Discussiones Mathematicae
Graph Theory 38 (2018) 477–490
doi:10.7151/dmgt.2023

ARC-DISJOINT HAMILTONIAN CYCLES IN ROUND DECOMPOSABLE LOCALLY SEMICOMPLETE DIGRAPHS

Ruijuan Li [1] and Tingting Han

School of Mathematical Sciences
Shanxi University
030006 Taiyuan, P.R. China

e-mail: ruijuanli@sxu.edu.cn
tingtinghan19@163.com

Abstract

Let $D = (V, A)$ be a digraph; if there is at least one arc between every pair of distinct vertices of D, then D is a semicomplete digraph. A digraph D is locally semicomplete if for every vertex x, the out-neighbours of x induce a semicomplete digraph and the in-neighbours of x induce a semicomplete digraph. A locally semicomplete digraph without 2-cycle is a local tournament. In 2012, Bang-Jensen and Huang [J. Combin Theory Ser. B 102 (2012) 701–714] concluded that every 2-arc-strong locally semicomplete digraph which is not the second power of an even cycle has two arc-disjoint strong spanning subdigraphs, and proposed the conjecture that every 3-strong local tournament has two arc-disjoint Hamiltonian cycles. According to Bang-Jensen, Guo, Gutin and Volkmann, locally semicomplete digraphs have three subclasses: the round decomposable; the non-round decomposable which are not semicomplete; the non-round decomposable which are semicomplete. In this paper, we prove that every 3-strong round decomposable locally semicomplete digraph has two arc-disjoint Hamiltonian cycles, which implies that the conjecture holds for the round decomposable local tournaments. Also, we characterize the 2-strong round decomposable local tournaments each of which contains a Hamiltonian path P and a Hamiltonian cycle arc-disjoint from P.

Keywords: locally semicomplete digraph, local tournament, round decomposable, arc-disjoint, Hamiltonian cycle, Hamiltonian path.

2010 Mathematics Subject Classification: 05C20.

---

[1]Research is supported partially by NNSFC under no. 11401353, NSF of Shanxi Province, China under no. 2016011005 and TYAL of Shanxi.

1. Terminology and Introduction

In this paper, we consider finite digraph without loops and multiple arcs. The main source for terminology and notation is [1].

For an integer n, $[n]$ will denote the set $\{1, 2, 3, \ldots, n\}$.

Let $D = (V, A)$ be a digraph; if there is an arc from a vertex x to y, we say that x dominates y and denote it by $x \rightarrow y$. If V_1 and V_2 are arc-disjoint subsets of vertices of D such that there is no arc from V_2 to V_1 and $a \rightarrow b$ for all $a \in V_1$ and $b \in V_2$, then we say that V_1 completely dominates V_2 and denote this by $V_1 \Rightarrow V_2$. We shall use the same notation when A and B are subdigraphs of D. Let $N^-(x)$ (respectively, $N^+(x)$) denote the set of vertices dominating (respectively, dominated by) x in D and say that $N^-(x)$ (respectively, $N^+(x)$) is the in-neighbours of x (respectively, the out-neighbours of x).

Let H be a subdigraph of D; if $V(D) = V(H)$, we say that H is a spanning subdigraph of D. If every arc of $A(D)$ with both end-vertices in $V(H)$ is in $A(H)$, we say that H is induced by $X = V(H)$ and denote this by $D\langle X\rangle$. We also use the notation $D - X$, where $X \subseteq V$, for digraph $D\langle V(D)\backslash V(X)\rangle$.

Let D_1, D_2 be two subdigraphs of a digraph D. The union $D_1 \cup D_2$ is the digraph D with vertex set $V(D_1) \cup V(D_2)$ and arc set $A(D_1) \cup A(D_2)$.

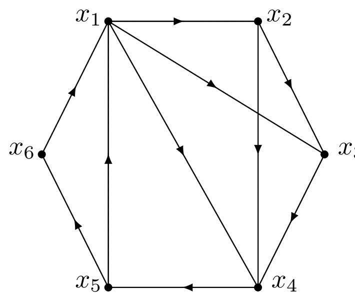
Paths and cycles in a digraph are always directed. Let P be a directed path of digraph D. If $V(P) = V(D)$, then P is a Hamiltonian path of D. Similarly, let C be a directed cycle of digraph D. If $V(C) = V(D)$, then C is a Hamiltonian cycle of D.

Let $P_1, P_2, \ldots, P_q$ be paths which are pairwise vertex-disjoint. If $\mathcal{F} = P_1 \cup P_2 \cup \cdots \cup P_q$ is a spanning subdigraph of D, then $\mathcal{F}$ is a q-path factor of D. Let $C_1, C_2, \ldots, C_q$ be cycles which are pairwise vertex-disjoint. If $\mathcal{F} = C_1 \cup C_2 \cup \cdots \cup C_q$ is a spanning subdigraph of D, then $\mathcal{F}$ is a q-cycle factor of D.

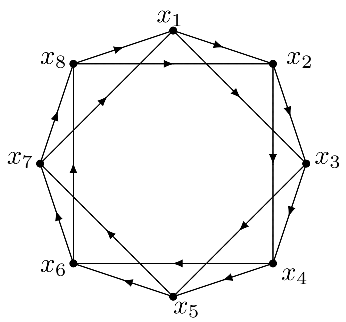
A digraph $D = (V, A)$ is called strongly connected (or just strong) if there exists a path from x to y and a path from y to x in D for every choice of distinct vertices x, y of D, and D is k-arc-strong (respectively, k-strong) if $D - X$ is strong for every subset $X \subseteq A$ (respectively, $X \subseteq V$) of size at most $k - 1$. Note that a digraph with only one vertex is strong.

