 \le j \le n$ (addition modulo n). Then do the same with C^2 and C^3. Finally, join all vertices of C^1 with a new n-vertex, and do the same for C^3.

The tightness of $(6, 6, 6, 6, 6)$ is confirmed by putting a 5-vertex in each face of the dodecahedron.

To confirm the tightness of $(\overline{5, 6, 6, 6, 15})$, we take the dodecahedron and insert the fragment shown in Figure 1 into each face. As a result, we have a 3-polytope with only $(5, 6, 6, 6, 15)$-vertices.

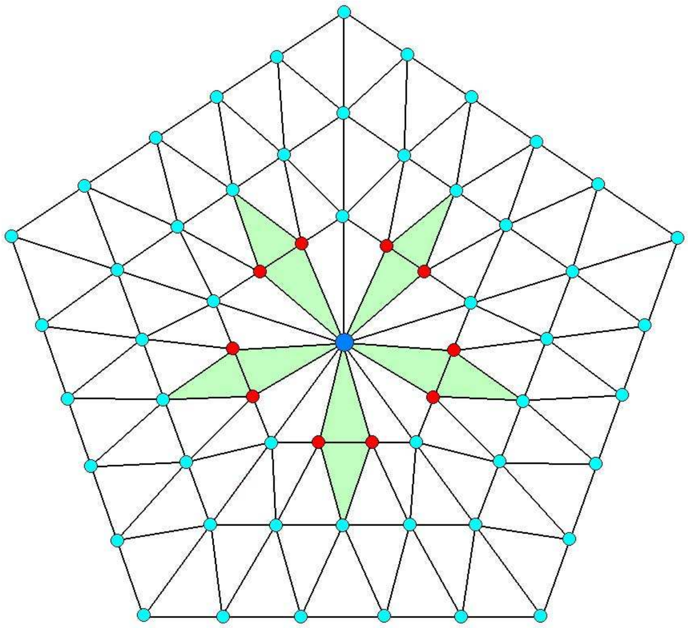

Figure 1. The insert in each face of the dodecahedron to produce a 3-polytope with 5-vertices only of type $(5, 6, 6, 6, 15)$.

Now suppose a 3-polytope P' is a counterexample to Theorem 5. Let P be a counterexample on the same number of vertices with maximum possible number of edges.

Remark 6. In P, each 4^+-face $f = v_1 \cdots v_{d(f)}$ with $d(v_1) = 5$ or $d(v_1) \ge 15$ satisfies $d(v_i) \ge 6$ whenever $3 \le i \le d(f) - 1$. Otherwise, we could put a diagonal $v_1 v_i$, which contradicts the maximality of P.

Corollary 7. *In P, each 4^+-face has at most two vertices with degree 5 and/or at least 15. Moreover, if there are precisely two such vertices, then they are adjacent to each other.*

2.1. Discharging

The sets of vertices, edges, and faces of P are denoted by V, E, and F, respectively. Euler's formula $|V| - |E| + |F| = 2$ for P implies

$$\sum_{v\in V}(d(v)-6)+\sum_{f\in F}(2r(f)-6)=-12. \tag{1}$$

We assign an *initial charge* $\mu(v) = d(v) - 6$ to every vertex v and $\mu(f) = 2d(f)-6$ to every face f, so that only 5^--vertices have negative charge. Using the properties of P as a counterexample, we define a local redistribution of charges, preserving their sum, such that the *new charge* $\mu'(x)$ is non-negative whenever $x \in V \cup F$. This will contradict the fact that the sum of the new charges is, by (1), equal to -12. The technique of discharging is often used in solving structural and coloring problems on plane graphs.

Let $v_1, \ldots, v_{d(v)}$ denote the neighbors of a vertex v in a cyclic order round v, and let $f_1, \ldots, f_{d(v)}$ be the faces incident with v in the same order.

We use the following rules of discharging (see Figure 2).

