Discussiones Mathematicae
Graph Theory 38 (2018) 515–524
doi:10.7151/dmgt.2022

SOME RESULTS ON THE INDEPENDENCE POLYNOMIAL OF UNICYCLIC GRAPHS

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Abstract

Let G be a simple graph on n vertices. An independent set in a graph is a set of pairwise non-adjacent vertices. The independence polynomial of G is the polynomial $I(G,x) = \sum_{k=0}^{n} s(G,k)x^k$, where $s(G,k)$ is the number of independent sets of G with size k and $s(G,0) = 1$. A unicyclic graph is a graph containing exactly one cycle. Let C_n be the cycle on n vertices. In this paper we study the independence polynomial of unicyclic graphs. We show that among all connected unicyclic graphs G on n vertices (except two of them), $I(G,t) > I(C_n,t)$ for sufficiently large t. Finally for every $n \geq 3$ we find all connected graphs H such that $I(H,x) = I(C_n,x)$.

Keywords: independence polynomial, independent set, unicyclic graphs.

2010 Mathematics Subject Classification: 05C30, 05C31, 05C38, 05C69.

1. Introduction

Throughout this paper we will consider only simple graphs, the graphs with no loops and multiple edges. Let $G = (V(G), E(G))$ be a simple graph. The *order* of G denotes the number of vertices of G. Let e be an edge of G. By $e = uv$ we mean that e is an edge between vertices u and v. For every vertex $v \in V(G)$, the *closed neighborhood* of v denoted by $N[v]$ is defined as $\{u \in V(G) \,|\, uv \in E(G)\} \cup \{v\}$. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *disjoint union* of G_1 and G_2 denoted by $G_1 + G_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The graph rG denotes the disjoint union of r copies of G. For every vertex $v \in V(G)$, the *degree* of v is the number of edges incident with v. A *pendant* vertex is a vertex of degree one. For a vertex $v \in V(G)$, $G \setminus v$ denotes the graph obtained from G by removing v. A *unicyclic graph* is a graph containing exactly one cycle. We denote the complete graph of order n, the complete bipartite graph with part sizes m, n, the cycle of order n, and the path of order n, by K_n, $K_{m,n}$, C_n, and P_n, respectively. Also $K_{1,n}$ is called a *star*.

A set $S \subseteq V(G)$ is an *independent set* if there is no edge between the vertices of S. If S is an independent set with $|S| = k$, then S is called a *k-independent set*. By $s(G,k)$ we mean the number of k-independent sets of G. The *independence number* of G, $\alpha(G)$, is the maximum cardinality of an independent set of G. The *independence polynomial* of G, $I(G,x)$, is defined as $I(G,x) = \sum_{k=0}^{\alpha(G)} s(G,k)x^k$, where $s(G,k)$ is the number of independent sets of G of size k and $s(G,0) = 1$. This polynomial was introduced by Gutman and Harary in [10]. For example for every $n \geq 1$, $\alpha(K_n) = 1$ and $s(K_n, 1) = n$. Thus $I(K_n, x) = 1 + nx$. The independence polynomial has very nice properties, see [5, 6, 13] for more details. There are many polynomials associated with graphs. For example chromatic polynomial, clique polynomial, domination polynomial, edge cover polynomial and matching polynomial, see [1]–[16]. One of the most important problems related to graph polynomials is the following:

Problem. Which graphs are uniquely determined by their graph polynomials?

In many papers, researchers study the problem defined above for graph polynomials. For example in [3] the authors show that the complete graphs, the cycles and some complete bipartite graphs are determined by their edge cover polynomials. In [2] it is proved that the cycles are determined by their domination polynomials. In this paper we study the independence polynomial of unicyclic graphs. We show that among all connected unicyclic graphs G on n vertices except the cycle C_n and the graph D_n (see Figure 3), $I(G,t) > I(C_n,t)$ for sufficiently large t. We show that for every $n \geq 4$ there is only one connected graph H such that $H \ncong C_n$ and $I(H,x) = I(C_n,x)$.

2. The Independence Polynomials of Unicyclic Graphs

In this section we study the independence polynomials of unicyclic graphs. We need the following basic properties of independence polynomials.