A digraph D is semicomplete if, for every pair x, y of vertices of D, either x dominates y or y dominates x (or both). A digraph D is locally semicomplete if for every vertex x, the out-neighbours of x induce a semicomplete digraph and the in-neighbours of x induce a semicomplete digraph. A semicomplete digraph without 2-cycle is a tournament and a locally semicomplete digraph without 2-cycle is a local tournament.

A digraph R on r vertices is round if we can label its vertices $x_1, x_2, \ldots, x_r$ so that for each i, we have $N_R^+(x_i) = \left\{x_{i+1}, x_{i+2}, \ldots, x_{i+d_R^+(x_i)}\right\}$ and $N_R^-(x_i) = \left\{x_{i-d_R^-(x_i)}, \ldots, x_{i-1}\right\}$ (all subscripts are taken modulo r). Note that every round digraph is locally semicomplete, a round digraph without 2-cycle is a local tournament. If a local tournament R is round then there exists a unique (up to cyclic permutations) labeling of vertices of R which satisfies the properties in the definition. We refer to this as the round labeling of R. See Figure 1(a) for an example of a round digraph R. Observe that the ordering $x_1, x_2, \ldots, x_6$ is a round labeling of R. The second power of a cycle C_n, denoted by C_n^2, is the digraph obtained from C_n by adding the arcs $\{x_i x_{i+2} : i \in [n]\}$, where $C_n = x_1 x_2 \cdots x_n x_1$ and subscripts are modulo n. See Figure 1(b) for the second power of an 8-cycle.



(a) R



(b) C_8^2

Figure 1. A round digraph and the second power of an 8-cycle.

Let R be a digraph with vertex set $\{x_i : i \in [r]\}$, and let $D_1, D_2, \ldots, D_r$ be digraphs which are pairwise vertex-disjoint. Let $D = R[D_1, D_2, \ldots, D_r], r \geq 2$, be the new digraph obtained from R by replacing x_i with D_i and adding arc from every vertex of D_i to every vertex of D_j if and only if $x_i \to x_j$ in R. If R is a round digraph and each D_i is a strong semicomplete digraph, it is easy to see that $D = R[D_1, D_2, \ldots, D_r]$ is a locally semicomplete digraph. We call D a round decomposable locally semicomplete digraph. If a round decomposable locally semicomplete digraph $D = R[D_1, D_2, \ldots, D_r]$ has no 2-cycle (i.e., the round digraph R has no 2-cycle and each $D_i, i \in [r]$, is a strong tournament or a single vertex), then we say that D is a round decomposable local tournament.

Locally semicomplete digraphs were introduced in 1990 by Bang-Jensen [2]. The following theorem, due to Bang-Jensen, Guo, Gutin and Volkmann, states a full classification of locally semicomplete digraphs.

Theorem 1.1 [3]. *Let D be a locally semicomplete digraph. Then exactly one of the following possibilities holds.*

(a) *D is round decomposable with a unique round decomposition $R[D_1, D_2, \ldots, D_r]$, where R is a round locally semicomplete on $r \geq 2$ vertices and D_i is a strong semicomplete digraph for $i = 1, 2, \ldots, r$;*

(b) *D is not round decomposable and not semicomplete;*
(c) *D is a semicomplete digraph which is not round decomposable.*

In [3], Bang-Jensen *et al.* also characterized the structure of locally semicomplete digraph which is not round decomposable and not semicomplete. If D is restricted to a local tournament, we have the following result.

Corollary 1.2. *Let D be a local tournament. Then exactly one of the following possibilities holds.*

(a) *D is round decomposable with a unique round decomposition $R[D_1, D_2, \ldots, D_r]$, where R is a round local tournament on $r \geq 2$ vertices and D_i is a strong tournament for $i = 1, 2, \ldots, r$;*
(b) *D is not round decomposable and not a tournament;*
(c) *D is a tournament which is not round decomposable.*

According to the classification of locally semicomplete digraphs, many nice properties of semicomplete digraphs (tournaments) are generalized to locally semicomplete digraphs (local tournaments), see [5–8]. Recently, some new problems on locally semicomplete digraphs, such as the out-arc pancyclicity, the number of Hamiltonian cycles, the kings, the H-force set and so on, were studied in [9–13]. In particular, Bang-Jensen and Huang investigated the decomposition of locally semicomplete digraphs and proved the theorem below.

Theorem 1.3 [4]**.** *A 2-arc-strong locally semicomplete digraph D has two arc-disjoint strong spanning subdigraphs if and only if D is not the second power of an even cycle.*

Meanwhile, they proposed the following conjecture.

Conjecture 1.4 [4]**.** *Every 3-strong local tournament has two arc-disjoint Hamiltonian cycles.*

In this paper, we prove the following theorem in Section 3 which implies that the conjecture holds for the subclass of local tournaments—the round decomposable local tournaments.

Theorem 1.5. *Every 3-strong round decomposable locally semicomplete digraph has two arc-disjoint Hamiltonian cycles.*

Also, in the following theorem, we give a characterization of the 2-strong round decomposable local tournaments each of which contains a Hamiltonian path P and a Hamiltonian cycle arc-disjoint from P. This theorem will be proved in Section 4.