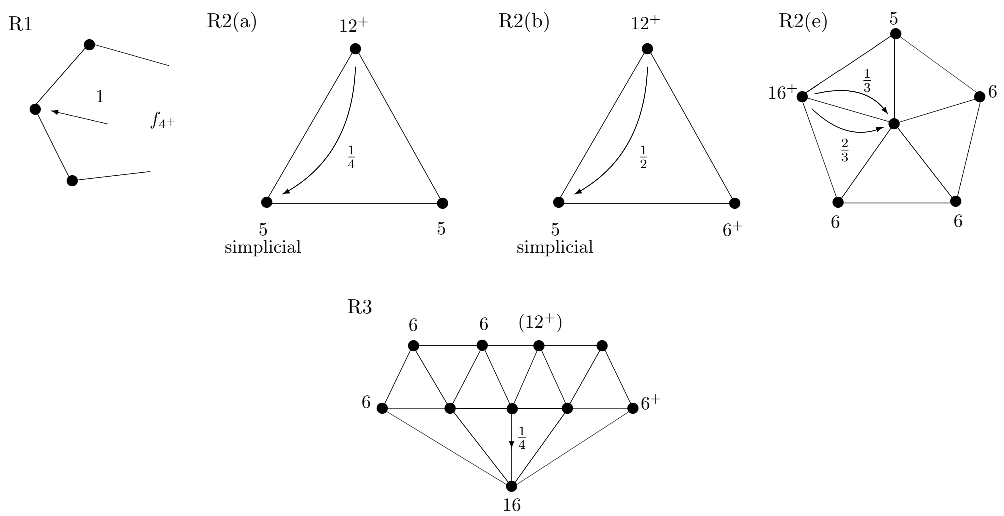

Figure 2. Rules of discharging.

R1. *Every* 4^+*-face gives* 1 *to every incident* 5*-vertex.*

R2. *Every* 12^+*-vertex* v *gives a simplicial* 5*-vertex* v_2 *the following charge through a face* $f = v_2vv_3$*:*

(a) $\frac{1}{4}$ *if* $d(v_3) = 5$,

(b) $\frac{1}{2}$ *if* $d(v_3) \geq 6$,

with the following exception.

(e) *If* $d(v) \geq 16$, $d(v_1) = 5$, $d(v_3) = d(x) = d(y) = 6$, *where* v_2 *is incident to face* v_2xy, *then* v *gives* $\frac{2}{3}$ *to* v_2 *through face* v_2vv_3 *and* $\frac{1}{3}$ *through face* v_1vv_2.

R3. *Suppose a simplicial* 5*-vertex* v *is adjacent to a* 16*-vertex* v_1, *simplicial* 5*-vertices* v_2 *and* v_5, *and* v_2 *is surrounded by* v_1, v, v_3, x, y, *where* $d(v_3) = d(x) = d(y) = 6$, (*consequently* $d(v_4) \geq 12$), *while* v_5 *is surrounded by* v_1, v, v_4, w, z, *where* $d(z) \geq 6$. *Then* v *gives* $\frac{1}{4}$ *to* v_1.

2.2. Proving $\mu'(x) \geq 0$ whenever $x \in V \cup F$

First consider a face f in P. If $d(f) = 3$, then f does not participate in discharging, and so $\mu'(v) = \mu(f) = 2 \times 3 - 6 = 0$. Note that every 4^+-face is incident with at most two 5-vertices due to Corollary 7, which implies that $\mu'(v) = 2d(f) - 6 - 2 \times 1 \geq 0$ by R1.

Now let v be a vertex in P.

Case 1. $d(v) = 5$. If v is incident with a 4^+-face, then $\mu'(v) \geq 5 - 6 + 1 = 0$ due to R1. In what follows we can assume that v is simplicial.

Subcase 1.1. v is incident only with 6^+-vertices. Then there is at least one v_i with $d(v_i) \geq 12$ due to the absence of $(6,6,6,6,6)$-vertices in P. Hence, $\mu'(v) \geq -1 + 2 \times \frac{1}{2} = 0$ by R2(b).