Theorem 1 [10, 11]**.** *Let G be a graph with connected components $G_1, \ldots, G_t$. Then $I(G,x) = \prod_{i=1}^{t} I(G_i,x)$.*

Theorem 2 [10, 11]**.** *Let G be a graph and v be a vertex of G. Then*

$$I(G,x) = I(G \setminus v, x) + xI(G \setminus N[v], x).$$

Remark 3. We remark that by independence polynomials one can find the number of vertices and the number of edges of graphs. More precisely, if G is a graph with n vertices and m edges, then $n = s(G,1)$ and $m = \binom{n}{2} - s(G,2)$.

Lemma 4. *Let T be a tree of order n. Then there exists a positive real number r_n such that for all $x \geq r_n$ we have*

$$I(T,x) > \begin{cases} x^{\lceil \frac{n}{2} \rceil}, & \textit{if } n \textit{ is odd;} \\ 2x^{\frac{n}{2}}, & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. Since T is a tree, T is bipartite. Assume that X and Y are partite sets of $V(T)$. Hence $\alpha(T) \geq |X|, |Y|$. This shows that $\alpha(T) \geq \lceil \frac{n}{2} \rceil$. First assume that n is odd. Thus for all $x \geq 1$, $x^{\alpha(T)} \geq x^{\lceil \frac{n}{2} \rceil}$. This shows that for all $x \geq 1$, $I(T,x) > x^{\lceil \frac{n}{2} \rceil}$. Now assume that n is even. If $\alpha(T) = \frac{n}{2}$, then $|X| = |Y| = \frac{n}{2}$. Thus $s(T, \frac{n}{2}) \geq 2$. Hence for all $x \geq 1$, $s(T, \alpha(T))x^{\alpha(T)} \geq 2x^{\frac{n}{2}}$ and so $I(T,x) > 2x^{\frac{n}{2}}$. Otherwise suppose that $\alpha(T) > \frac{n}{2}$. Thus $I(T,x) - 2x^{\frac{n}{2}}$ is a polynomial with positive leading coefficient. Therefore for sufficiently large x, $I(T,x) - 2x^{\frac{n}{2}} > 0$. This completes the proof. ■

Let G be a graph of order n with vertex set $\{v_1, \ldots, v_n\}$. Let $H_1, \ldots, H_n$ be some disjoint graphs. Assume that $u_1 \in V(H_1), \ldots, u_n \in V(H_n)$. By $G(H_1, \ldots, H_n; u_1, \ldots, u_n)$ we mean the graph that is obtained by identifying the vertices u_i and v_i for $i = 1, \ldots, n$. Note that the order of $G(H_1, \ldots, H_n; u_1, \ldots, u_n)$ is $|V(H_1)| + \cdots + |V(H_n)|$, see Figure 1. In particular, suppose that $H_1, \ldots, H_n$ are some stars, say $H_1 = K_{1,m_1}, \ldots, H_n = K_{1,m_n}$, where $m_1, \ldots, m_n$ are some non-negative integers (by $K_{1,0}$ we mean the single vertex K_1). In addition let u_i be the vertex of K_{1,m_i} with degree m_i. Then we use $G(m_1, \ldots, m_n)$ instead of $G(K_{1,m_1}, \ldots, K_{1,m_n}; u_1, \ldots, u_n)$. Note that the order of $G(m_1, \ldots, m_n)$ is $m_1 + \cdots + m_n + n$ and $G(0, \ldots, 0) \cong G$. See Figure 2.

Lemma 5. *Let $k \geq 3$ be an integer. Let $V(C_k) = \{v_1, \ldots, v_k\}$ and $E(C_k) = \{v_1v_2, \ldots, v_{k-1}v_k, v_kv_1\}$. Let $G = C_k(n_1, \ldots, n_k)$ and $n = n_1 + \cdots + n_k + k$, where $n_1, \ldots, n_k$ are some non-negative integers. If $G \not\cong C_n$ and $n \geq 5$, then for sufficiently large x we have $I(G,x) > I(C_n,x)$.*

Proof. First we note that if $n = 3$, then $G \cong C_3$. Also if $n = 4$, then $G \cong C_4$ or $G \cong C_3(1,0,0)$. Since $I(C_3(1,0,0),x) = I(C_4,x) = 1 + 4x + 2x^2$, $I(G,x) = I(C_4,x)$. We note that $C_k(n_1, \ldots, n_k) \cong C_n$ if and only if $n = k$. Now assume that $n \geq 5$ and $G \not\cong C_n$ ($G \not\cong C_k$). We have one of the following cases.