Theorem 1.6. *Every 2-strong round decomposable local tournament has a Hamiltonian path and a Hamiltonian cycle which are arc-disjoint if and only if it is not the second power of an even cycle.*

To show the main results, we introduce the following definition and theorem due to Thomassen. A tournament is called transitive if it contains no cycle. It is easy to see that, for a transitive tournament T, there is a unique vertex ordering $v_1, v_2, \ldots, v_n$ of T, such that $v_i \to v_j$ for all $1 \leq i < j \leq n$. A tournament is almost transitive if it is obtained from the transitive tournament T by reversing the arc $v_1 v_n$.

Theorem 1.7 [11]**.** *Every tournament which is strong and which is not an almost transitive tournament of odd order has two arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices.*

We also use the following facts several times.

Theorem 1.8 [1]**.** *Every strong semicomplete digraph is vertex-pancyclic.*

Theorem 1.9 [7]**.** *A tournament is strong if and only if it has a Hamiltonian cycle.*

2. Preliminaries

In this section we start with the following three lemmas which imply that every strong semicomplete digraph with at least 3 vertices contains a Hamiltonian path Q and a 2-path-factor $P' \cup P''$ arc-disjoint from Q such that the paths Q, P' and P'' have distinct initial vertices and distinct terminal vertices.

Lemma 2.1. *Let D be a strong semicomplete digraph with at least 3 vertices. Then D contains a spanning subdigraph which is a strong tournament.*

Proof. By Theorem 1.8, assume that C is a Hamiltonian cycle of D. Notice that at most one arc of each 2-cycle of D is in C. For each 2-cycle of D, by deleting exactly one arc which is not in C, we obtain a tournament T which is a spanning subdigraph of D and contains a Hamiltonian cycle. Note that a tournament is strong if and only if it has a Hamiltonian cycle. The proof is complete. ∎

Lemma 2.2. *Let T be a strong tournament which is not an almost transitive tournament of odd order. Then T contains a Hamiltonian path Q and a 2-path-factor $P' \cup P''$ arc-disjoint from Q such that the paths Q, P' and P'' have distinct initial vertices and distinct terminal vertices.*

Proof. Let $|V(T)| = n$. Since T is a strong tournament which is not an almost transitive tournament of odd order, if $n = 3$, then T is an almost transitive tournament of odd order. So $n \geq 4$. By Theorem 1.7, T contains a pair of arc-disjoint Hamiltonian paths P and Q such that P and Q have distinct initial vertices and distinct terminal vertices. Denote $P = v_1v_2 \cdots v_n$, $Q = u_1u_2 \cdots u_n$. Then $v_1 \neq u_1$, $v_n \neq u_n$. Let the vertex u_1 of Q correspond to the vertex v_i of P, and the vertex u_n of Q correspond to the vertex v_j of P. Note that $i > 1$, $j < n$ and $i \neq j$. Now we will construct a 2-path-factor $P' \cup P''$ arc-disjoint from Q such that Q, P' and P'' have distinct initial vertices and distinct terminal vertices.

Case 1. $i < j$. Let $P' = v_1v_2 \cdots v_i$ and $P'' = v_{i+1}v_{i+2} \cdots v_n$.

Case 2. $i > j, j \neq 1$. Let $P' = v_1v_2 \cdots v_{j-1}$ and $P'' = v_jv_{j+1} \cdots v_n$.

Case 3. $i > j, j = 1$ and $i \neq n$. Let $P' = v_1v_2 \cdots v_i$ and $P'' = v_{i+1}v_{i+2} \cdots v_n$.

Case 4. $i > j, j = 1$ and $i = n$. Let $P' = v_1v_2$ and $P'' = v_3v_4 \cdots v_n$.

Figure 2 shows the construction of the Hamiltonian path Q and the 2-path factor $P' \cup P''$ in the four cases. Notice that in all cases the paths Q, P' and P'' have distinct initial vertices and distinct terminal vertices, respectively, i.e., T contains a Hamiltonian path Q and a 2-path factor $P' \cup P''$ arc-disjoint from Q such that Q, P' and P'' have distinct initial vertices and distinct terminal vertices. We are done. ■

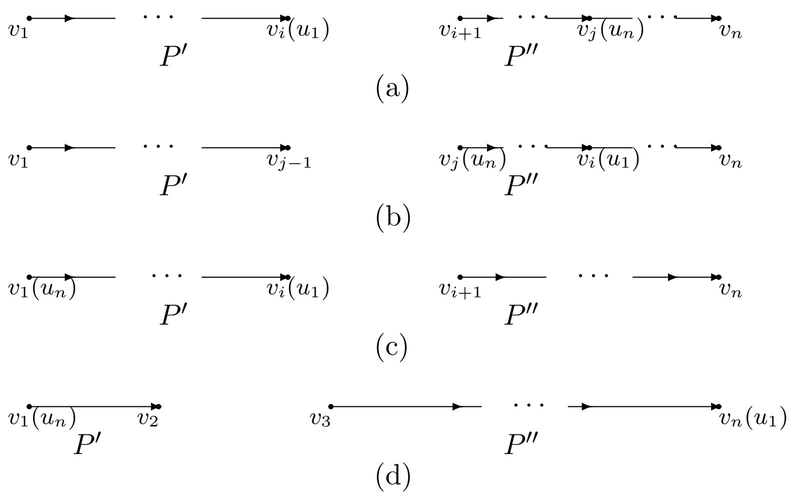


Figure 2. The 2-path factor constructed in the proof of Lemma 2.2.