Subcase 1.2. v is incident with precisely one 5-vertex. Since there is no $(5,6,6,6,15)$-vertex in P, we can assume that v has either at least two 12^+-neighbors, or precisely one 16^+-neighbor. So we have either $\mu'(v) \geq -1 + 2 \times \frac{1}{2} + 2 \times \frac{1}{4} > 0$ by R2(a),(b), or $\mu'(v) = -1 + \frac{2}{3} + \frac{1}{3} = 0$ by R2(e), respectively.

Subcase 1.3. v is incident with at least two 5-vertices. Note that now R2(e) is not applicable to v. Also note that v cannot be incident with more than three 5-vertices due to the absence of $(5,5,6,6,\infty)$-vertices in P, which implies that v has at least two 12^+-neighbors. If v is incident with precisely three 5-vertices, then we have $\mu'(v) \geq -1 + 4 \times \frac{1}{4} = 0$ by R2(a),(b).

Suppose v is incident with precisely two 5-vertices. If v does not participate in R3, then $\mu'(v) \geq -1 + 3 \times \frac{1}{4} + \frac{1}{2} > 0$ by R2(a),(b). Note that if v participates in R3, then it gives $\frac{1}{4}$ only to one 16-neighbor, hence $\mu'(v) \geq -1 + 3 \times \frac{1}{4} + \frac{1}{2} - \frac{1}{4} = 0$.

Case 2. $d(v) = 6$. Since v does not participate in discharging, we have $\mu'(v) = \mu'(v) = 6 - 6 = 0$.

Case 3. $12 \leq d(v) \leq 15$. Now R2(e) is not applicable to v, so v sends at most $\frac{1}{2}$ through each face by R2(a),(b), which implies that $\mu'(v) \geq d(v) - 6 - d(v) \times \frac{1}{2} = \frac{d(v)-12}{2} \geq 0$.

Case 4. $16 \le d(v) \le 17$. Note that v gives at most $\frac{2}{3}$ through each 3-face and only to a simplicial 5-vertex. If v gives nothing through at least one incident face, then $\mu'(v) \ge 16 - 6 - 15 \times \frac{2}{3} = 0$ by R1, R2. Further, we can assume that v is simplicial and each face takes away some positive charge from v, which implies that each face at v is incident with a 5-vertex, and all 5-vertices adjacent to v are simplicial. Thus, $\mu'(v) \ge d(v) - 6 - d(v) \times \frac{2}{3} = \frac{d(v)-18}{3}$, and we have the deficiency $\frac{1}{3}$ for a 17-vertex and $\frac{2}{3}$ for a 16-vertex with respect to donating $\frac{2}{3}$ per face.

Suppose $S_k = v_1, \ldots, v_k$ is a sequence of neighbors of v with $d(v_1) \ge 6$, $d(v_k) \ge 6$, while $d(v_i) = 5$ whenever $2 \le i \le k-1$ and $k \ge 3$, and $f_1, \ldots, f_{k-1}$ are the corresponding faces. (It is not excluded that $S_k = S_{d(v)}$, which happens when v has precisely one 6^+-neighbor.) We say that the sequence of faces $f_1, \ldots, f_{k-1}$ *saves* ε with respect to the level of $\frac{2}{3}$ if these faces take away the total of $(k-1) \times \frac{2}{3} - \varepsilon$ from v.

Remark 8. Only v_2 and v_{k-1} in S_k can receive the charge $\frac{2}{3}$ from v by R2(e), while each of the other 5-vertices v_i receives precisely $\frac{1}{4}$ from v through each incident face. So, if $k \ge 5$, then v_2 receives at most 1, and v_3 receives $\frac{1}{2}$ from v through incident faces.

Remark 9. If v is completely surrounded by 5-vertices, then $\mu'(v) \ge d(v) - 6 - \frac{d(v)}{2} = \frac{d(v)-12}{2} > 0$, and hence we can assume from now on that the neighborhood of v is partitioned into S_ks.