(i) For some $i \in \{1, \ldots, k\}$, $n_i \geq 2$. Without losing the generality assume that $n_1 \geq 2$. Note that $\alpha(G) \geq n_1 + \alpha(P_{k-1}(n_2, \ldots, n_k))$, where $V(P_{k-1}) = \{v_2, \ldots, v_k\}$ and $E(P_{k-1}) = \{v_2v_3, \ldots, v_{k-1}v_k\}$. Since $P_{k-1}(n_2, \ldots, n_k)$ is a tree of order $n-n_1-1$ (by the proof of Lemma 4), $\alpha(P_{k-1}(n_2,\ldots,n_k)) \geq \lceil \frac{n-n_1-1}{2} \rceil$. Hence

$$\alpha(G) \geq n_1 + \left\lceil \frac{n-n_1-1}{2} \right\rceil = \left\lceil \frac{n+n_1-1}{2} \right\rceil \geq \left\lceil \frac{n+1}{2} \right\rceil > \left\lfloor \frac{n}{2} \right\rfloor = \alpha(C_n).$$

Thus $\alpha(G) > \alpha(C_n)$. Since the coefficients of independence polynomials are positive, for sufficiently large x we have $I(G,x) > I(C_n,x)$.

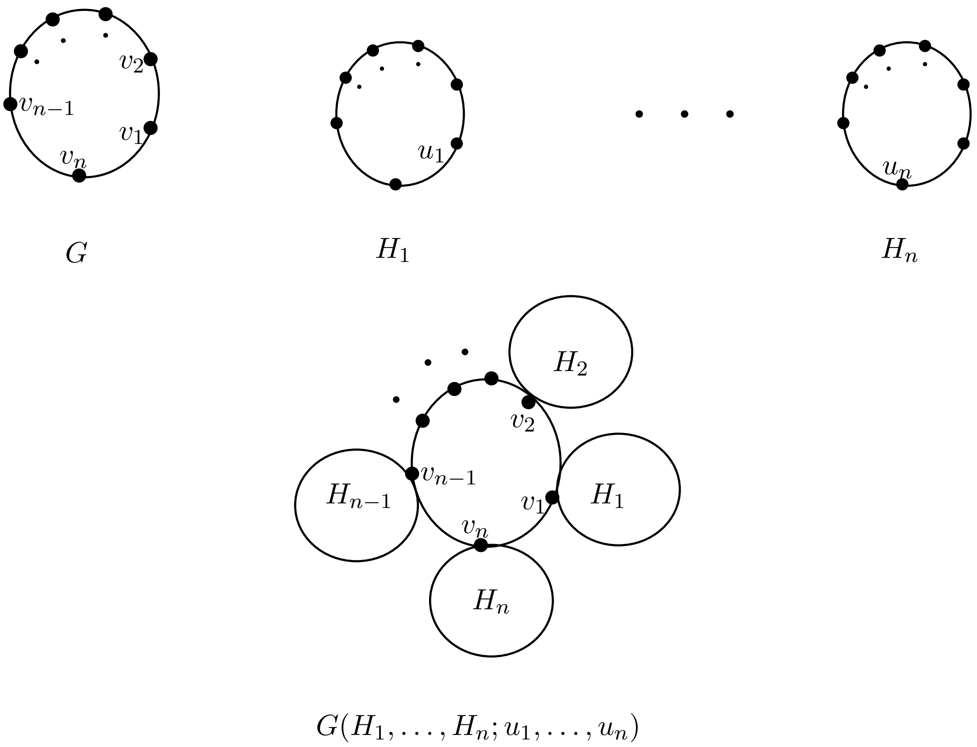
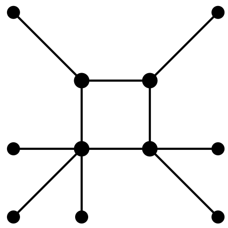


Figure 1. The graph $G(H_1,\ldots,H_n;u_1,\ldots,u_n)$.