Lemma 2.3. *Let T be a strong tournament which is an almost transitive tournament of odd order. Then T contains a Hamiltonian path Q and a 2-path factor $P' \cup P''$ arc-disjoint from Q such that the paths Q, P' and P'' have distinct initial vertices and distinct terminal vertices.*

Proof. Let $|V(T)| = n$ and $V(T) = \{v_1, v_2, \ldots, v_n\}$. Obviously, $n \geq 3$. Since T is an almost transitive tournament of odd order, assume without loss of generality that $v_i \to v_j$ for all $1 \leq i < j \leq n$ except for $v_n \to v_1$. Hence, for arbitrary $i \leq n-2$, we must have $v_i \to v_{i+1}$ and $v_i \to v_{i+2}$.

Case 1. $n = 3$. Let $P' = v_2v_3, P'' = v_1, Q = v_3v_1v_2$. It is clear that $P' \cup P''$ is a 2-path factor of T which is arc-disjoint from Q. And the paths Q, P' and P'' have distinct initial vertices and distinct terminal vertices.

Case 2. $n > 3$. Let $n = 2k+1$. Suppose that $P' = v_1v_3 \cdots v_{2k-1}v_{2k+1}, P'' = v_2v_4 \cdots v_{2k-2}v_{2k}, Q = v_3v_4 \cdots v_nv_1v_2$. Obviously, $P' \cup P''$ is a 2-path-factor of T which is arc-disjoint from Q. The paths Q, P' and P'' have distinct initial vertices and distinct terminal vertices, respectively.

Figure 3 shows the construction of the Hamiltonian path Q and the 2-path factor $P' \cup P''$ in the two cases of the proof. ∎

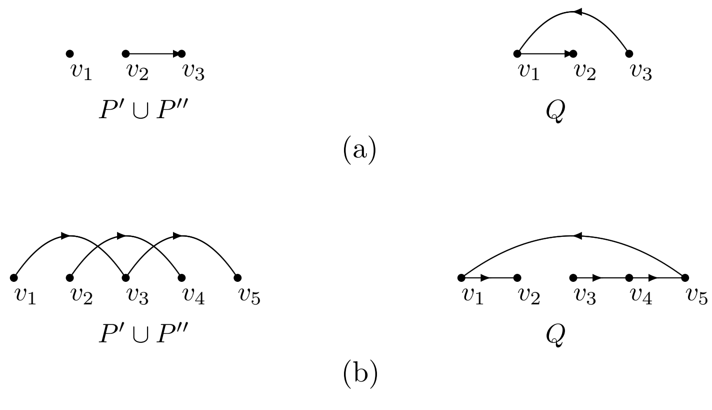


Figure 3. The arc-disjoint 2-path factor $P' \cup P''$ and the Hamiltonian path Q in an almost transitive tournament. In (a), $P' = v_2v_3, P'' = v_1, Q = v_3v_1v_2$. In (b), $P' = v_1v_3v_5, P'' = v_2v_4, Q = v_3v_4v_5v_1v_2$.

The following lemma is also useful in our proof of main results.

Lemma 2.4. *Every 2-strong semicomplete digraph with at least 3 vertices contains two arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices.*

Proof. Let D be a 2-strong semicomplete digraph. By Lemma 2.1, D contains a strong tournament T as a spanning subdigraph. If T is not an almost transitive tournament of odd order, by Lemma 1.7, we are done. Otherwise, T is an almost transitive tournament of odd order. Assume without loss of generality that $V(T) = \{v_1, v_2, \ldots, v_n\}$ and $v_i \to v_j$ for all $1 \leq i < j \leq n$ except $v_n \to v_1$ in T. In the following, we will construct two arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices in D.

Since D is 2-strong, there must exist some arc of the form $v_jv_i, i < j$ besides v_nv_1 in D. For all arcs of the form $v_jv_i, i < j$ except for v_nv_1 in D, we shall consider the following two cases.

Case 1. There is one arc of the form $v_jv_i, j > i + 1$ besides v_nv_1 in D. Let $v_jv_i, j > i + 1$, which is not v_nv_1, be an arc of D. Now we replace the arc v_iv_j with v_jv_i in T. Then we can get a tournament T' which is a spanning subdigraph of D. Recall that T is an almost transitive tournament of odd order. Then T' is not an almost transitive tournament of odd order. Notice that $C = v_1v_2 \cdots v_n$ is still a Hamiltonian cycle of T'. So T' is a strong tournament. By Theorem 1.7, we are done.

Case 2. There is no arc of the form $v_jv_i, j > i + 1$ besides v_nv_1 in D. This means that if $v_jv_i, j > i$ is an arc of D, then $j = i + 1$. Note that there must exist two arc-disjoint paths from v_n to v_1 in D since D is 2-strong. Then we have $v_{k+1}v_k \in D$ for any $k \in [n - 1]$ since otherwise there exists only one path from v_n to v_1 in D, a contradiction. Obviously, $v_1v_2 \cdots v_n$ and $v_nv_{n-1} \cdots v_1$ are two arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices. ■

For 3-strong round decomposable locally semicomplete digraphs, the following result is clear.

Lemma 2.5. *Let D be a 3-strong round decomposable locally semicomplete digraph. $D = R[D_1, D_2, \ldots, D_r], r \geq 2$ is the round decomposition of D, where R is a round digraph and for each $i \in [r]$, D_i is either a strong semicomplete digraph or a single vertex. Then*

(a) *when $r = 2$, we have $D_1 \Rightarrow D_2 \Rightarrow D_1$;*
(b) *when $r \geq 3$, for any $i \in [r]$ with $|V(D_i)| \leq 2$, we have $D_{i-1} \Rightarrow D_{i+1}$ (subscripts are modulo r).*

For 2-strong round decomposable local tournaments, we have the similar result.