(P1) *If $k = 3$, then $\varepsilon = \frac{1}{3}$.* Indeed, here v_2 receives $\frac{1}{2}$ through each of the faces v_1vv_2 and v_2vv_3 by R2(b), whence $\varepsilon = 2 \times \frac{2}{3} - 2 \times \frac{1}{2} = \frac{1}{3}$.

(P2) *If $k = 4$, then $\varepsilon = 0$.* Now each of v_2 and v_3 receives at most 1 from v by Remark 8, so $\varepsilon = 3 \times \frac{2}{3} - 2 = 0$.

(P3) *If $k = 5$, then $\varepsilon = \frac{2}{3}$.* Suppose $w_1, \ldots, w_4$ are the neighbors of $v_1, \ldots, v_5$ such that there are the faces $v_iw_iv_{i+1}$, where $1 \le i \le 4$.

If v_2 receives 1 by R2(e), then $d(w_1) = d(w_2) = 6$. Hence, $d(w_3) \ge 12$ due to the absence of a $(5,5,6,6,\infty)$-vertex in P, which implies that v_4 is adjacent to two 12^+-vertices, whence it receives $\frac{1}{2}$ from v through f_4 and $\frac{1}{4}$ through f_3. Moreover, v_3 gives $\frac{1}{4}$ to v by R3. Hence, $\varepsilon = 4 \times \frac{2}{3} - 1 - \frac{1}{2} - \frac{3}{4} + \frac{1}{4} = \frac{2}{3}$.

If R2(e) is not applicable to v, then $\varepsilon = 4 \times \frac{2}{3} - 4 \times \frac{1}{2} = \frac{2}{3}$.

(P4) *If $k = 6$, then $\varepsilon = \frac{1}{3}$.* Here, each of v_2 and v_5 receives at most 1, while each of v_3 and v_4 receives $\frac{1}{2}$ from v by Remark 8, so $\varepsilon = 5 \times \frac{2}{3} - 2 \times 1 - 2 \times \frac{1}{2} = \frac{1}{3}$.

(P5) *If $k = 7$, then $\varepsilon = \frac{1}{2}$.* Now we have $\varepsilon = 6 \times \frac{2}{3} - 2 \times 1 - 3 \times \frac{1}{2} = \frac{1}{2}$ by Remark 8.

(P6) *If* $k \geq 8$, *then* $\varepsilon \geq \frac{2}{3}$. Now we have $\varepsilon = (k-1) \times \frac{2}{3} - 2 \times 1 - (k-4) \times \frac{1}{2} = \frac{k-4}{6} \geq \frac{2}{3}$.

If $d(v) = 17$, then it suffices to assume that the neighborhood of v consists of pairs of 5-vertices separated from each other by 6^+-vertices by (P1)–(P6) (since otherwise we pay off the deficiency), which is impossible due to the fact that 17 is not divisible by 3.

Suppose that $d(v) = 16$ and $\mu'(v) < 0$. As follows from (P1)–(P6), the neighborhood of v can have at most one of the paths S_{t+2} of t vertices of degree 5, where $t \in \{1, 4, 5\}$, while all other vertices are partitioned into pairs of 5-vertices separated from each other by 6-vertices. Indeed, if there are either two paths with $t \in \{1, 4, 5\}$, or at least one path with $t = 3$ or $t \geq 6$, then we can pay off the deficiency $\frac{2}{3}$, a contradiction. But none of these cases is possible due to the divisibility by 3. Namely, if $t = 1$ we have $16 - 2 = 14$ faces to be divided into triplets of faces with a sequence S_4 of neighbors of v as in (P2), or $16 - 5 = 11$ and $16 - 6 = 10$ faces for $t = 4$ and $t = 5$, respectively; a contradiction.

Case 6. $d(v) \geq 18$. Now $\mu'(v) \geq d(v) - 6 - d(v) \times \frac{2}{3} = \frac{d(v)-18}{3} \geq 0$ by R2.

Thus we have proved $\mu'(x) \geq 0$ for every $x \in V \cup F$, which contradicts (1) and completes the proof of Theorem 5.

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