(ii) For $i=1,\ldots,k$, $n_i \in \{0,1\}$ and n is odd. Since $G \ncong C_n$, for some i, $n_i = 1$. Without losing the generality let $n_1 = 1$. Since n is odd, similar to part (i), $\alpha(G) \geq 1 + \lceil \frac{n-2}{2} \rceil = \lceil \frac{n}{2} \rceil > \lfloor \frac{n}{2} \rfloor = \alpha(C_n)$. Thus the result follows.

(iii) For $i=1,\ldots,k$, $n_i \in \{0,1\}$ and n is even. Since $G \ncong C_n$, for some t, $n_t = 1$. First suppose that there is only one i such that $n_i = 1$. Without losing the generality assume that $n_1 = \cdots = n_{k-1} = 0$ and $n_k = 1$. Hence $k = n-1$. In other words, $G \cong C_{n-1}(0,\ldots,0,1)$. Hence $\alpha(G) = \frac{n}{2}$. Let $V(G) = \{v_1,\ldots,v_n\}$ and $E(G) = \{v_1v_2,\ldots,v_{n-2}v_{n-1},v_{n-1}v_1,v_{n-1}v_n\}$. Since $n \geq 6$, $\{v_1,v_3,\ldots,v_{n-3},v_n\}$, $\{v_1,v_3,\ldots,v_{n-5},v_{n-2},v_n\}$ and $\{v_2,v_4,\ldots,v_{n-2},v_n\}$ are three independent sets of G with cardinality $\frac{n}{2}$. Hence $s(G,\frac{n}{2}) \geq 3$. On the



$C_4(1,1,2,3)$

Figure 2. The graph $C_4(1,1,2,3) = G(H_1, H_2, H_3, H_4; u_1, u_2, u_3, u_4)$, where $G = C_4$ and u_1 is the vertex of $H_1 = K_{1,1}$ of degree one, u_2 is the vertex of $H_2 = K_{1,1}$ of degree one, u_3 is the vertex of $H_3 = K_{1,2}$ of degree two and u_4 is the vertex of $H_4 = K_{1,3}$ of degree three.

other hand, since n is even, $\alpha(C_n) = \frac{n}{2}$ and $s(C_n, \frac{n}{2}) = 2$. By the fact that $\alpha(G) = \alpha(C_n) = \frac{n}{2}$ and $s(G, \frac{n}{2}) > s(C_n, \frac{n}{2})$, for sufficiently large x we obtain $I(G,x) > I(C_n, x)$. Now assume that there are some $i \neq j$ such that $n_i = 1$ and $n_j = 1$. This shows that G has at least two vertices of degree one (G has two pendant vertices). Let u and v be two pendant vertices of G. Applying Theorem 2 for vertex u we obtain $I(G,x) = I(G \setminus u, x) + xI(T_1, x)$, where T_1 is a tree of order $n-2$. Using Theorem 2 for v and $G \setminus u$ we have

$$I(G,x) = I(G \setminus \{u,v\}, x) + xI(T_2, x) + xI(T_1, x),$$

where T_2 is a tree of order $n-3$. Hence for $x \geq 0$, $I(G,x) > xI(T_2,x) + xI(T_1,x)$. Using Lemma 4 for trees T_1 and T_2 we obtain that for sufficiently large x,

$$I(G,x) > xx^{\lceil \frac{n-3}{2} \rceil} + 2xx^{\frac{n-2}{2}} = 3x^{\frac{n}{2}}.$$

On the other hand, $\alpha(C_n) = \frac{n}{2}$ and $s(C_n, \frac{n}{2}) = 2$. Hence for sufficiently large x, $3x^{\frac{n}{2}} > I(C_n, x)$. Thus for sufficiently large x, $I(G,x) > 3x^{\frac{n}{2}} > I(C_n, x)$. The proof is complete. ■

3. Graphs Whose Independence Polynomials Coincide with Independence Polynomials of Cycles

In this section we study the graphs G such that $I(G,x) = I(C_n, x)$, where $n \geq 3$. We show that there is only one connected graph $G \not\cong C_n$ satisfying $I(G,x) = I(C_n,x)$. Let $n \geq 4$ be an integer. By D_n we mean the graph with vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\} \cup \{v_{n-2}v_n\}$, see Figure 3. In addition by D_3 we mean the cycle C_3. The next result shows that the independence polynomials of C_n and D_n are the same.