Lemma 2.6. *Let D be a 2-strong round decomposable local tournament. $D = R[D_1, D_2, \ldots, D_r], r \geq 2$ is the round decomposition of D, where R is a round* *digraph and for each $i \in [r]$, D_i is either a strong tournament or a single vertex. Then $r \geq 3$ and for any $i \in [r]$ with $|V(D_i)| = 1$, we have $D_{i-1} \Rightarrow D_{i+1}$ (subscripts are modulo r).*

3. Proof of Theorem 1.5

Let D be a 3-strong round decomposition locally semicomplete digraph, and let $D = R[D_1, D_2, \ldots, D_r]$ be the round decomposition of D. In this section, we shall prove Theorem 1.5 in three classes: there exists at least one component D_i that has more than 2 vertices; each component D_i for $i \in [r]$ is either a 2-cycle or a single vertex and there exists at least one component D_i that is a 2-cycle; each component D_i for $i \in [r]$ is a single vertex.

Theorem 3.1. *Let D be a 3-strong round decomposable locally semicomplete digraph. $D = R[D_1, D_2, \ldots, D_r]$ is the round decomposition of D, where R is a round digraph and for each $i \in [r]$, D_i is either a strong semicomplete digraph or a single vertex. If there is a component D_i that has more than 2 vertices, then D contains two arc-disjoint Hamiltonian cycles.*

Proof. Suppose that $x_1, x_2, \ldots, x_r$ is a round labeling of R. When $|V(D_i)| \geq 3$, by Lemma 2.1, D_i contains a spanning subdigraph T_i which is a strong tournament. Combining Lemma 2.2 and Lemma 2.3, we know that D_i contains a Hamiltonian path Q_i and a 2-path-factor $P_i' \cup P_i''$ arc-disjoint from Q_i such that Q_i, P_i' and P_i'' have distinct initial vertices and distinct terminal vertices. Let u_i, u_i', u_i'' be the initial vertices of Q_i, P_i', P_i'' and v_i, v_i', v_i'' be the terminal vertices of Q_i, P_i', P_i'', respectively. When $|V(D_i)| = 2$, let $Q_i = u_iv_i, P_i' = P_i'' = v_iu_i$. When $|V(D_i)| = 1$, suppose that u_i is the only vertex in D_i. Let $Q_i = P_i' = P_i'' = u_i$. We will consider two cases below.

Case 1. $r = 2$. By Lemma 2.5, we know that $D_1 \Rightarrow D_2 \Rightarrow D_1$. Without loss of generality, assume that $|V(D_1)| \geq 3$. When $|V(D_2)| \geq 3$, let $C_1 = Q_1Q_2u_1, C_2 = P_1'P_2'P_1''P_2''u_1'$. When $|V(D_2)| = 2$, let $C_1 = Q_1Q_2u_1, C_2 = P_1'u_2P_1''v_2u_1'$. When $|V(D_2)| = 1$, notice that D_1 is a 2-strong semicomplete digraph since $D_1 = D - u_1$ and D is 3-strong. By Lemma 2.4, assume that $\widehat{P}_1$ and $\widehat{Q}_1$ are two arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices. Let $C_1 = u_2\widehat{Q}_1u_2, C_2 = u_2\widehat{P}_1u_2$. It is easy to check that C_1 and C_2 are two arc-disjoint Hamiltonian cycles of D.

Case 2. $r \geq 3$. We can easily obtain a Hamiltonian cycle $C_1 = Q_1Q_2 \cdots Q_ru_1$. An example is shown in Figure 4(a), where $C_1 = Q_1Q_2Q_3Q_4Q_5Q_6u_1$. In the following, we shall find the other Hamiltonian cycle C_2 such that C_1 and C_2 are arc-disjoint.

Step 1. Build a 2-cycle factor $C' \cup C''$ of D.

Let $C' = P'_1 P'_2 \cdots P'_r u'_1, C'' = P''_1 P''_2 \cdots P''_r u''_1$.

If $|V(D_i)| \geq 3$ for each $i \in [r]$, then $C' \cup C''$ is a 2-cycle factor of D. We are done.

If there exist several subscripts k's such that $|V(D_k)| \leq 2$, then $C' \cup C''$ is not a 2-cycle factor. We will obtain the desired 2-cycle factor by modifying $C' \cup C''$. For convenience, if there exist i, j satisfying $|V(D_i)| \geq 3, |V(D_j)| \geq 3$ and $|V(D_k)| \leq 2$ for each $i < k < j$ (possibly, $D_i = D_j$), we call $D_{i+1} D_{i+2} \cdots D_{j-1}$ a maximal singular segment. Here and below the subscripts are taken modulo r. For every pair of i, j such that $D_{i+1} D_{i+2} \cdots D_{j-1}$ is a maximal singular segment, we do the following:

If $j - i \equiv 0 (\text{mod } 2)$, denote $j = i + 2k$. In C', replace $v'_i P'_{i+1} P'_{i+2} \cdots P'_{i+(2k-1)} u'_{i+2k}$ with $v'_i P'_{i+1} P'_{i+3} \cdots P'_{i+(2k-1)} u'_{i+2k}$. In C'', replace $v''_i P''_{i+1} P''_{i+2} \cdots P''_{i+(2k-1)} u''_{i+2k}$ with $v''_i P''_{i+2} P''_{i+4} \cdots P''_{i+(2k-2)} u''_{i+2k}$. If $j - i \equiv 1 (\text{mod } 2)$, denote $j = i + 2k + 1$. In C', replace $v'_i P'_{i+1} P'_{i+2} \cdots P'_{i+(2k-1)} u'_{i+2k}$ with $v'_i P'_{i+1} P'_{i+3} \cdots P'_{i+(2k-1)} u'_{i+(2k+1)}$. In C'', replace $v''_i P''_{i+1} P''_{i+2} \cdots P''_{i+(2k-1)} u''_{i+2k}$ with $v''_i P''_{i+2} P''_{i+4} \cdots P''_{i+(2k)} u''_{i+(2k+1)}$. See Figure 4(b), D_2 and $D_4 D_5$ are all maximal singular segments of D. Replace $v'_1 u_2 u'_3$ with $v'_1 u_2 u'_3$, $v''_1 u_2 u''_3$ with $v''_1 u''_3$, $v'_3 v_4 u_4 u_5 u'_6$ with $v'_3 v_4 u_4 u'_6$, and $v''_3 v_4 u_4 u_5 u''_6$ with $v''_3 u_5 u''_6$, respectively. Hence, $C' = P'_1 u_2 P'_3 u_4 P'_6 u'_1$, $C'' = P''_1 P''_3 u_5 P''_6 u''_1$. Clearly, $C' \cup C''$ is a 2-cycle factor of D.

Step 2. Build a 2-path factor $P' \cup P''$ based on the 2-cycle factor $C' \cup C''$.

Since there is a component D_i that has more than 2 vertices for some $i \in [r]$, without loss of generality, assume that $|V(D_r)| \geq 3$. Let w' be the successor of v'_r in C', and w'' be successor of v''_r in C''. By the construction process of $C' \cup C''$, if $|V(D_1)| \leq 2$, we have $w' \in D_1, w'' \in D_2$, and if $|V(D_1)| \geq 3$ we have $w', w'' \in D_1$. We obtain P', P'' by deleting arc $v'_r w', v''_r w''$ of C', C'', respectively. It is easy to check that $P' \cup P''$ is a 2-path factor of D. See Figure 4(c). We obtain $P' = P'_1 u_2 P'_3 v_4 u_4 P'_6$ by deleting arc $v'_6 u'_1$ of C', and $P'' = P''_1 P''_3 u_5 P''_6$ by deleting $v''_6 u''_1$.

Step 3. Build a Hamiltonian cycle C_2 based on the 2-path factor $P' \cup P''$.

If $|V(D_1)| \leq 2$, then we have $w' \in D_1$ and $w'' \in D_2$. By Lemma 2.5, since D is 3-strong, D_r must completely dominate D_2. This implies that there exist the arcs $v'_r w''$ and $v''_r w'$. If $|V(D_1)| \geq 3$, then we have $w', w'' \in D_1$. Since D_r completely dominates D_1, there also exist the arcs $v'_r w''$ and $v''_r w'$. Now the initial vertices of P', P'' are w', w'', respectively. The terminal vertices of P', P'' are v'_r, v''_r, respectively. Hence, add the arcs $v'_r w''$ and $v''_r w'$ into the 2-path factor $P' \cup P''$, and we obtain the Hamiltonian cycle $C_2 = P' P'' w'$. It is easy to check that C_1 is arc-disjoint from C_2. See Figure 4(d). $C_2 = P'_1 u_2 P'_3 v_4 u_4 P'_6 P''_1 P''_3 u_5 P''_6 u'_1$ is a Hamiltonian cycle arc-disjoint from C_1. ∎