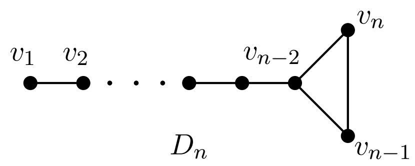


Figure 3. The graph D_n.

Lemma 6. *Let $n \geq 4$ be an integer. Then*

(i) $I(C_n, x) = I(C_{n-1}, x) + xI(C_{n-2}, x)$, *where C_2 is the path P_2.*

(ii) $I(C_n, x) = I(D_n, x)$.

Proof. It is easy to check the result for $n = 4$. Thus let $n \geq 5$. Using Theorem 2 for one of the vertices of C_n we obtain that

$$I(C_n, x) = I(P_{n-1}, x) + xI(P_{n-3}, x). \tag{1}$$

On the other hand, by Theorem 2 for one of the pendant vertices of P_t we have

$$I(P_t, x) = I(P_{t-1}, x) + xI(P_{t-2}, x), \text{ for } t \geq 2, \text{ where } I(P_0, x) = 1. \tag{2}$$

Using equations (1) and (2) one can see that

$$I(C_n, x) = I(P_{n-2}, x) + xI(P_{n-4}, x) + x(I(P_{n-3}, x) + xI(P_{n-5}, x)).$$

So by equation (1) the first part is proved. Now we prove the second part. Using Theorem 2 for the vertex v_n of D_n (see Figure 3) we obtain that $I(D_n, x) = I(P_{n-1}, x) + xI(P_{n-3}, x)$. Hence by equation (1), $I(D_n, x) = I(C_n, x)$. The proof is complete. ■

We recall that a unicyclic graph is a graph with exactly one cycle. The next result shows that among all connected unicyclic graphs the cycles have the smallest independence polynomials.

Theorem 7. *Let G be a connected unicyclic graph of order n. Assume that $G \ncong C_n$ and $G \ncong D_n$. Then for sufficiently large x we have $I(G, x) > I(C_n, x)$.*

Proof. Assume that H is a connected unicyclic graph of order n. Thus $n \geq 3$. If $n = 3$, then $H \cong C_3$. If $n = 4$, then $H \cong C_4$ or $H \cong D_4$. If $n = 5$, then $H \cong C_5$ or $H \cong D_5$ or $H \cong C_4(1, 0, 0, 0)$ or $H \cong C_3(2, 0, 0)$ or $H \cong C_3(1, 1, 0)$. So by the fact that G is unicyclic and $G \ncong C_n$ and $G \ncong D_n$ we obtain that $n \geq 5$. We use induction on n to prove the result. If $n = 5$, then $G \cong C_4(1, 0, 0, 0)$ or $G \cong C_3(2, 0, 0)$ or $G \cong C_3(1, 1, 0)$. One can see that $I(C_4(1, 0, 0, 0), x) = 1 + 5x + 5x^2 + x^3$, $I(C_3(2, 0, 0), x) = 1 + 5x + 5x^2 + 2x^3$ and

$I(C_3(1,1,0),x) = 1+5x+5x^2+x^3$. On the other hand $I(C_5,x) = 1+5x+5x^2$. Thus the result holds for $n = 5$.

Now assume that $n \geq 6$. Suppose that the length of the unique cycle of G is k. Assume that $v_1,\ldots,v_k$ are the vertices of this cycle. Since G is unicyclic there are some trees $T_1,\ldots,T_k$ such that $G = C_k(T_1,\ldots,T_k; v_1,\ldots,v_k)$. If each tree $T_1,\ldots,T_k$ is a star, then by Lemma 5 the result follows. Now without losing the generality assume that T_1 is not a star. Let u_1 be a pendant vertex of T_1 which has the maximum distance from v_1 among all pendant vertices of T_1. We consider the three following cases for $G \setminus u_1$.

(i) Assume that $G \setminus u_1$ is the cycle C_{n-1}. Hence $G = C_{n-1}(1,0,\ldots,0)$ and $T_1 = P_2$, a contradiction (since T_1 is not a star). Thus this case does not happen.

(ii) Assume that $G \setminus u_1$ is the graph D_{n-1}. Hence $G \cong D_n$ or $G \cong H$, where H is obtained by identifying the pendant vertex of D_{n-2} with the non-pendant vertex of P_3. Thus it suffices to check the result for H. Let z be a pendant vertex of H. Thus $H \setminus z \cong D_{n-1}$ and $H \setminus N[z] \cong D_{n-3} + K_1$. Hence by Theorems 1 and 2, $I(H,x) = I(H \setminus z, x) + xI(H \setminus N[z], x) = I(D_{n-1},x) + x(1+x)I(D_{n-3},x)$. So by the second part of Lemma 6 we obtain

$$I(H,x) = I(C_{n-1},x) + x(1+x)I(C_{n-3},x). \tag{3}$$

On the other hand, by the first part of Lemma 6 for $n \geq 7$, $I(C_{n-3},x) = I(C_{n-4},x) + xI(C_{n-5},x)$. This shows that for $x > 0$, $I(C_{n-3},x) > I(C_{n-4},x)$ (this inequality also holds for $n = 6$, where C_2 is the path P_2). Hence for $x > 0$, $xI(C_{n-3},x) > xI(C_{n-4},x)$. Thus for every $x > 0$ we have

$$(1+x)I(C_{n-3},x) = I(C_{n-3},x) + xI(C_{n-3},x) > I(C_{n-3},x) + xI(C_{n-4},x).$$

Therefore by the first part of Lemma 6 we obtain that

$$\text{for } x > 0,\ (1+x)I(C_{n-3},x) > I(C_{n-2},x). \tag{4}$$

The equations (3) and (4) show that for $x > 0$, $I(H,x) > I(C_{n-1},x)+xI(C_{n-2},x)$. Hence by the first part of Lemma 6 for every $x > 0$, $I(H,x) > I(C_n,x)$.

(iii) Suppose that $G\setminus u_1 \ncong C_{n-1}$ and $G\setminus u_1 \ncong D_{n-1}$. Since $G\setminus u_1$ is a connected unicyclic graph of order $n-1$, by the induction hypothesis for sufficiently large x, $I(G\setminus u_1,x) > I(C_{n-1},x)$. As we defined above, u_1 is a pendant vertex of T_1 which has the maximum distance from v_1 among all pendant vertices of T_1. Assume that w_1 is the neighbor of u_1. Since T_1 is not a star, $d(u_1,v_1) \geq 2$. We note that $w_1 \neq v_1$. Let $deg(w_1) = t+1$. Thus $t \geq 1$. By the definition of u_1, exactly t neighbors of w_1 have degree one. Hence $G \setminus N[u_1]$ is the union of a unicyclic graph of order $n-t-1$, say L, with exactly $t-1$ isolated vertices. In other words, $G\setminus N[u_1] = L+(t-1)K_1$. Hence by Theorem 1, $I(G\setminus N[u_1],x) = I(L,x)(1+x)^{t-1}$. On the other hand, by the induction hypothesis for sufficiently large x, $I(L,x) \geq$

$I(C_{n-t-1},x)$ (if $L \neq C_{n-t-1}$ and $L \neq D_{n-t-1}$, $I(L,x) > I(C_{n-t-1},x)$ for large x). Since $n \geq t+4$, similar to the previous part one can see that for $x > 0$, $(1+x)I(C_{n-t-1},x) > I(C_{n-t},x)$. Hence for $x > 0$, $(1+x)^2 I(C_{n-t-1},x) > (1+x)I(C_{n-t},x)$. Similarly for $x > 0$, $(1+x)I(C_{n-t},x) > I(C_{n-t+1})$. By applying this method $t-1$ times, we obtain that if $t \geq 2$, then

$$\text{for } x > 0,\ (1+x)^{t-1} I(C_{n-t-1},x) > I(C_{n-2},x). \tag{5}$$

Hence for $t \geq 1$ we conclude that

$$\text{for } x > 0,\ (1+x)^{t-1} I(C_{n-t-1},x) \geq I(C_{n-2},x). \tag{6}$$

The equation (6) shows that for sufficiently large x,

$$I(G \setminus N[u_1],x) = I(L,x)(1+x)^{t-1} \geq (1+x)^{t-1} I(C_{n-t-1},x) \geq I(C_{n-2},x).$$