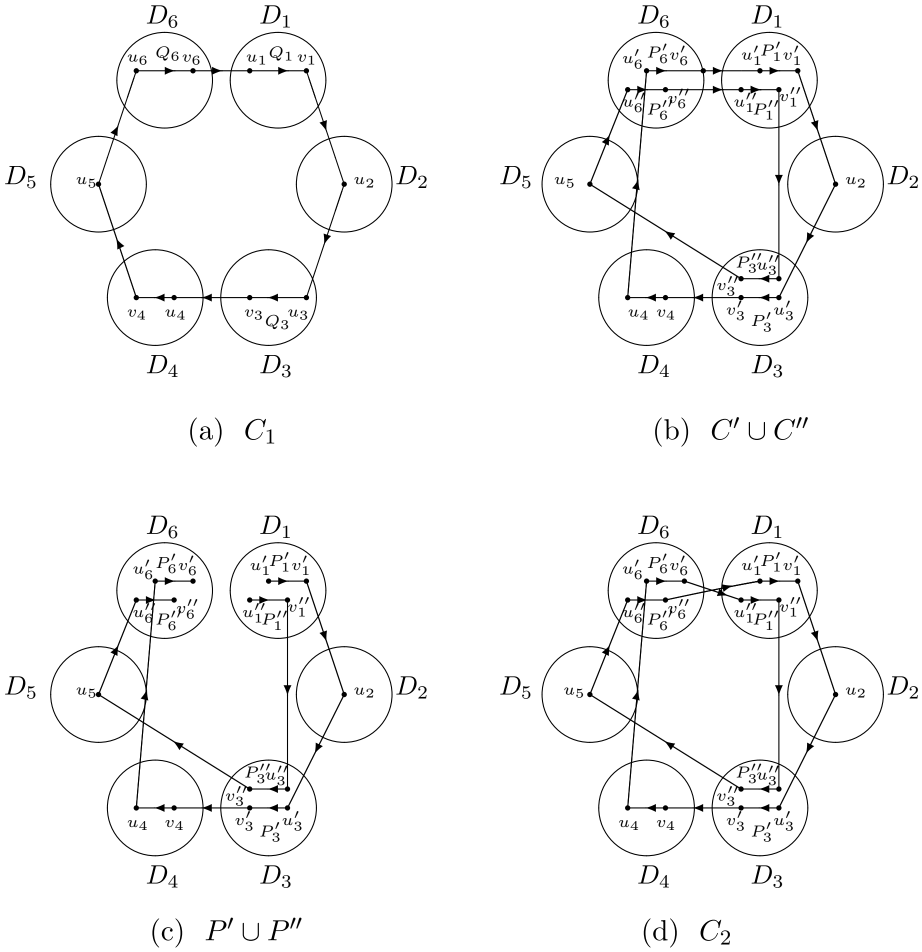


Figure 4. (a) The Hamiltonian cycle C_1. (b) The 2-cycle factor $C' \cup C''$. (c) The 2-path factor $P' \cup P''$. (d) The Hamiltonian cycle C_2.

Theorem 3.2. *Let D be a 3-strong round decomposable locally semicomplete digraph. $D = R[D_1, D_2, \ldots, D_r]$ is the round decomposition of D, where R is a round digraph and for each $i \in [r]$, D_i is either a 2-cycle or a single vertex. If there is a component D_i that is a 2-cycle, then D contains two arc-disjoint Hamiltonian cycles.*

Proof. When $|V(D_i)| = 2$, let $Q_i = u_i v_i, P_i = v_i u_i$. When $|V(D_i)| = 1$, suppose that u_i is the only vertex in D_i. Let $Q_i = P_i = u_i$. Obviously, $C_1 = Q_1 Q_2 \cdots Q_r u_1$ is a Hamiltonian cycle of D. Assume without loss of generality that $|V(D_1)| = 2$. If r is even, then let $C_2 = v_1 P_3 P_5 \cdots P_{r-1} u_1 P_2 P_4 \cdots P_r v_1$. If r is odd, then let

$C_2 = P_1P_3\cdots P_rP_2P_4\cdots P_{r-1}v_1$. It is easy to check that C_1 and C_2 are two arc-disjoint Hamiltonian cycles. ■

Theorem 3.3. *Let R be a 3-strong round digraph. Then R contains two arc-disjoint Hamiltonian cycles.*

Proof. Let $x_1, x_2, \ldots, x_r$ be the unique (up to cyclic permutations) round labeling of R. Since R is 3-strong round digraph, the vertex x_i dominates the vertices x_{i+1}, x_{i+2} and x_{i+3} for each $i \in [r]$ (subscripts are modulo r).

If r is odd, denote $r = 2k+1$. Then R contains two arc-disjoint Hamiltonian cycles $C_1 = x_1x_2x_3\cdots x_{2k+1}x_1$ and $C_2 = x_1x_3\cdots x_{2k+1}x_2x_4\cdots x_{2k}x_1$.

If r is even, we consider two cases, $r = 4m+2$ or $r = 4m$.

Case 1. $r = 4m+2$. R contains two arc-disjoint Hamiltonian cycles $C_1 = x_1x_2x_4x_6\cdots x_{4m+2}x_3x_5x_7\cdots x_{4m+1}x_1$ and $C_2 = x_1x_4x_5x_8x_9\cdots x_{4m-4}x_{4m-3}x_{4m}x_{4m+1}x_2x_3x_6x_7\cdots x_{4m-6}x_{4m-5}x_{4m-2}x_{4m-1}x_{4m+2}x_1$.

Case 2. $r = 4m$. If $r = 4m$, R contains two arc-disjoint Hamiltonian cycles $C_1 = x_1x_2x_4x_6\cdots x_{4m}x_3x_5x_7\cdots x_{4m-1}x_1$ and $C_2 = x_1x_3x_4x_7x_8\cdots x_{4m-3}x_{4m-1}x_{4m}x_2x_5x_6x_9x_{10}\cdots x_{4m-7}x_{4m-6}x_{4m-3}x_{4m-2}x_1$.

The theorem holds. ■

Combining with Theorem 3.1, Theorem 3.2 and Theorem 3.3, the proof of Theorem 1.5 is complete.

4. Proof of Theorem 1.6

Let D be a 2-strong round decomposable local tournament, and let $D = R[D_1, D_2, \ldots, D_r]$ be the round decomposition of D, where R is a round digraph and for each $i \in [r]$, D_i is either a strong tournament or a single vertex. We prove Theorem 1.6 by dividing into two cases: there is a strong component D_i that is not a single vertex; each strong component D_i for $i \in [r]$ is a single vertex, i.e., $D = R$ is a round diraph.

In the proof of Theorem 3.1, the condition that D is 3-strong is necessary only when $r = 2$ or when $r \geq 3$ and $|V(D_i)| = 2$ for some $i \in [r]$. In other cases, the condition that D is 2-strong is sufficient. When $D = R[D_1, D_2, \ldots, D_r]$ is a round decomposable local tournament, we always have $r \geq 3$ and $|V(D_i)| \neq 2$ for each $i \in [r]$. Thus the proof of Theorem 3.1 can be used to prove the following theorem.