Since for large x, $I(G \setminus u_1,x) > I(C_{n-1},x)$, by Theorem 2, the equation (5) and the first part of Lemma 6, we find that for large x,

$$I(G,x) = I(G \setminus u_1,x) + xI(G \setminus N[u_1],x) > I(C_{n-1},x) + xI(C_{n-2},x) = I(C_n,x).$$

The proof is complete. ∎

Now we are in a position to prove the main result of this section.

Theorem 8. *Let $n \geq 3$ be an integer. Assume that G is a connected graph such that $I(G,x) = I(C_n,x)$. Then $G \cong C_n$ or $G \cong D_n$.*

Proof. Since $I(G,x) = I(C_n,x)$ and C_n has n vertices and n edges, by Remark 3 we find that G has exactly n vertices and n edges. Since the number of vertices and the number of edges of G are the same and G is connected, G is unicyclic. If $G \ncong C_n$ or $G \ncong D_n$, then by Theorem 7 for large x we have $I(G,x) > I(C_n,x)$, a contradiction. This completes the proof. ∎

Let $n \geq 3$ be an integer. One might ask whether there is a disconnected graph G satisfying $I(G,x) = I(C_n,x)$. We check this question for $n \leq 9$.

Remark 9. Let $3 \leq n \leq 9$ and G be a graph of order n. Assume that $I(G,x) = I(C_n,x)$. We find that if $n \in \{3,4,5,7,8\}$, then $G \cong C_n$ or $G \cong D_n$ (see Theorem 8). We obtain that $I(G,x) = I(C_6,x)$ if and only if $G \in \{C_6, D_6, K_2 + K_4 \setminus e\}$, where e is an edge of K_4. We find that $I(G,x) = I(C_9,x)$ if and only if $G \in \{C_9, D_9, H_1, H_2, H_3\}$, where H_1, H_2 and H_3 have been shown in Figure 4. In fact $I(C_6,x) = 1+6x+9x^2+2x^3 = (1+4x+x^2)(1+2x) = I(K_4 \setminus e,x)I(K_2,x)$ and $I(C_9,x) = 1+9x+27x^2+30x^3+9x^4 = (1+6x+9x^2+3x^3)(1+3x)$. These examples show that the structure of all non-connected graphs G with $I(G,x) = I(C_m,x)$ is not clear, where $m \geq 10$.

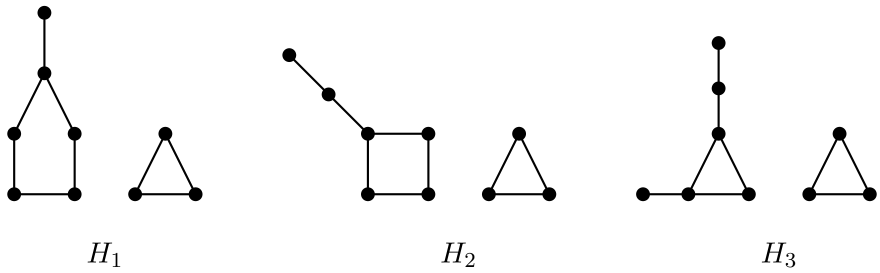


Figure 4. All non-connected graphs G such that $I(G,x) = I(C_9,x)$.

We finish the paper by the following problem.

Problem. Let $n \geq 10$ be an integer. Find all non-connected graphs G such that $I(G,x) = I(C_n,x)$.

Acknowledgements

The author is grateful to the referees for their helpful comments. This research was in part supported by a grant (No. 96050011) from School of Mathematics, Institute for Research in Fundamental Sciences (IPM).

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Received 14 September 2016
Revised 10 January 2017
Accepted 10 January 2017