Theorem 4.1. *Let D be a 2-strong round decomposable local tournament, and let $D = R[D_1, D_2, \ldots, D_r]$ be the round decomposition of D, where R is a round digraph and for each $i \in [r]$, D_i is either a strong tournament or a single vertex.*

If there is a component D_i that is not a single vertex, then D contains two arc-disjoint Hamiltonian cycles.

Theorem 4.2. *Let R be a 2-strong round digraph. Then R contains a Hamiltonian cycle and a Hamiltonian path which are arc-disjoint if and only if R is not the second power of an even cycle.*

Proof. Firstly, we show the 'only if' part. Let R be a digraph with the vertex set $\{x_1, x_2, \ldots, x_r\}$ and the ordering $x_1, x_2, \ldots, x_r$ be the unique (up to cyclic permutations) round labeling of vertices of R. Suppose to the contrary that R is the second power of an even cycle. Obviously, $C = x_1 x_2 \cdots x_r x_1$ is the unique Hamiltonian cycle of R. We obtain two vertex-disjoint $\frac{r}{2}$-cycles by deleting arcs of C from R. Hence, R will not contain a Hamiltonian path P arc-disjoint from the Hamiltonian cycle C, a contradiction. Thus R is not the second power of an even cycle.

To show the 'if' part, let R be a 2-strong round digraph. This means that x_i dominates x_{i+1} and x_{i+2} for each $i \in [r]$ (all subscripts are modulo r). Then R contains C_r^2 as a spanning subdigraph of R. Since R is not the second power of an even cycle, we discuss two cases below.

Case 1. $r = 2k+1$. It is obvious that C_{2k+1}^2 can be decomposed into two arc-disjoint Hamiltonian cycles $C_1 = x_1 x_2 x_3 \cdots x_{2k} x_{2k+1} x_1$ and $C_2 = x_1 x_3 x_5 \cdots x_{2k+1} x_2 x_4 x_6 \cdots x_{2k} x_1$. It is certain that R contains a Hamiltonian cycle and a Hamiltonian path which are arc-disjoint.

Case 2. $r = 2k$. Since R is not the second power of an even cycle, there exists a vertex x_i dominating x_{i+3}. Without loss of generality, assume that x_1 dominates x_4. Thus R can be decomposed into a Hamiltonian cycle $C_1 = x_1 x_2 x_3 \cdots x_{2k-1} x_{2k} x_1$ and a Hamiltonian path $P_2 = x_3 x_5 x_7 \cdots x_{2k-1} x_1 x_4 x_6 \cdots x_{2k} x_2$. ■

Combining with Theorem 4.1 and Theorem 4.2, the proof of Theorem 1.6 is complete.

References

[1] J. Bang-Jensen and G. Gutin, Digraphs: Theory, Algorithms and Applications (Springer Monographs in Mathematics, Spring-Verlag, London, 2009). doi:10.1007/978-1-84800-998-1

[2] J. Bang-Jensen, *Locally semicomplete digraph: A generalization of tournaments*, J. Graph Theory **14** (1990) 371–390. doi:10.1002/jgt.3190140310

[3] J. Bang-Jensen, Y. Guo, G. Gutin and L. Volkmann, *A classification of locally semicomplete digraphs*, Discrete Math. **167-168** (1997) 101–114. doi:10.1016/S0012-365X(96)00219-1

[4] J. Bang-Jensen and J. Huang, *Decomposing locally semicomplete digraphs into strong spanning subdigraphs*, J. Combin. Theory Ser. B **102** (2012) 701–714. doi:10.1016/j.jctb.2011.09.001

[5] Y. Guo, Locally Semicomplete Digraphs (Ph.D. Thesis, RWTH Aachen University, 1995).

[6] Y. Guo, *Strongly Hamiltonian-connected locally semicomplet digraphs*, J. Graph Theory **21** (1996) 65–73. doi:10.1002/(SICI)1097-0118(199605)22:1⟨65::AID-JGT9⟩3.0.CO;2-J

[7] F. Harary and L. Moser, *The theory of round robin tournaments*, Amer. Math. Monthly **73** (1966) 231–246. doi:10.2307/2315334

[8] J. Huang, *On the structure of local tournaments*, J. Combin. Theory Ser. B **63** (1995) 200–221. doi:10.1006/jctb.1995.1016

[9] S. Li, W. Meng, Y. Guo and G. Xu, *A local tournament contains a vertex whose out-arc are pseudo-girth-pancyclic*, J. Graph Theory **62** (2009) 346–361. doi:10.1002/jgt.20404

[10] D. Meierling, *Local tournaments with the minimum number of Hamiltonian cycles or cycles of length three*, Discrete Math. **310** (2010) 1940–1948. doi:10.1016/j.disc.2010.03.003

[11] C. Thomassen, *Edge-disjoint Hamiltonian paths and cycles in tournaments*, Proc. Lond. Math. Soc. (3) **45** (1982) 151–168. doi:10.1112/plms/s3-45.1.151

[12] R. Wang, A. Yang and S. Wang, *Kings in locally semicomplete digraphs*, J. Graph Theory **63** (2010) 279–283. doi:10.1002/jgt.20426

[13] X.H. Zhang, R.J. Li and S.J. Li, *H-force sets of locally semicomplete digraphs*, Discrete Appl. Math. **160** (2012) 2491–2496. doi:10.1016/j.dam.2012.06.014

Received 17 August 2016
Revised 1 November 2016
Accepted 23 December 2016